University of Toronto Colloquium 15 November 2012

Quantum Chaos and the Foundations of Statistical Mechanics

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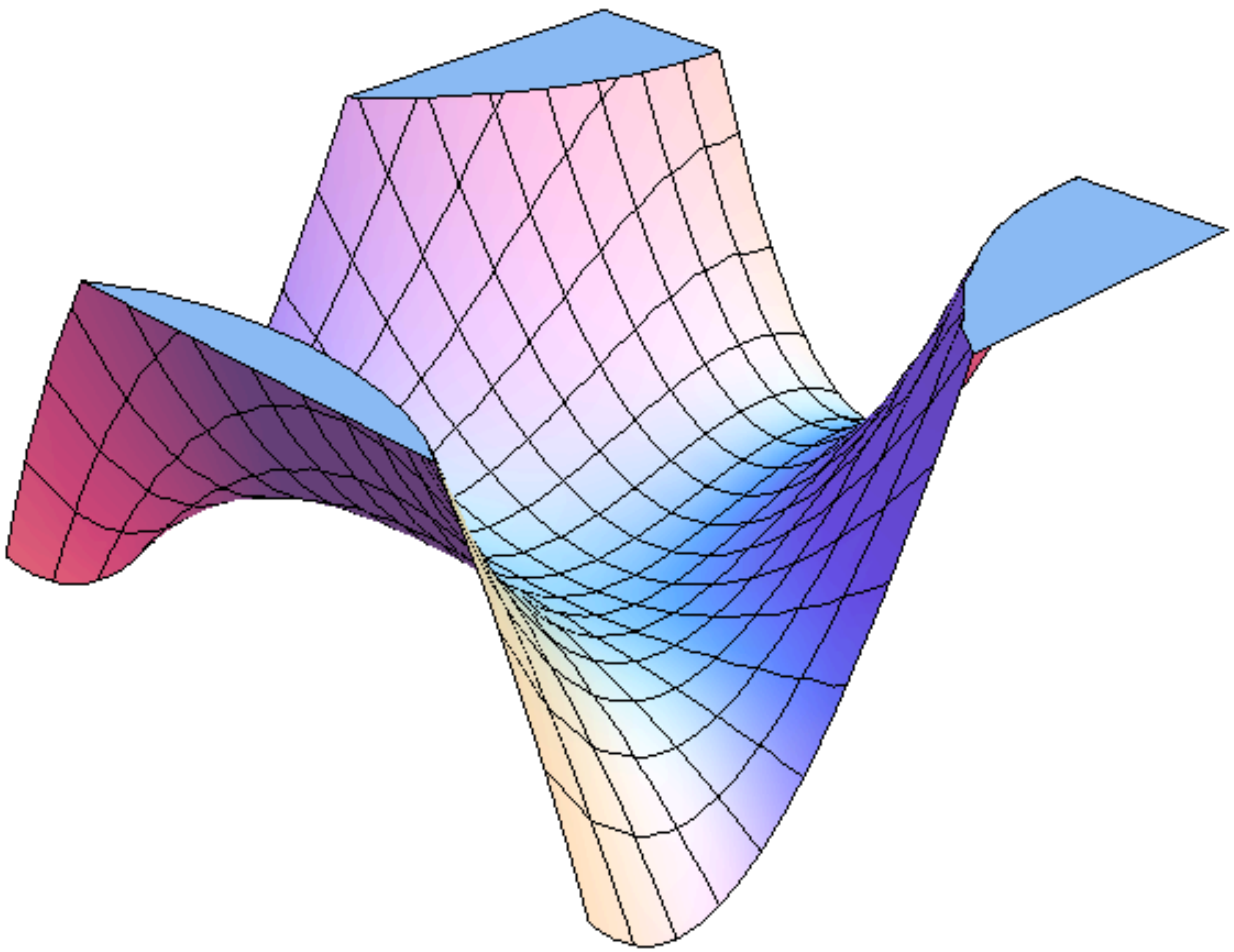
UCSB

Classical chaotic motion

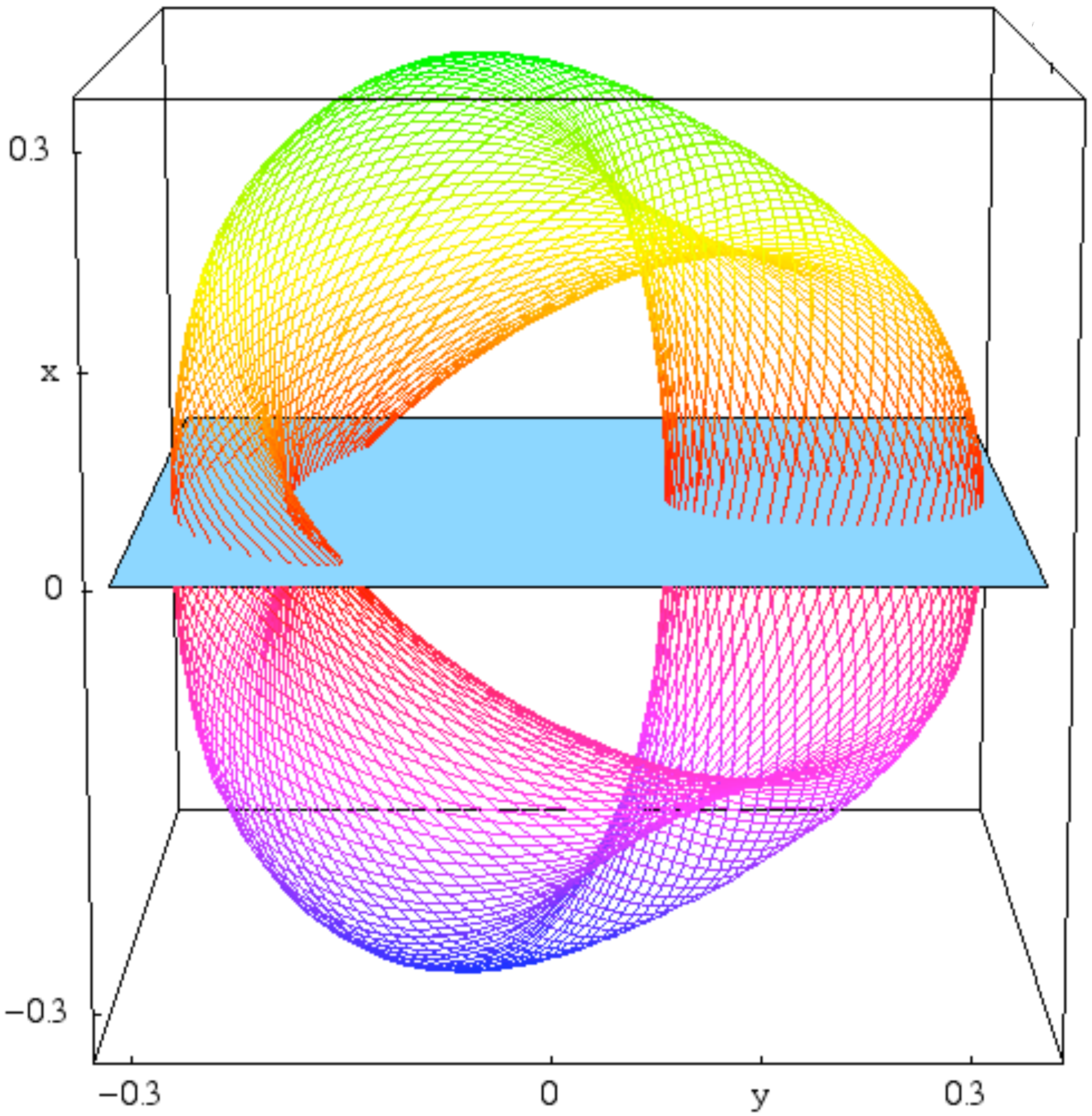
Classical chaotic motion

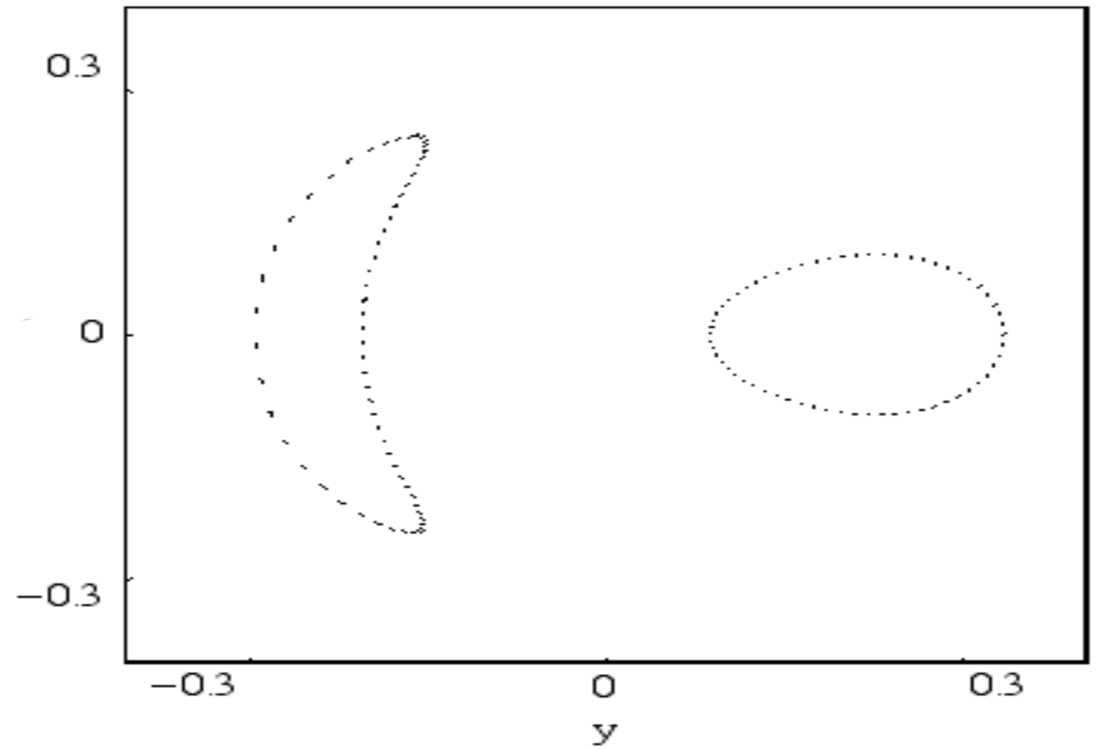
Hénon-Heiles potential:

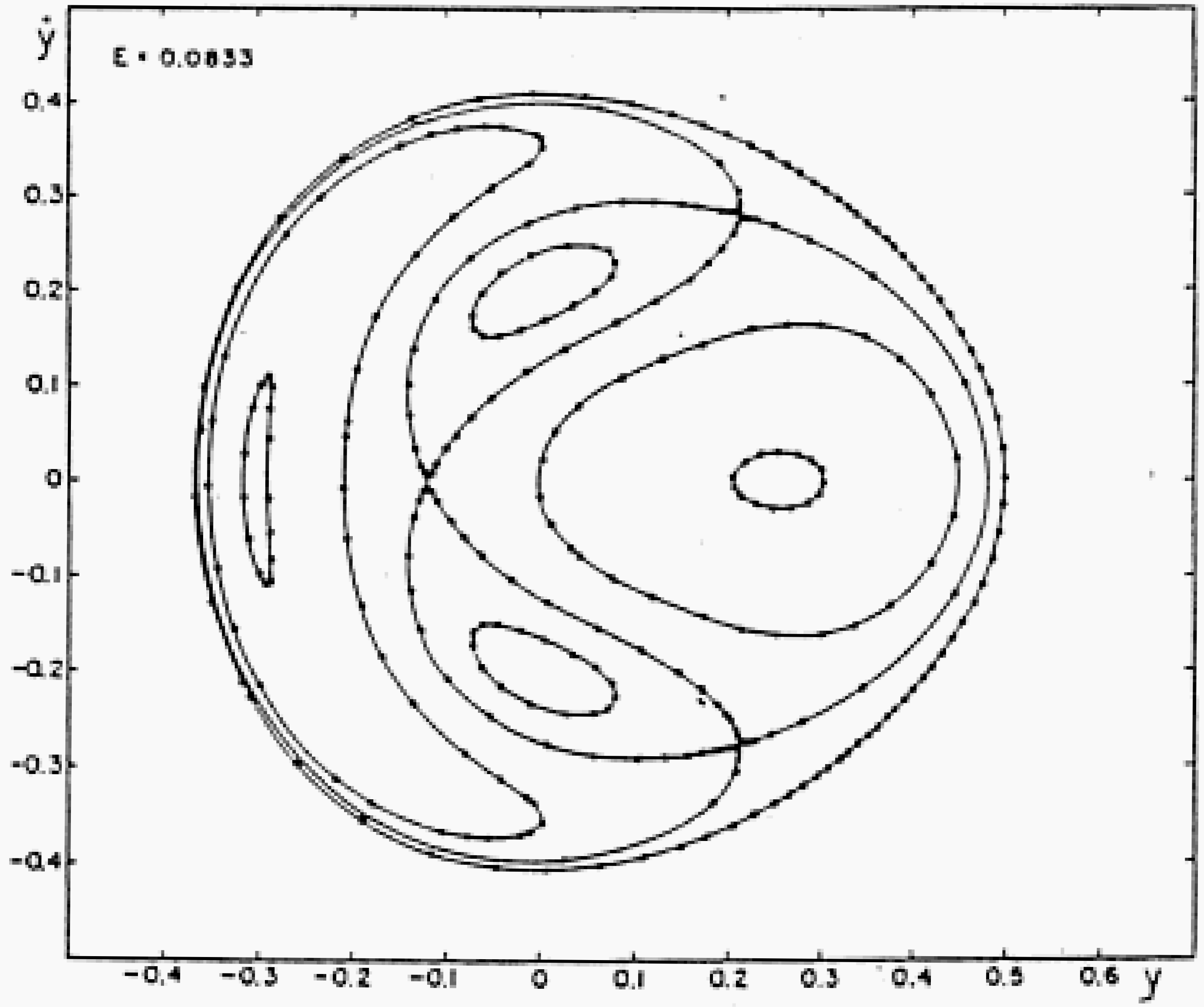
$$V(x,y) = \frac{1}{2}(x^2 + y^2) + x^2 y - \frac{1}{3}y^3 = \frac{1}{2}r^2 + \frac{1}{3}r^3 \sin(3\theta)$$



