Geometric mechanics: from the atomic to the tectonic

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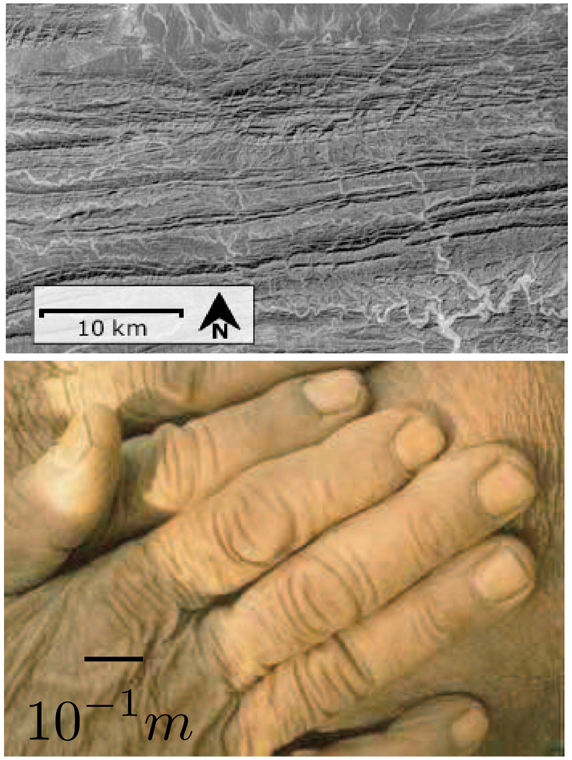
Harvard University

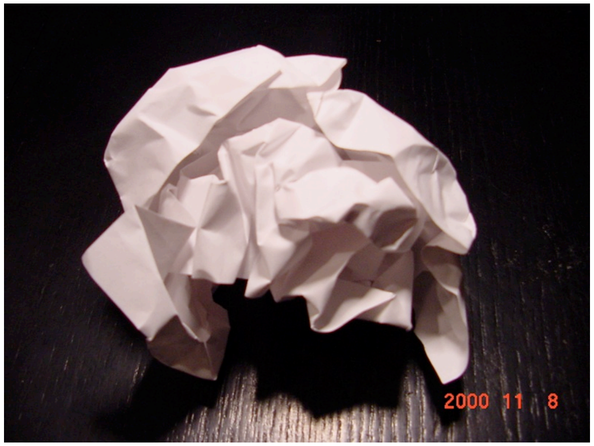
wrinkling



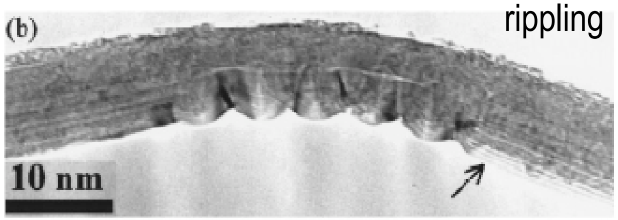
draping

Leonardo da Vinci, ca. 1500

crumpling



rippling



Questions: patterns, fine scales, singularities ?

Outline:

- A little geometry and physics
- An example - wrinkling of a multi-walled nanotube
- A generalized theory - of skins and elephant trunks ...
- The art of the coutourier - the elements of a drape
- The elements of crumpling - Gauss’ Theorema Egregium, Monge Ampere ...
- Dynamics, rheology and fluctuations - more questions ...

theory:

- E. Cerda
- S. Rica
- H. Liang

experiments:

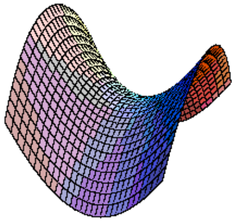
- J. Bico
- F. Melo
- J. Genzer
- R. Bendick

Geometry

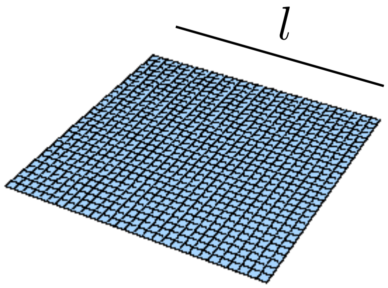
κ_1, κ_2 principal curvatures

Mean curvature $\kappa_M = \frac{1}{2}(\kappa_1 + \kappa_2)$

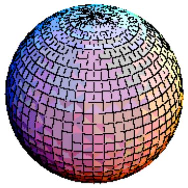
Gauss curvature $\kappa_G = \kappa_1 \kappa_2$



$\kappa_G < 0$



$\kappa_G = 0$



$\kappa_G > 0$

κ_G

intrinsic - isometric invariant

Physics

thickness t

$t/l \ll 1$
$t\kappa \ll 1$

long wavelength deformations

Stretching (tangential) mode

strain $\gamma \sim \frac{\Delta l}{l}$

energy/area $U_s \sim E\underline{t}\gamma^2$

Expensive

Bending (transverse) mode

curvature $\kappa \sim \frac{\Delta\theta}{l}$

strain $h\kappa$

energy/area $U_b \sim Et(t\kappa)^2 \sim E\underline{t}^3\kappa^2$

Cheap

E - modulus

- Coupling of stretching to bending ? - via geometry $\kappa_G \neq 0$

Q ?