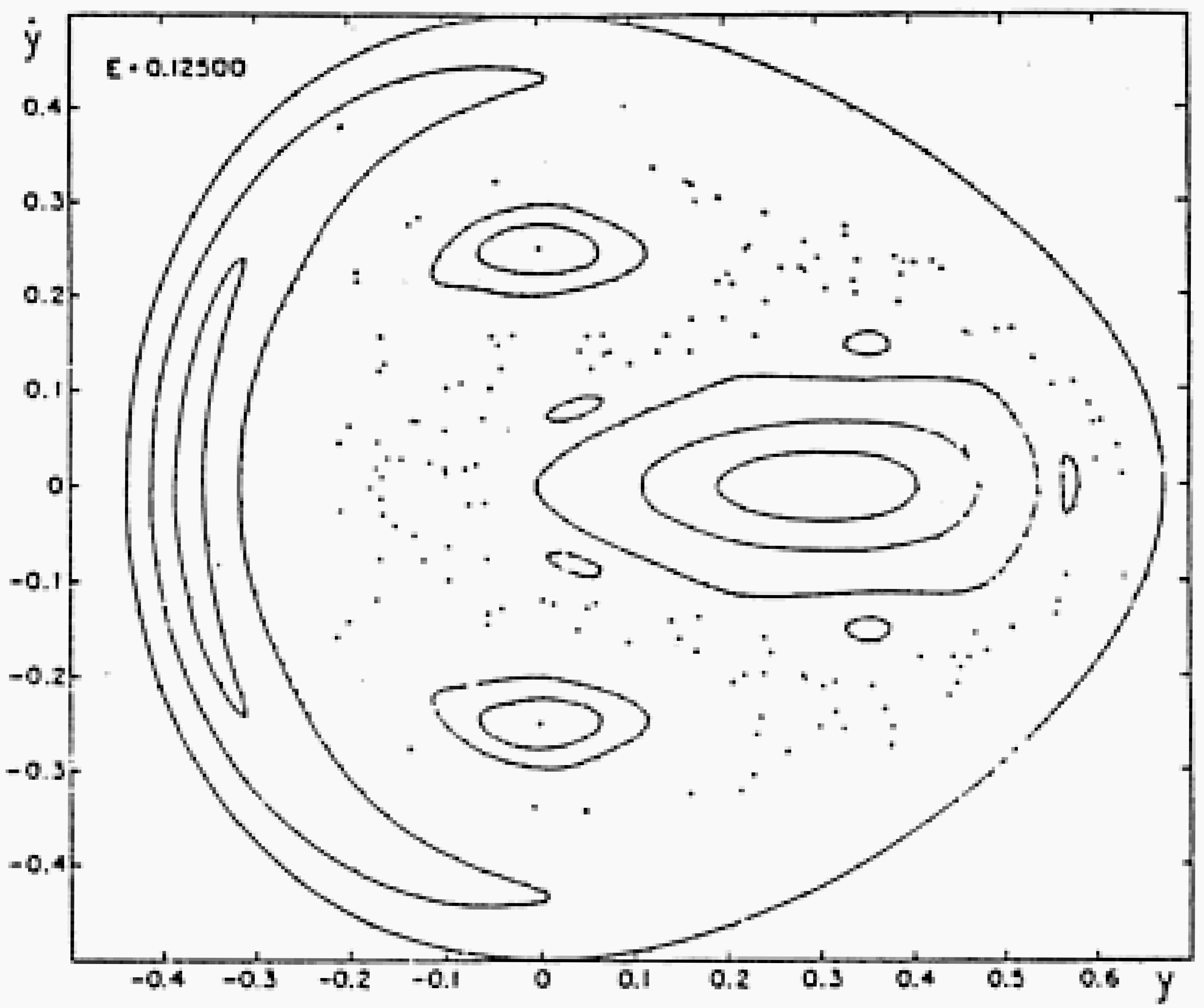
Poincaré section:
values of y & $\dot{y}$ when x=0



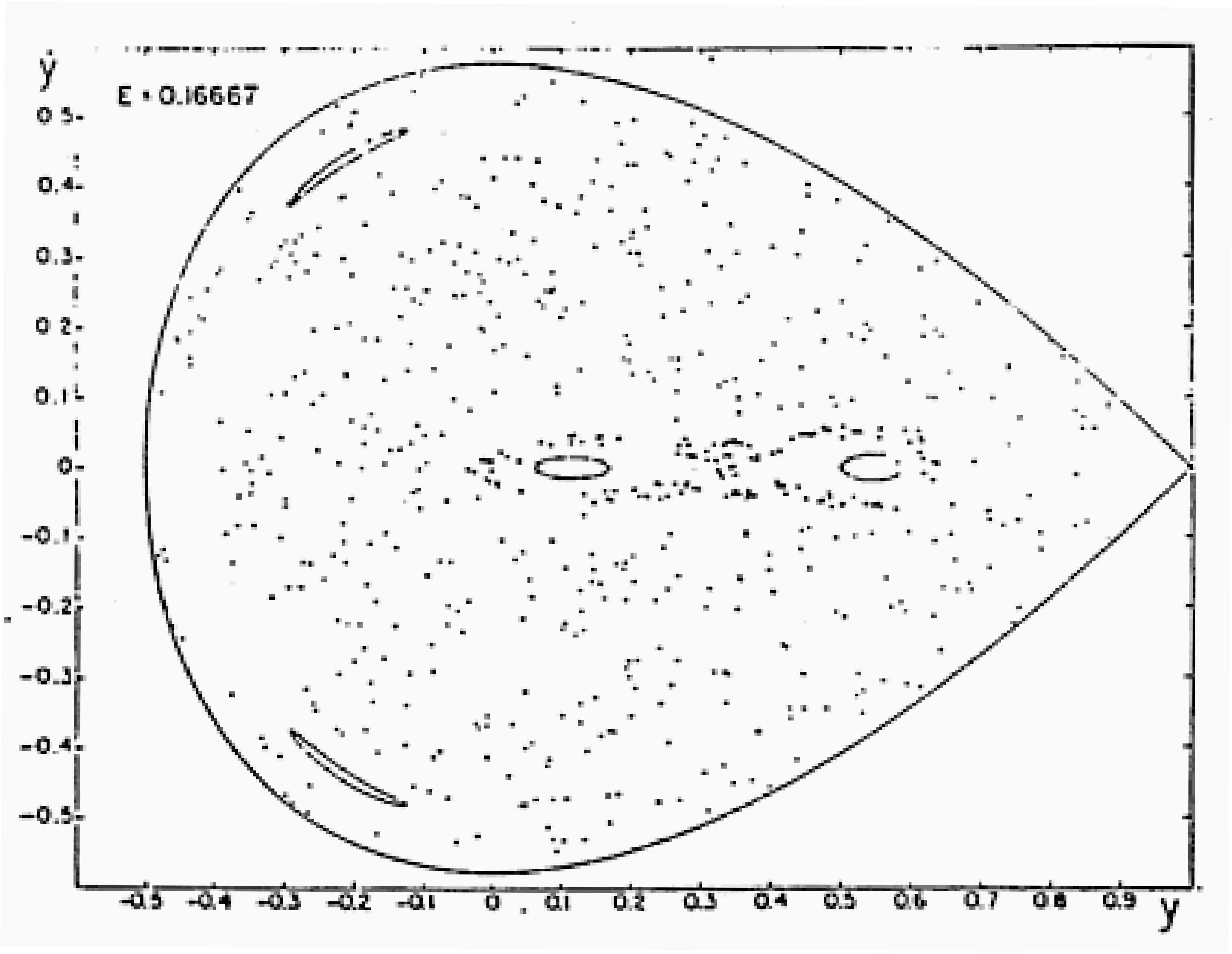




ẏ
E = 0.0833
0.4
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
-0.4
-0.3
-0.2
-0.1
0
0.1
0.2
0.3
0.4
0.5
0.6
y



ẏ
E = 0.12500
0.4
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
-0.4
-0.3
-0.2
-0.1
0
0.1
0.2
0.3
0.4
0.5
0.6
y

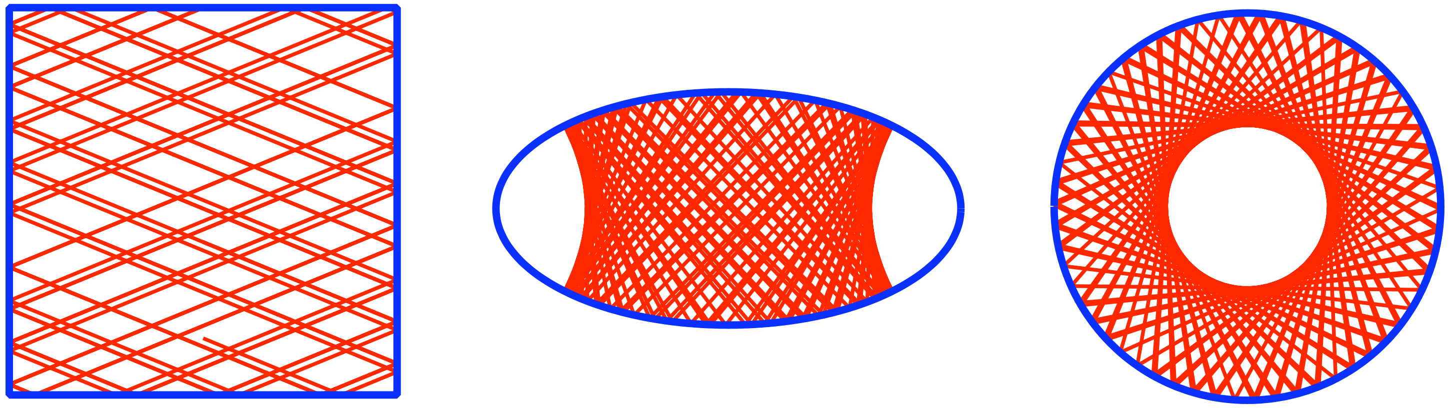


ẏ
E = 0.16667
0.5
0.4
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
-0.5
-0.5
-0.4
-0.3
-0.2
-0.1
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
y

Billiard systems

Billiard systems

Integrable:



Billiard systems

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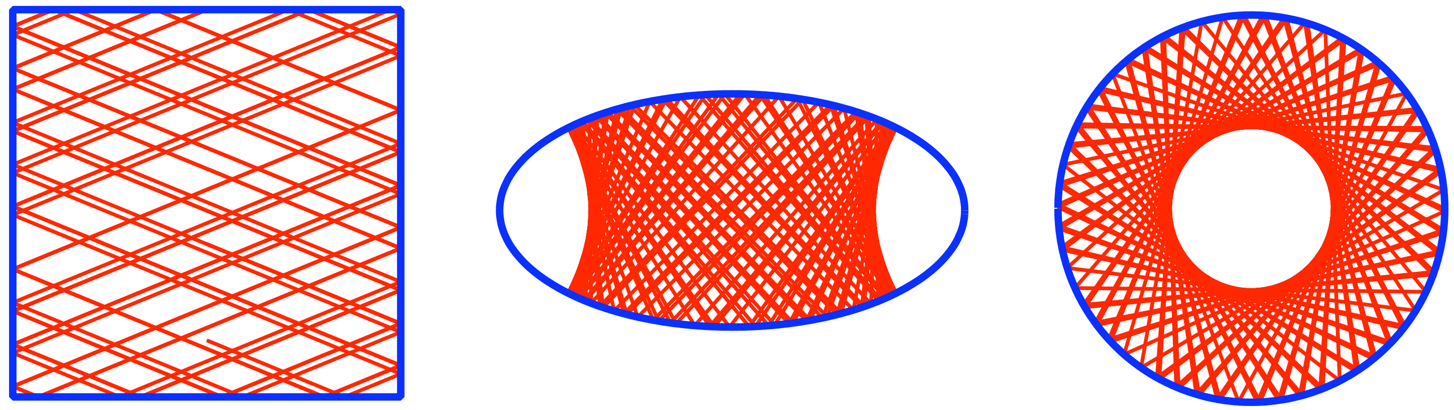
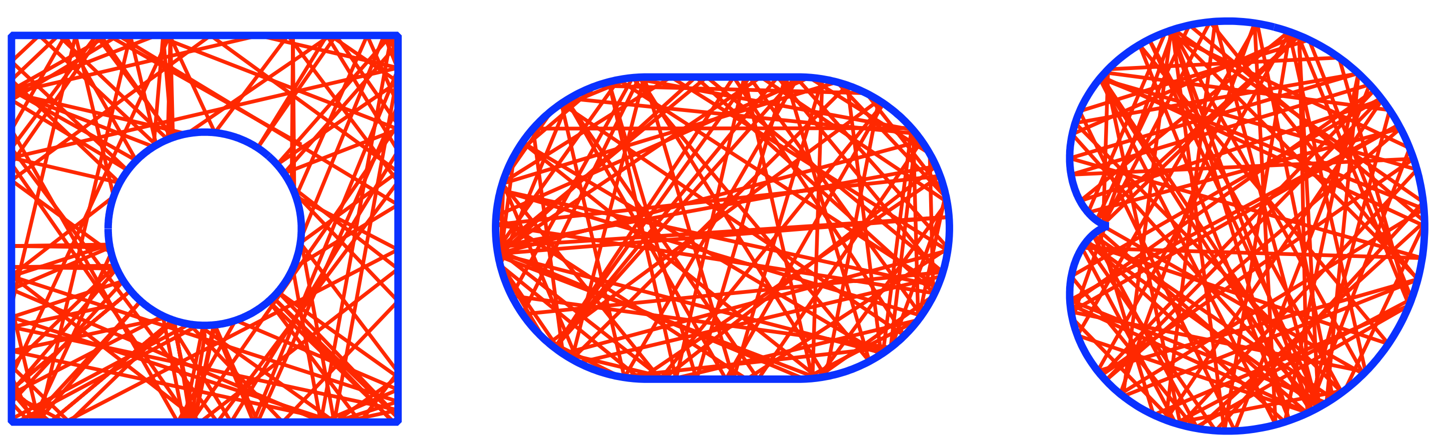


Chaotic:



graphs by Arnd Baecker

Quantum Mechanics

Quantum Mechanics

Given a hamiltonian *H*,

we find its

EIGENVALUES

and

EIGENSTATES:

$$H|\alpha\rangle = E_\alpha|\alpha\rangle$$

QUANTUM MECHANICS

Given a hamiltonian *H*,

we find its

EIGENVALUES

and

EIGENSTATES:

$$H|\alpha\rangle = E_\alpha|\alpha\rangle$$

Then

$$|\psi_t\rangle = \sum_\alpha c_\alpha e^{-iE_\alpha t/\hbar}|\alpha\rangle$$

Do **chaotic** systems have any QUANTUM signatures

?

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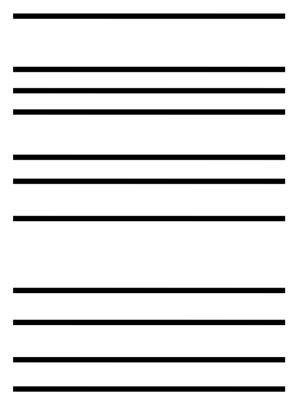
have any

QUANTUM signatures

?

Yes!

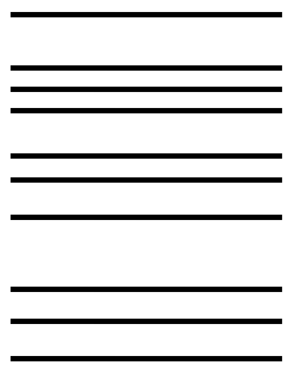
Energy eigenvalue statistics:



s = nearest neighbor spacing

$P(s) = ?$

Energy eigenvalue statistics:



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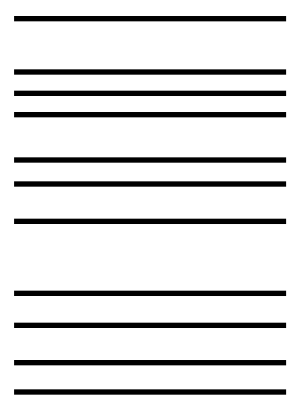
Integrable system: $P(s) \sim e^{-s}$

Energy eigenvalue statistics:



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$P(s) = ?$

Integrable system: $P(s) \sim e^{-s}$

Chaotic system: $P(s) \sim s^{\beta} e^{-s^2}$

$\beta = 1, 2, 4$

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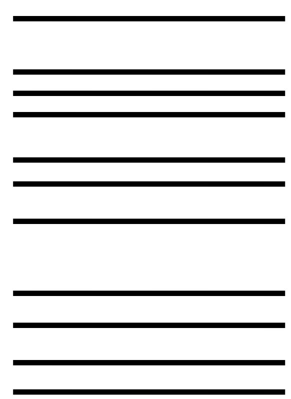

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Wigner-Dyson distribution
Random Matrix Theory
GOE, GUE, GSE

Example:

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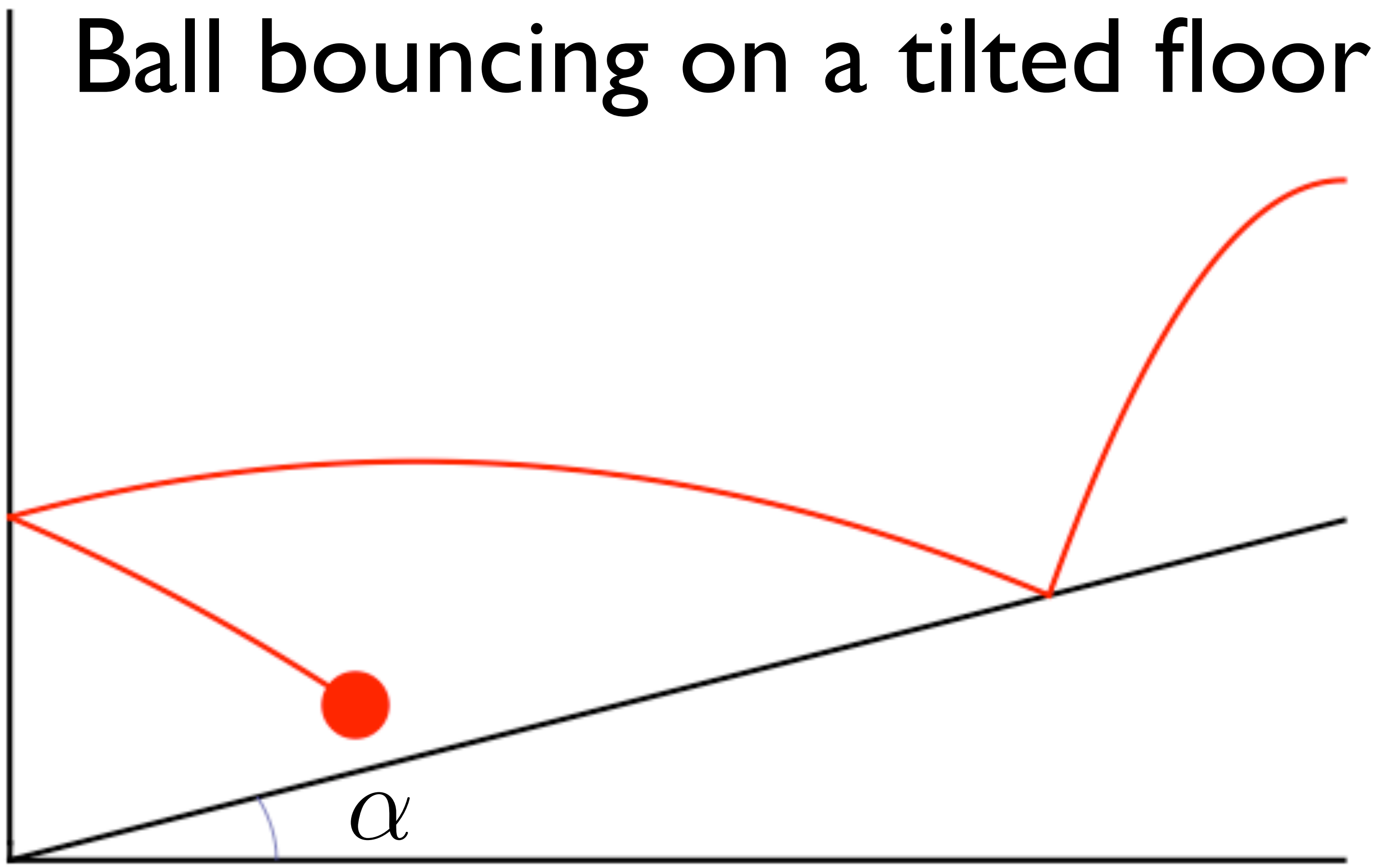
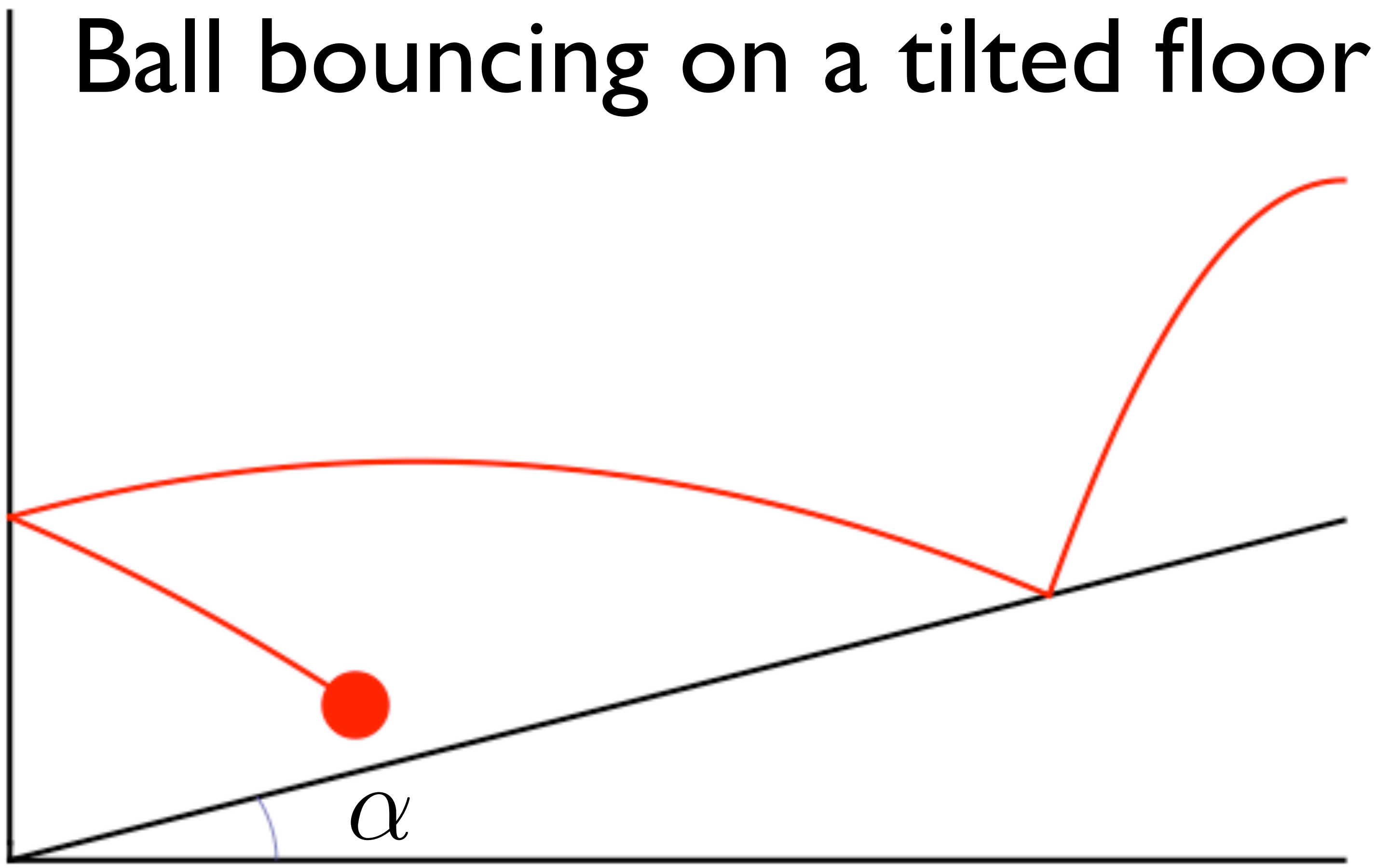


Example:

Ball bouncing on a tilted floor

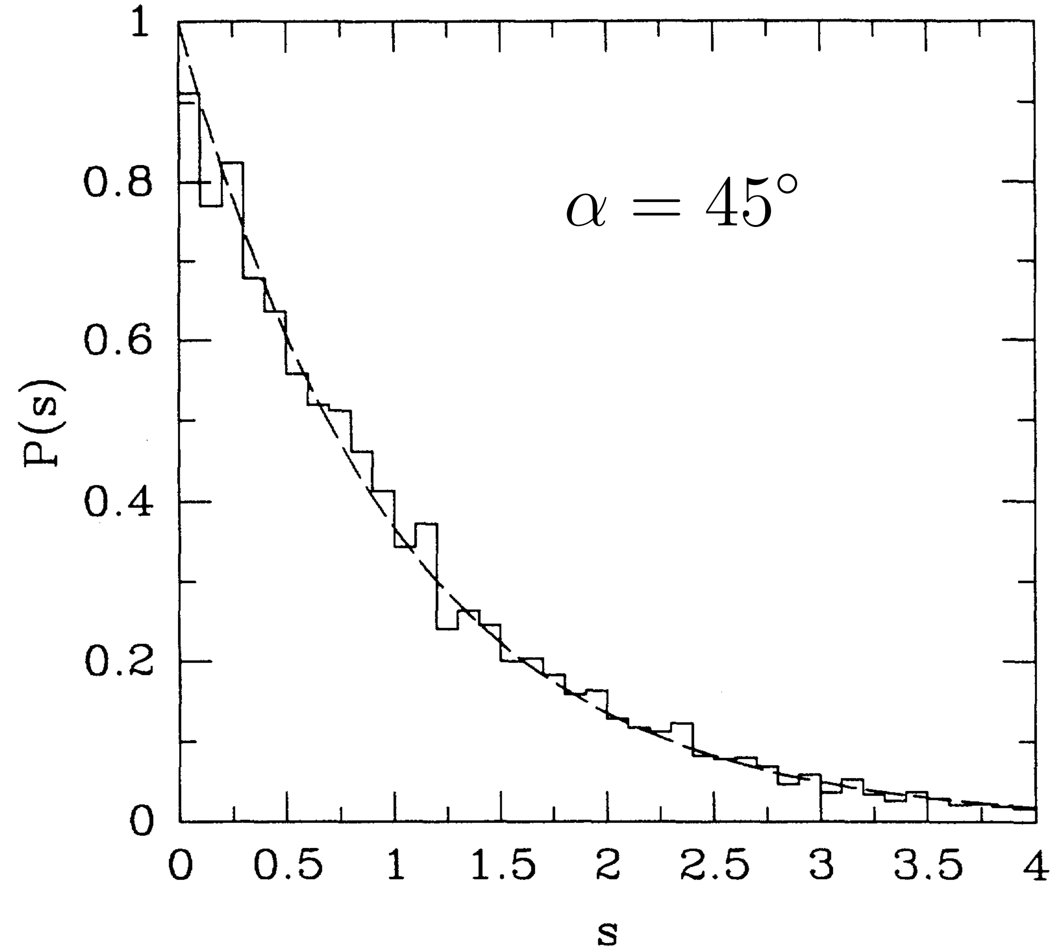


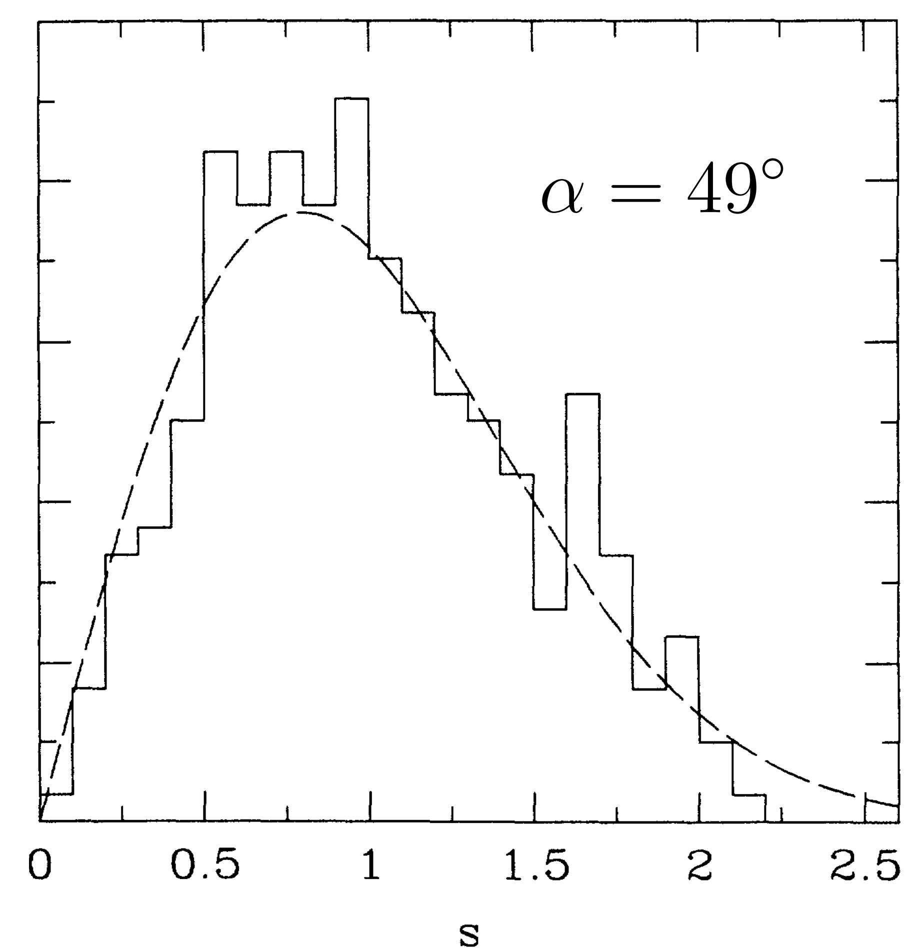
$$\underline{\text{integrable}} \text{ if } \alpha \leq 45^\circ$$

$$\underline{\text{chaotic}} \text{ if } \alpha > 45^\circ$$

Szeredi & Goodings '93

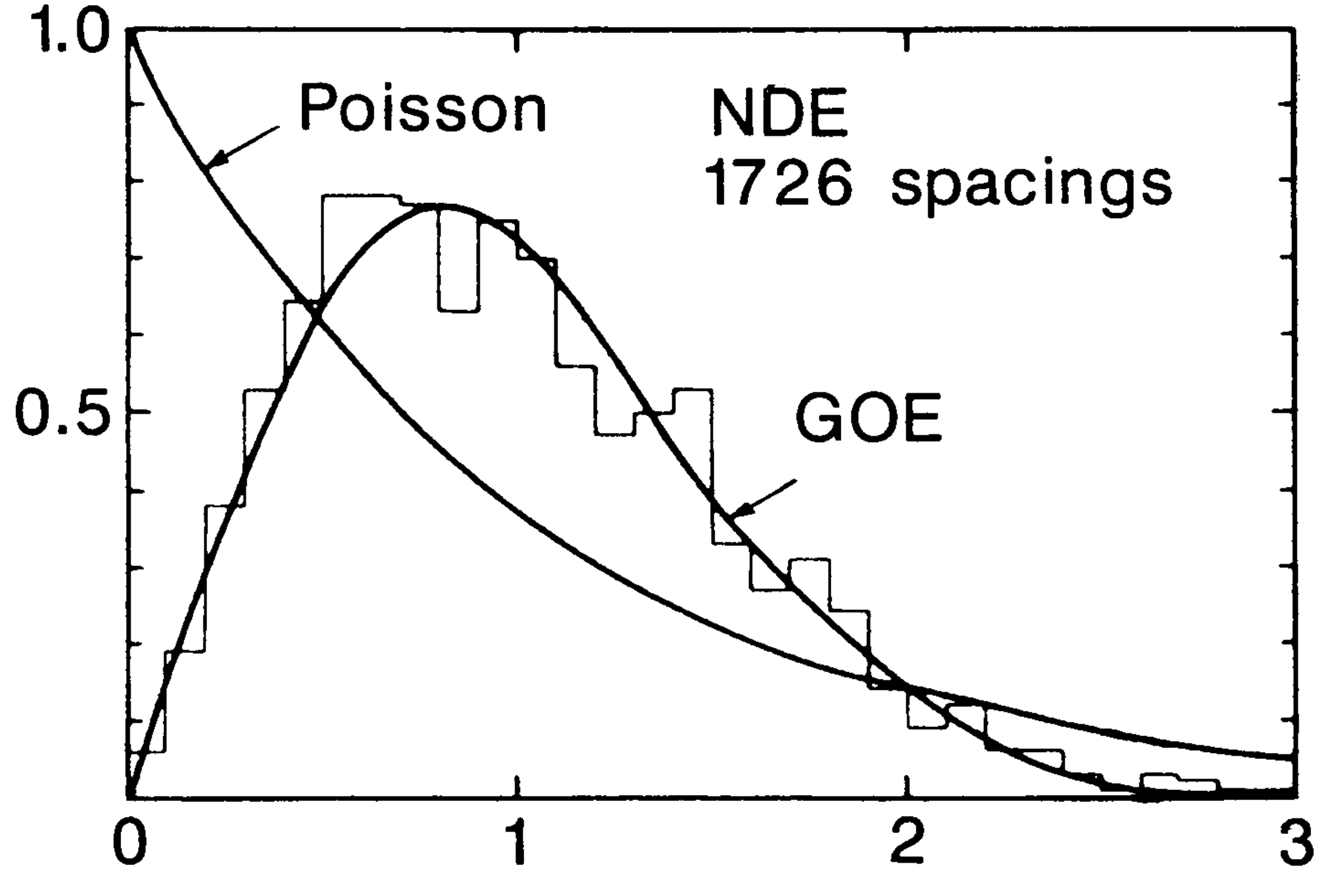
Quantum level spacings:





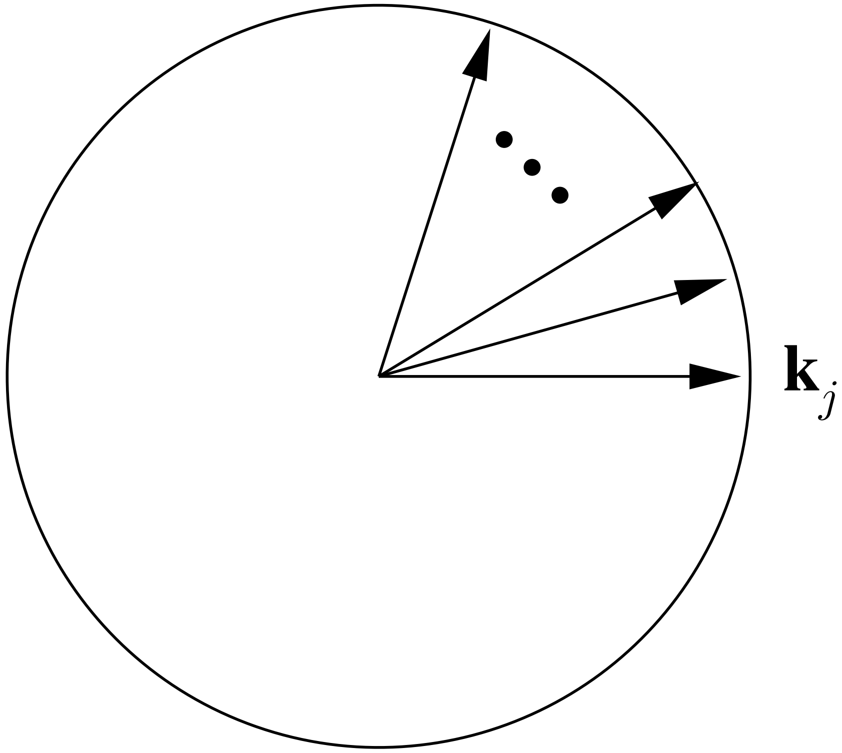
Szeredi & Goodings '93

Level spacings in nuclei:



Quantum energy eigenfunctions

$$\psi(\mathbf{x}) = \sum_{j=1}^{N} A_j e^{i\mathbf{k}_j \cdot \mathbf{x}}$$



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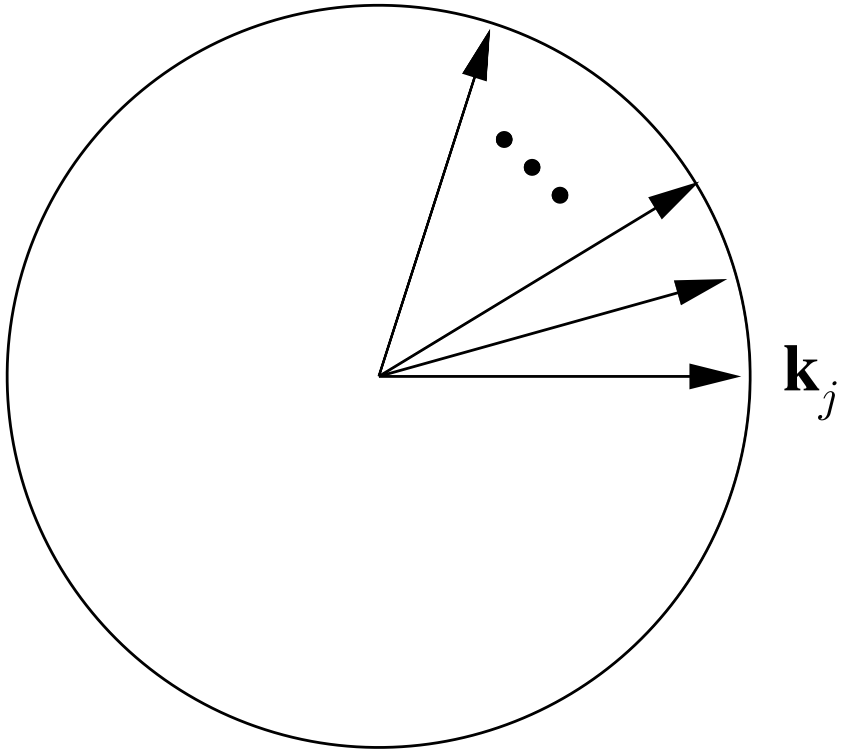
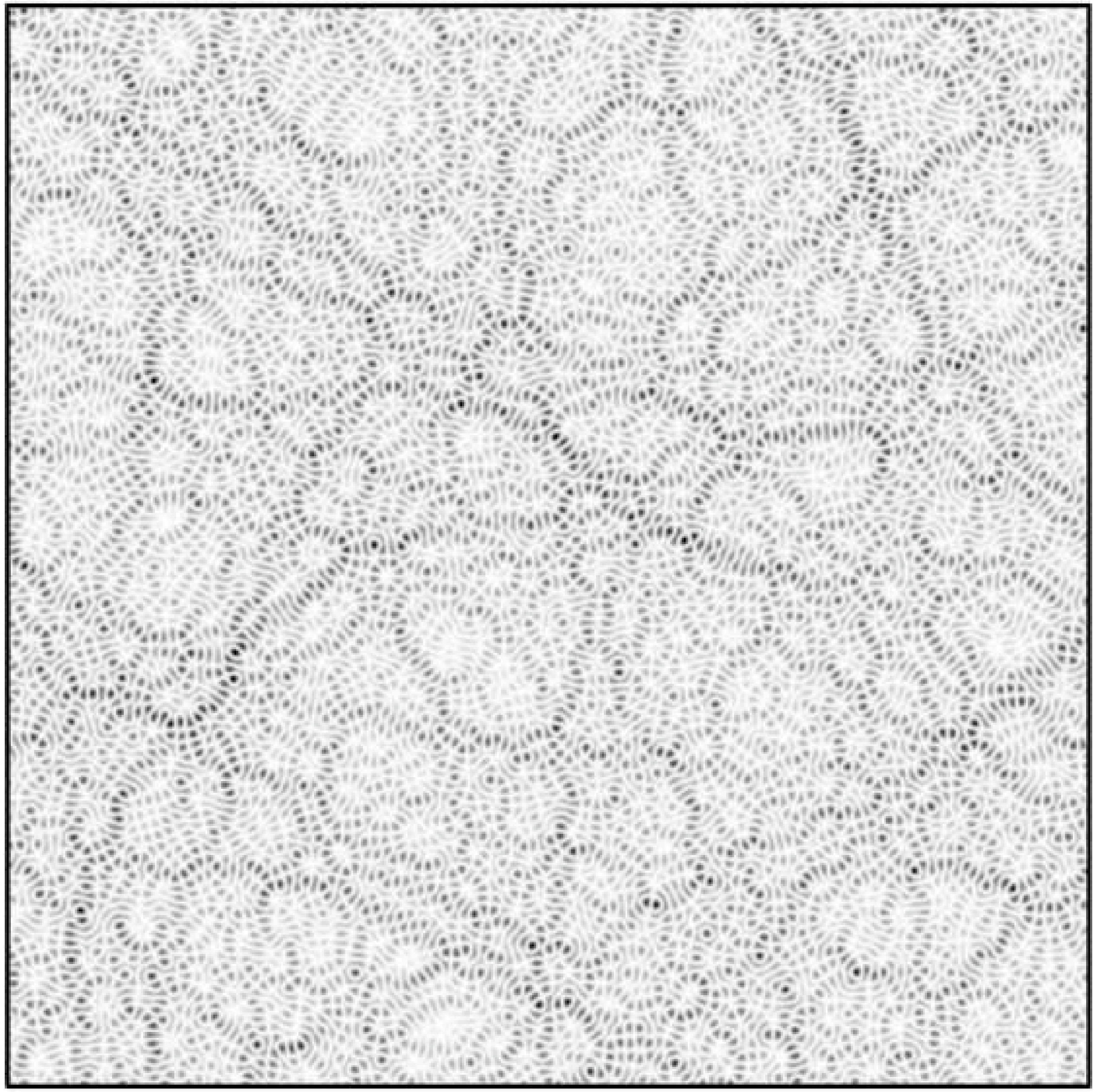


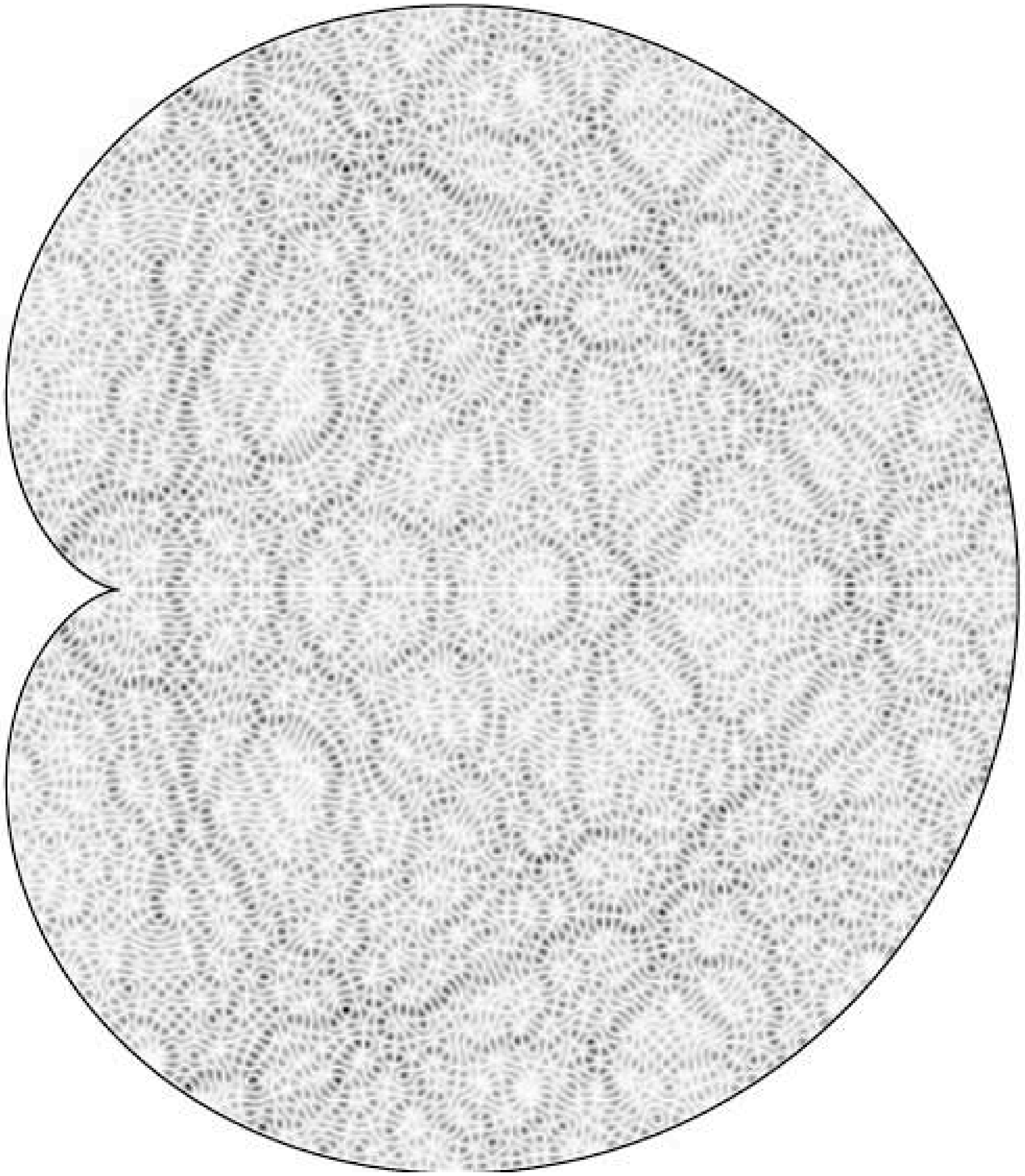
Berry's random wave conjecture:
A_j's are gaussian random

Berry '77

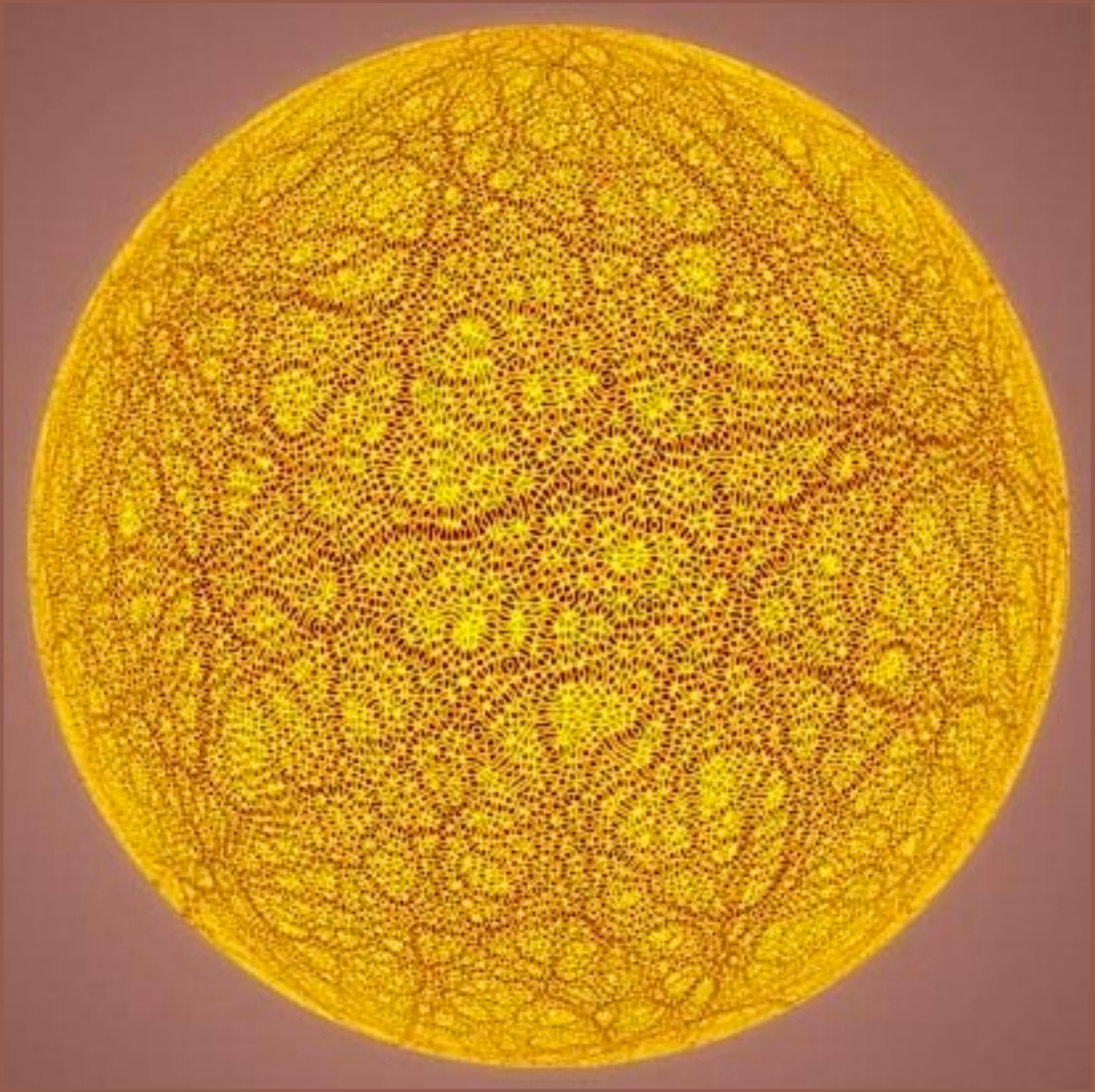
Random waves:



Cardioid billiard eigenfunction:



Baecker '03



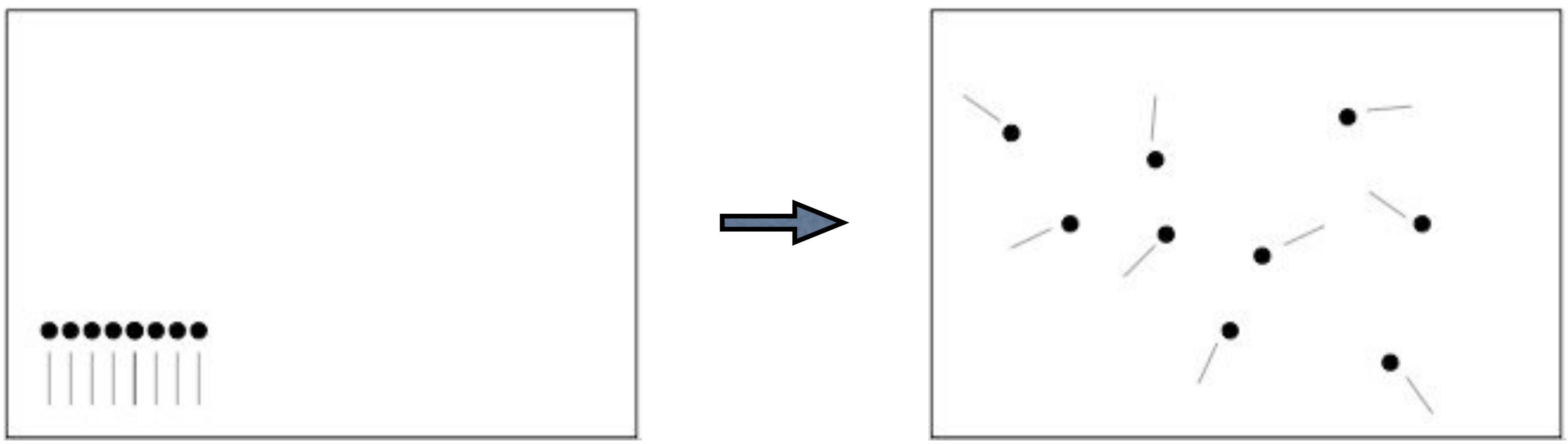
Random waves on a sphere
(Eric Heller)

QUANTUM CHAOS

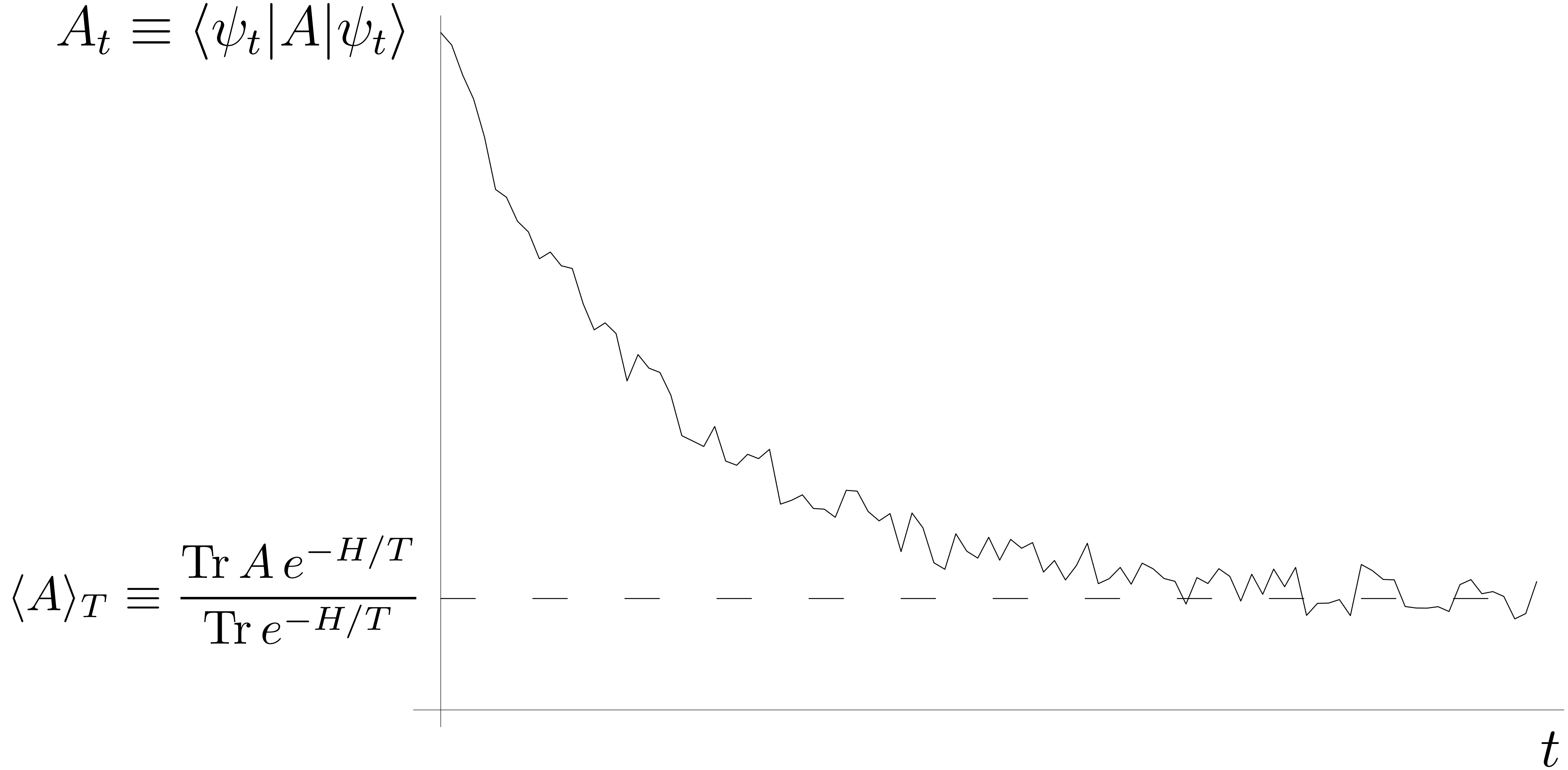
and

STATISTICAL MECHANICS

Dilute gas in a box:



Let $A = a_{\mathrm{p}}^\dagger a_{\mathrm{p}}$



Let's compute

$$A_t \equiv \langle \psi_t | A | \psi_t \rangle$$

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$$\overline{A} \equiv \lim_{\tau\to\infty} \frac{1}{\tau} \int_0^\tau dt \; A_t$$

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$$= \sum_\alpha |c_\alpha|^2 A_{\alpha\alpha}$$

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$$= \langle A \rangle_T \quad ???$$

Shnirelman's theorem:

$$\langle\alpha|A|\alpha\rangle \rightarrow \text{classical, microcanonical average of } A$$
$$+ O(\hbar^{1/2})$$

if system is chaotic and $A = \hbar$-independent operator

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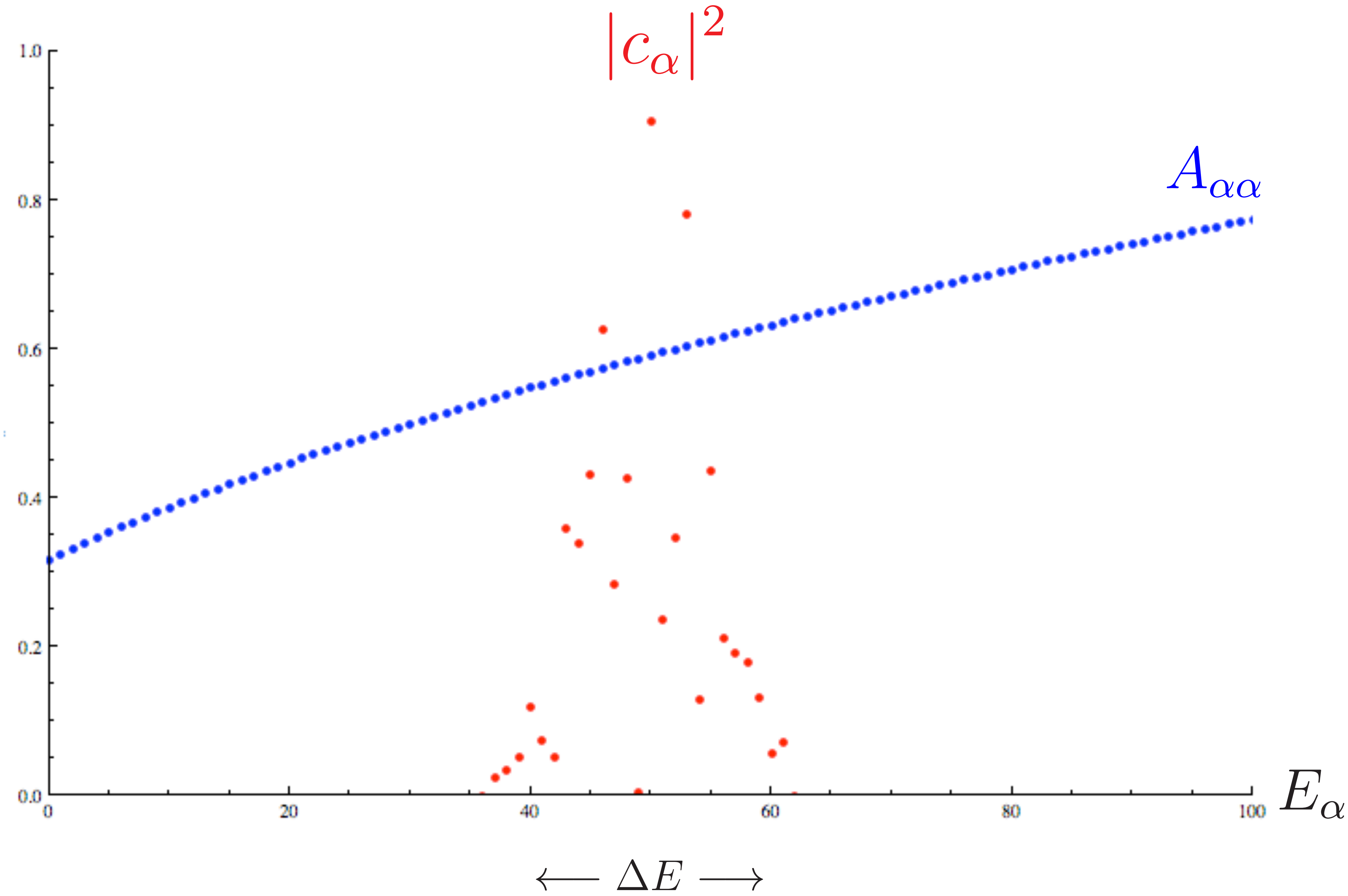
$$\langle\alpha|A|\alpha\rangle \to \text{classical, microcanonical average of } A$$
$$+\,O(\hbar^{1/2})$$

if system is chaotic and $A = \hbar$-independent operator

But all we will need is

$$\boxed{\langle\alpha|A|\alpha\rangle = O(\hbar^0),\ \text{varies smoothly with } E_\alpha}$$

which follows from the random-wave conjecture



$|c_\alpha|^2$
$A_{\alpha\alpha}$
1.0
0.8
0.6
0.4
0.2
0.0
0
20
40
60
80
100
E_α
$\longleftarrow \Delta E \longrightarrow$

If ΔE is "small", then

$$\overline{A} = \sum_{\alpha} |c_\alpha|^2 A_{\alpha\alpha} \quad \underline{\text{is independent of the } c_\alpha\text{'s}}$$

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So let $|c_\alpha|^2 =$ Boltzmann weight

$$\implies \quad \Delta E \sim N^{-1/2} E$$

$$\implies \quad \overline{A} = \langle A \rangle_T$$

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$\implies \overline{A} = \langle A \rangle_T$ for *all* c_α's with "small" ΔE

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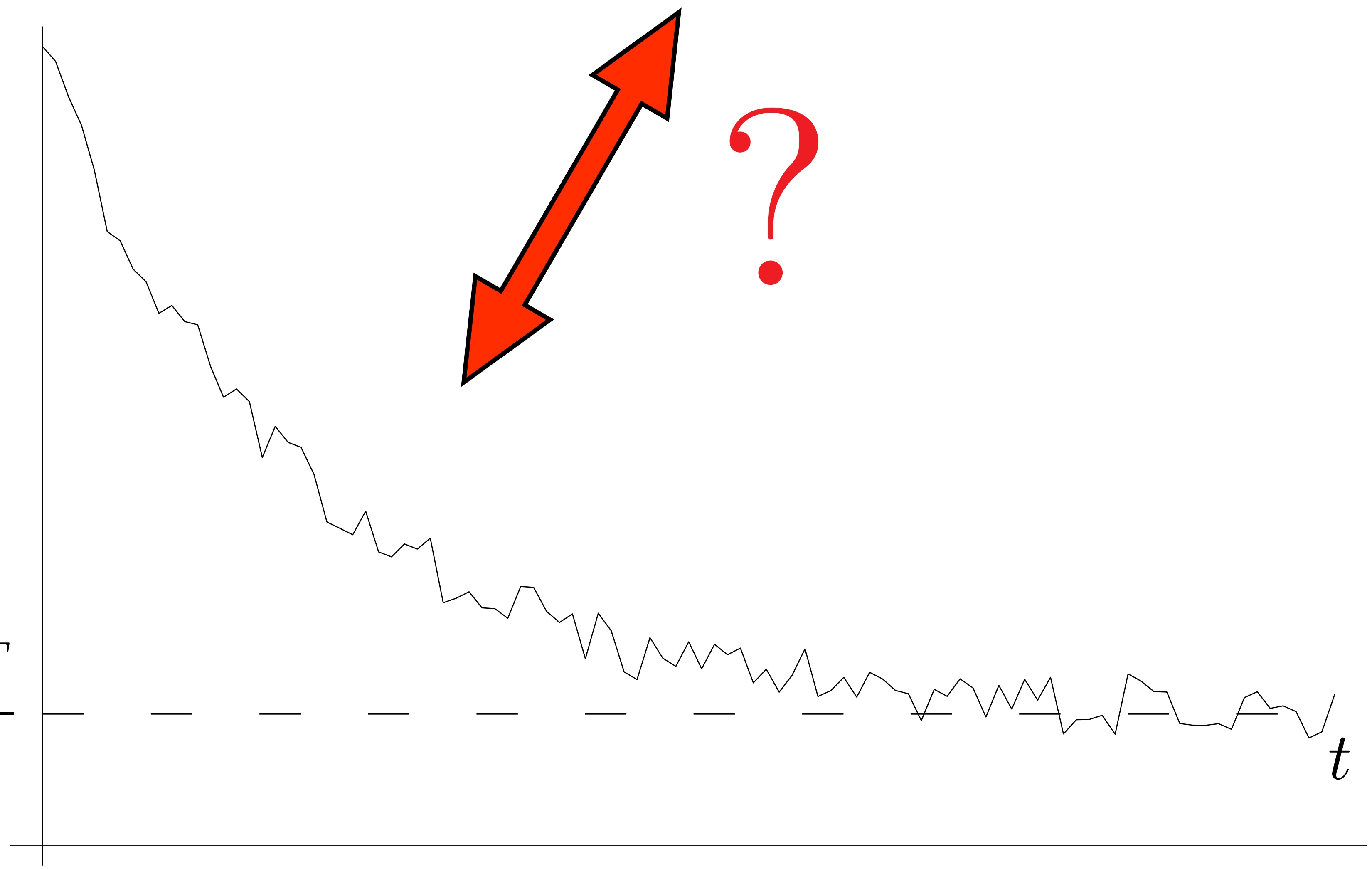
→ "eigenstate thermalization"

Deutsch '91, M.S. '94

Review:

$$A_t = \sum_{\alpha} |c_\alpha|^2 A_{\alpha\alpha} + \sum_{\beta \neq \alpha} c_\alpha^* c_\beta \, e^{i(E_\alpha - E_\beta)t} A_{\alpha\beta}$$

$$A_t \equiv \langle \psi_t | A | \psi_t \rangle$$



$$\langle A \rangle_T \equiv \frac{\mathrm{Tr}\, A\, e^{-H/T}}{\mathrm{Tr}\, e^{-H/T}}$$

Random wave conjecture $\Longrightarrow$

$A_{\alpha\beta} = \langle\alpha|A|\beta\rangle$ varies erratically with α and β

$$O(\hbar^0) = \langle\alpha|(A - \langle A\rangle)^2|\alpha\rangle = \sum_{\beta\neq\alpha} \langle\alpha|A|\beta\rangle\langle\beta|A|\alpha\rangle$$

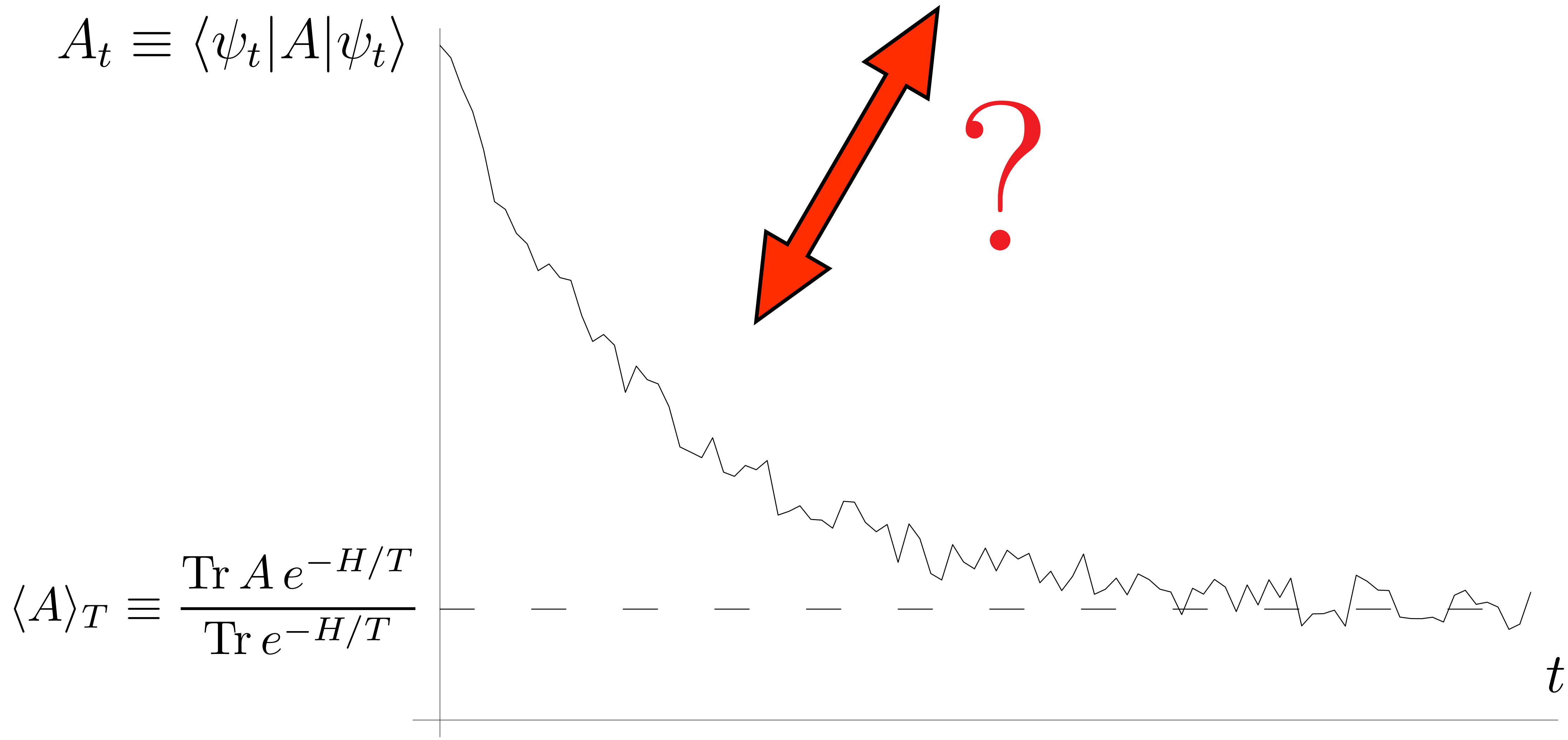
$$= \sum_{\beta\neq\alpha} |A_{\alpha\beta}|^2$$

$$\sim \rho(\bar{E})\ \overline{|A_{\alpha\beta}|^2}$$

$$\sim \hbar^{-(f-1)}\ \overline{|A_{\alpha\beta}|^2}$$

$$\Longrightarrow \quad \boxed{A_{\alpha\beta} \sim \hbar^{(f-1)/2} \sim e^{-S(\bar{E})/2}}$$

$$A_t = \sum_{\alpha} |c_\alpha|^2 A_{\alpha\alpha} + \sum_{\beta \neq \alpha} c_\alpha^* c_\beta \, e^{i(E_\alpha - E_\beta)t} A_{\alpha\beta}$$



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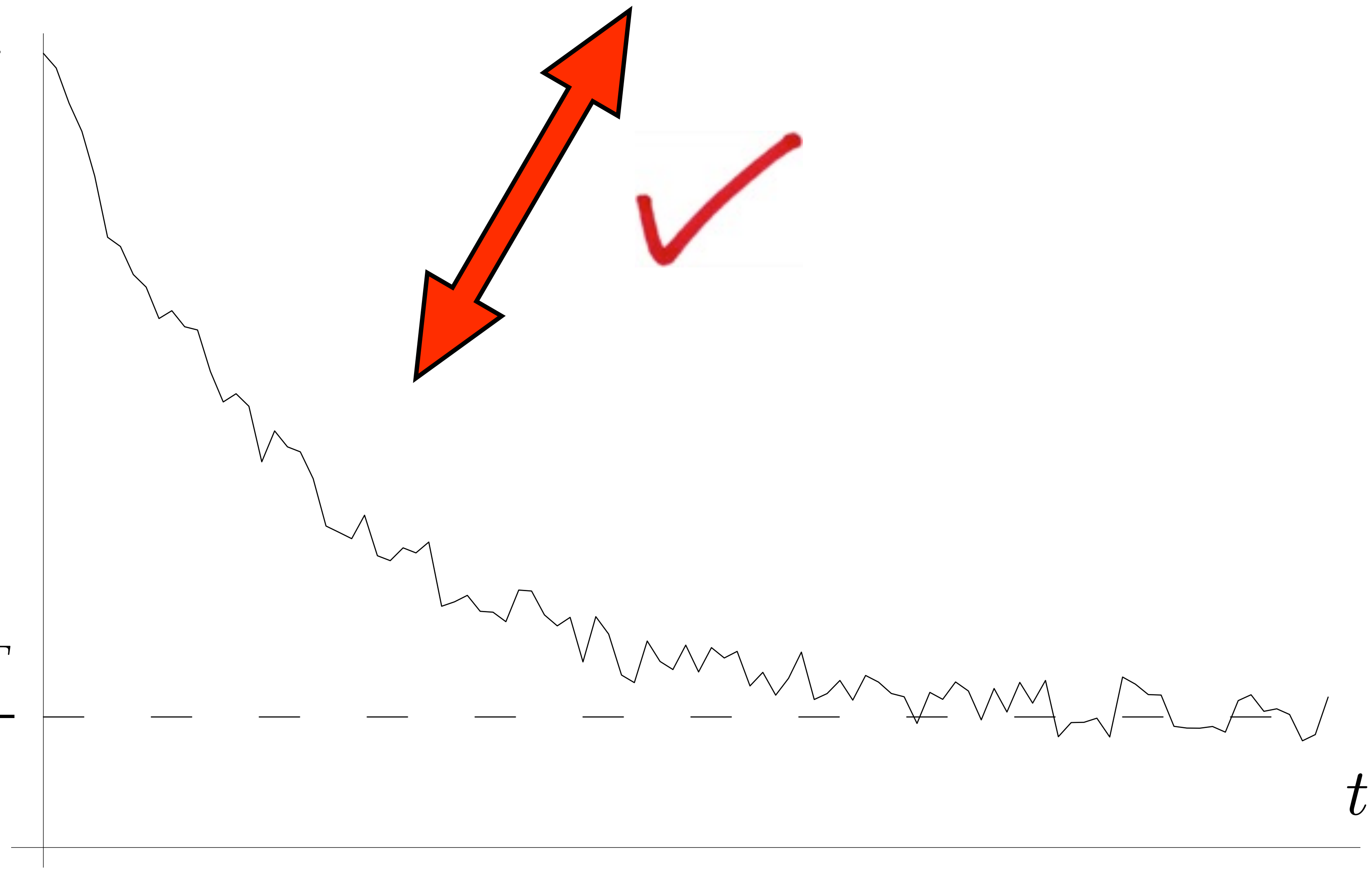

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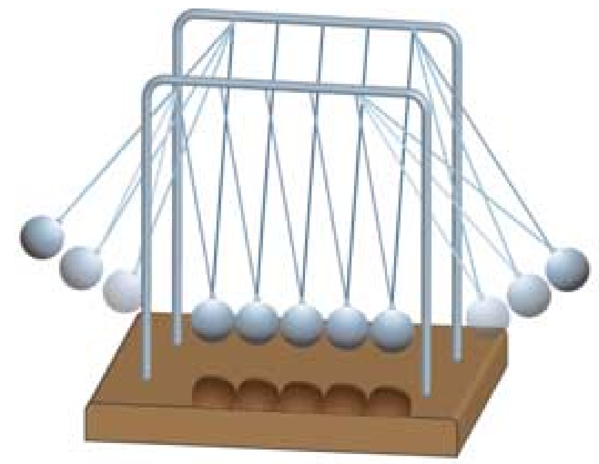
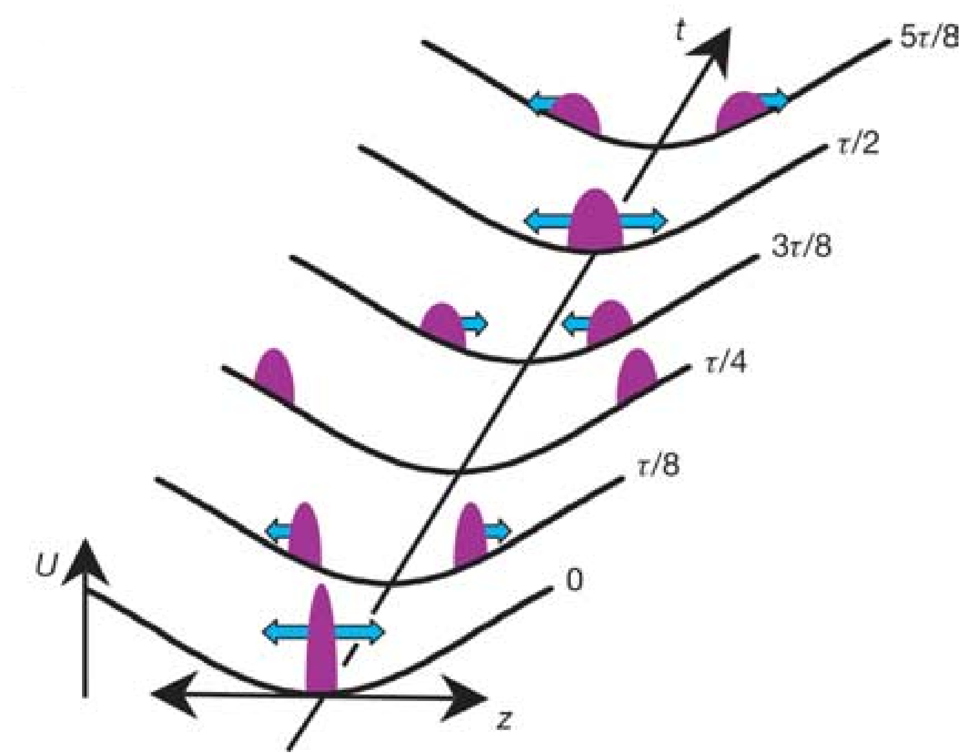
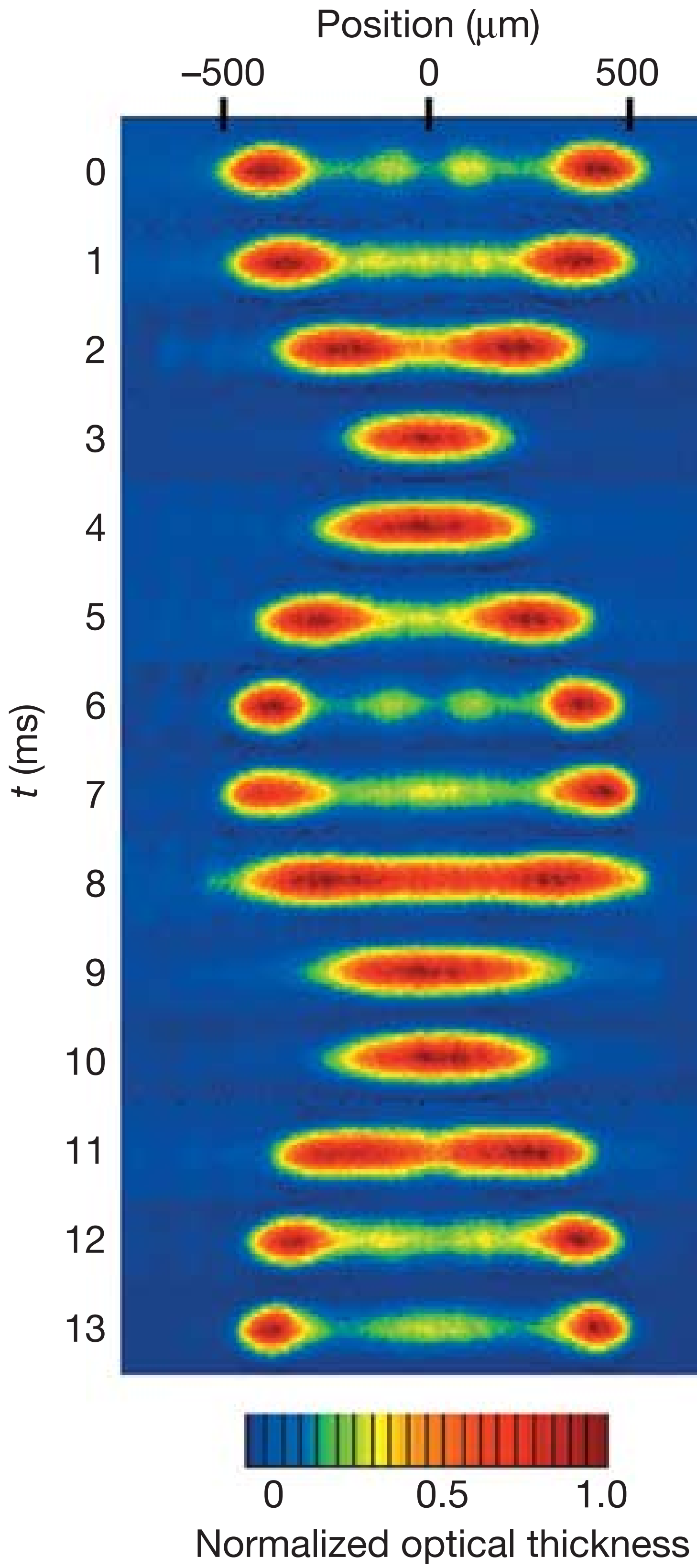
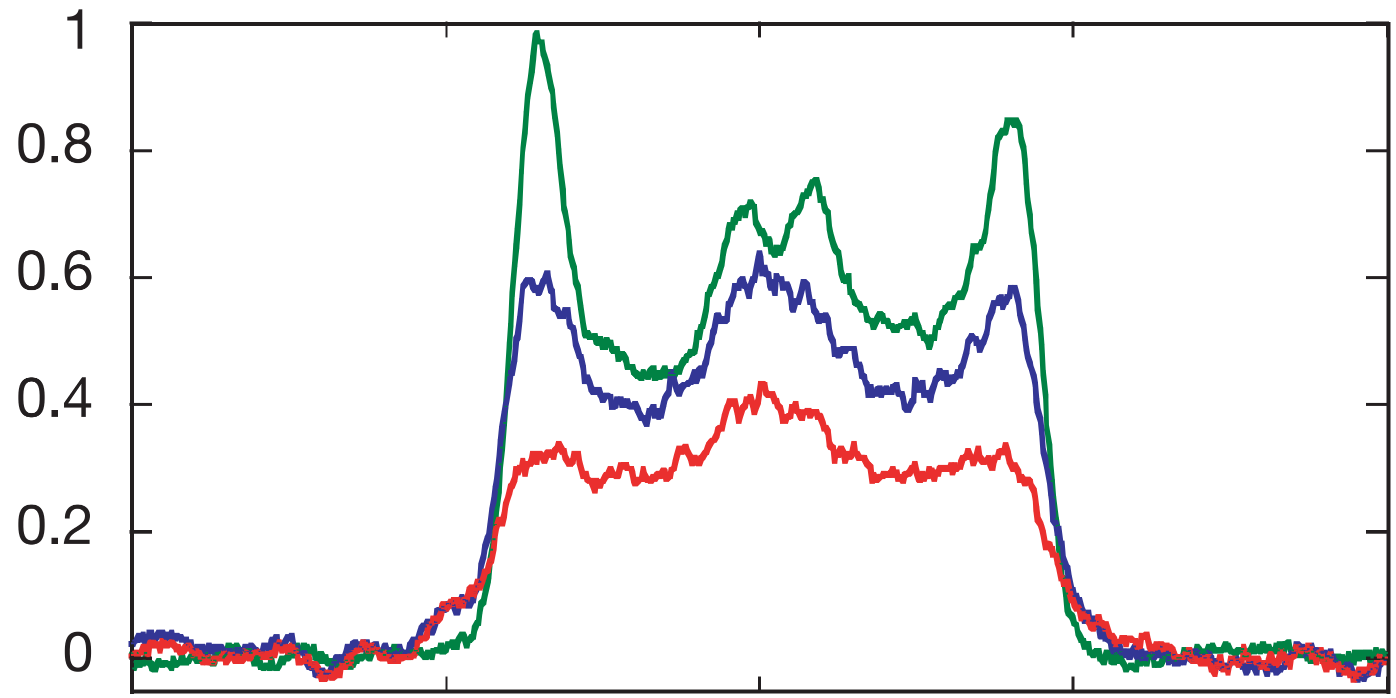
“A quantum Newton’s cradle”:





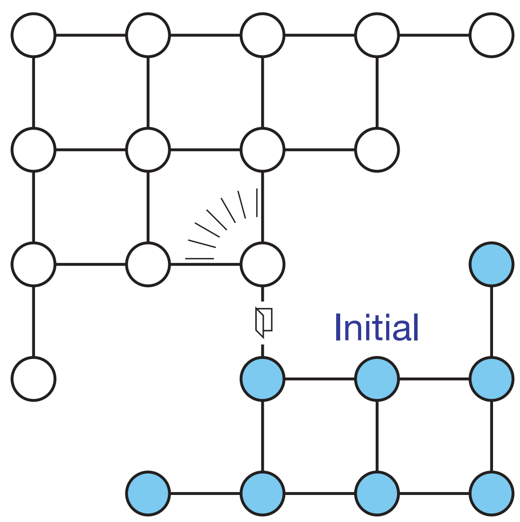
Kinoshita, Wenger, & Weiss (2006)

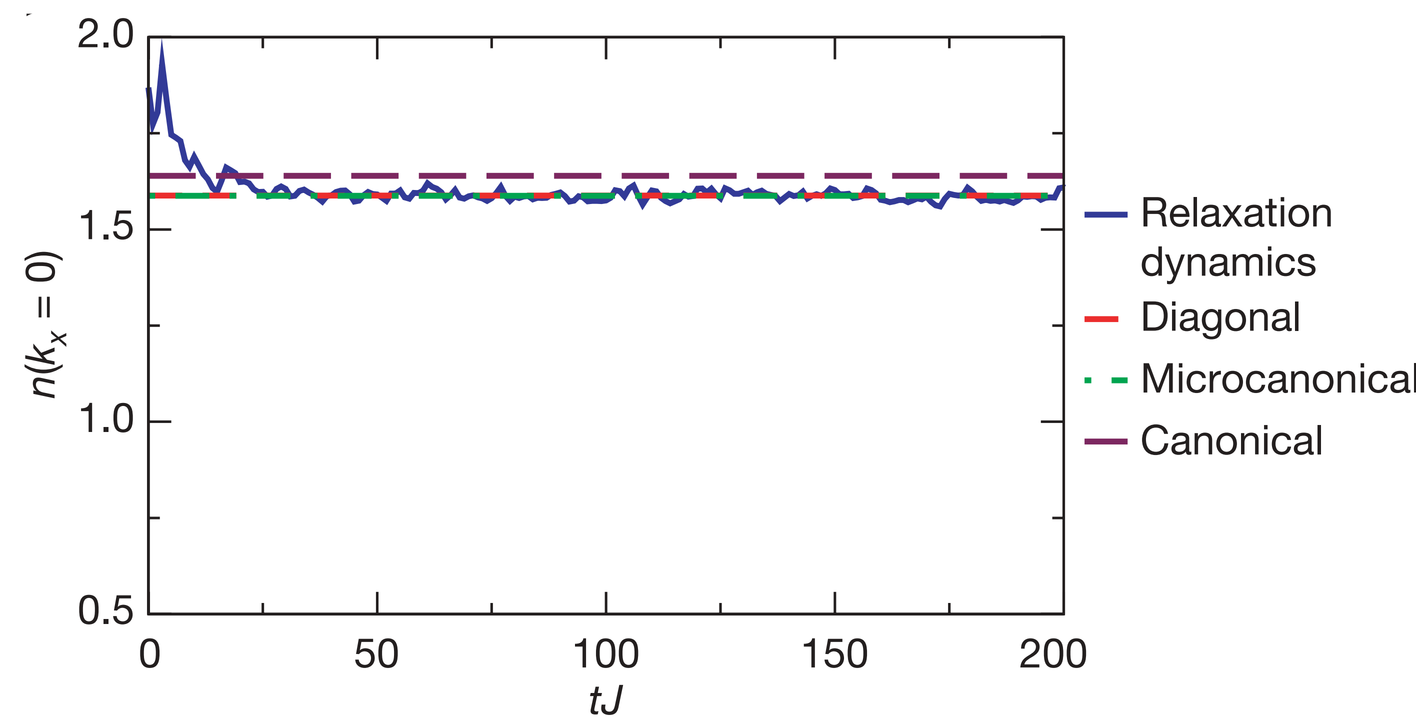




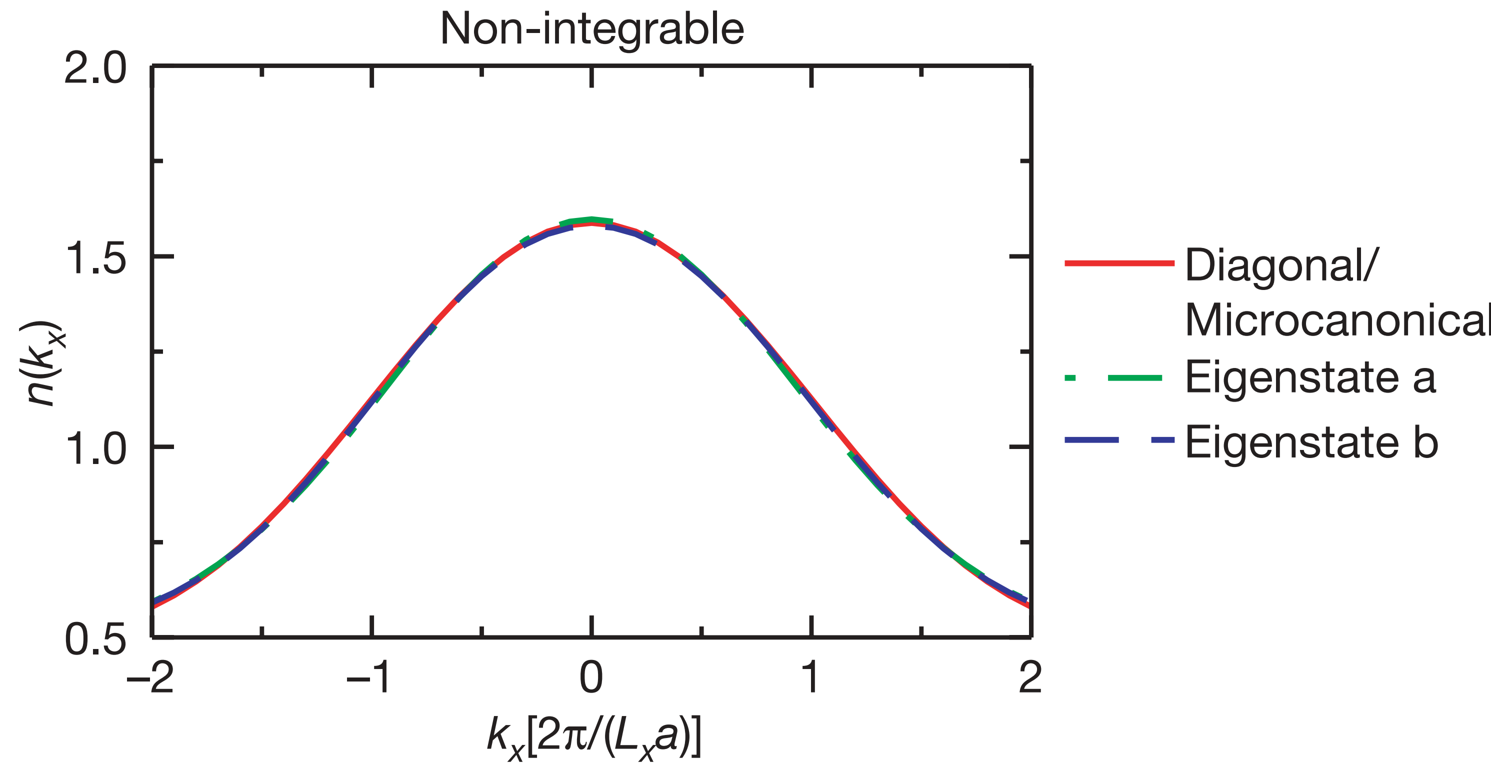
Kinoshita, Wenger, & Weiss (2006)

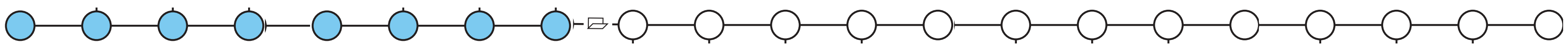
Numerical investigations:

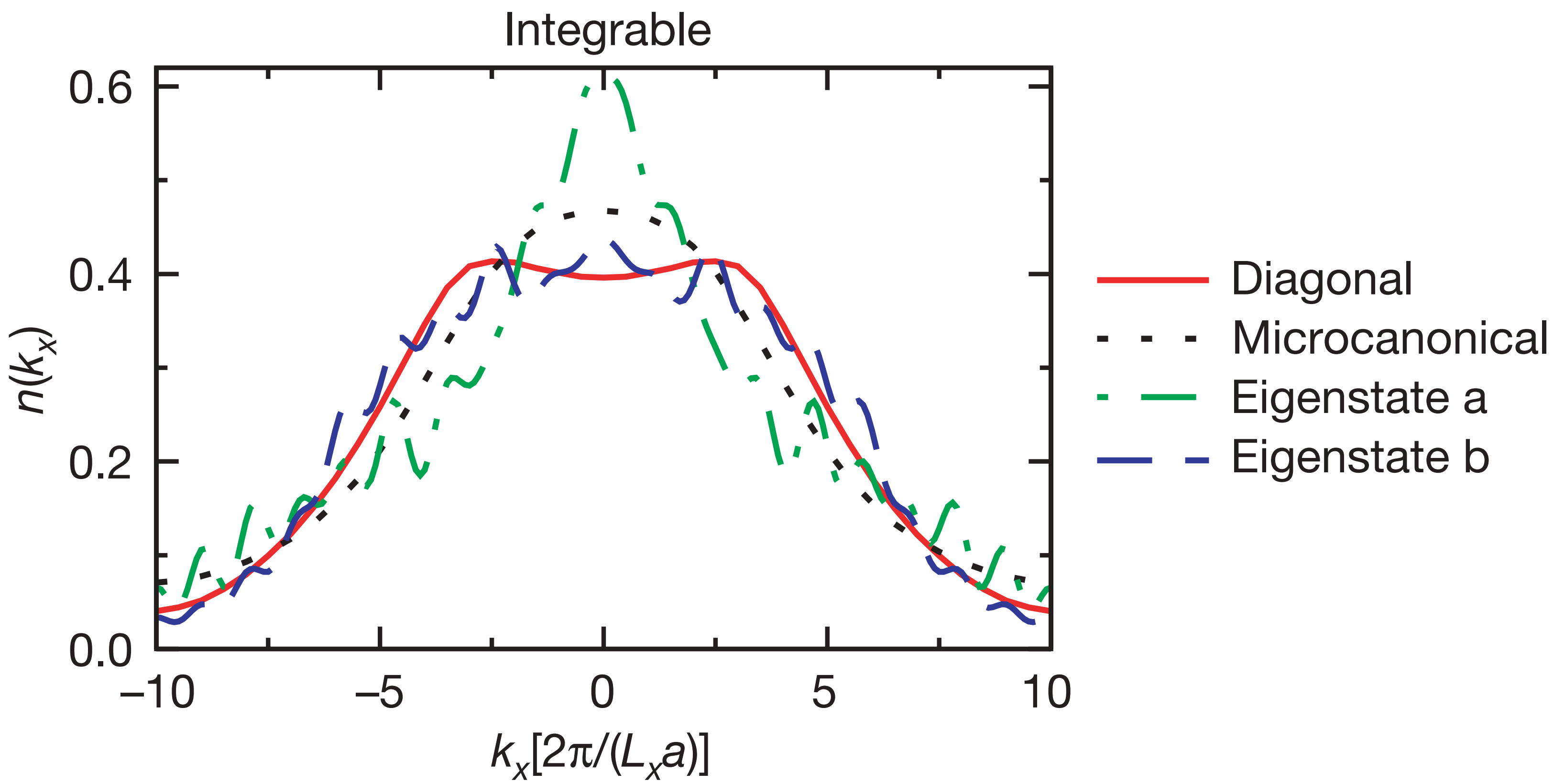




Rigol, Dunjko,
& Olshanii (2008)







Rigol, Dunjko, & Olshanii (2008)

Conclusions:

QUANTUM CHAOS

→

"eigenstate thermalization"

→

STATISTICAL MECHANICS !

Many open problems!

Behavior of near-integrable systems ?

Eigenstate thermalization threshold ?

Alternatives to eigenstate thermalization ?

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