- Stretching dominated ?
- Bending dominated ?
- Inhomogeneous ....

Packing constraint

$$Min[\int U_e dA + \int b[w(x,y) - R]dA]$$

$$U_e = C_1(\text{Tr } \kappa)^2 + C_2\text{Det } \kappa + C_3(\text{Tr } \gamma)^2 + C_4\text{Det } \gamma$$

Elastic energy

Equations of equilibrium ?

Geometry $[x, y, 0] \rightarrow [x + u(x,y), y + v(x,y), w(x,y)]$ e.g. Monge gauge

Physics $\sigma = \mathbf{C}\gamma$ linear stress-strain law

$$\frac{Et^3}{12(1-\nu^2)}\nabla^2(\underset{\kappa_M}{\text{Tr } \kappa}) = \text{Tr}(\sigma\ \kappa)$$

normal force balance

$$\nabla^2(\text{Tr } \sigma) = -Et(\underset{\kappa_G}{\text{Det } \kappa})$$

geometric compatibility

Foppl, von Karman (1907)

+ B.C.

- No 2-d analytical solutions !
- Scaling/asymptotic analysis ?

$$\epsilon = t/l \ll 1$$

2 limits :

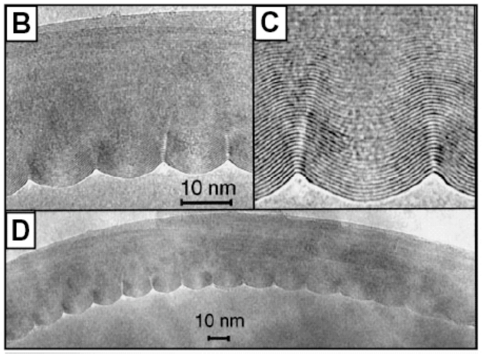
$\text{Tr}(\sigma\ \kappa) \approx 0$ almost planar (except for wrinkles)

$\text{Det } \kappa \approx 0$ almost isometric (except for crumples)

Wrinkling

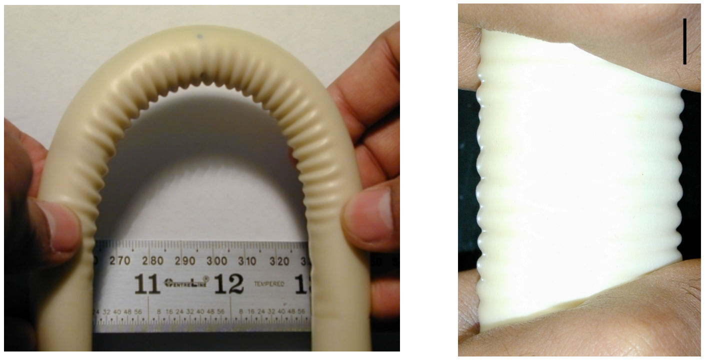
LM, Bico et al (EPL 2003)

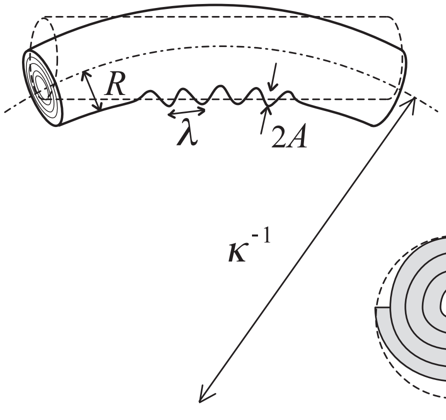
Multi-walled carbon nanotube



Poncharal et al. (2002)

Rubber scroll





t thickness (1 atom !)

n number of layers

$R \sim nt$

$\lambda = tf(n)$

$\kappa_w \sim A/\lambda^2$

$\epsilon \sim A/R$

Energy/area

stretching

$$U_B + U_S \sim [Et^3\kappa_w^2 + Et\epsilon^2]RL$$

bending

$+$

Inextensibility constraint

$$[\lambda(1 - R\kappa)]^2 + A^2 \approx \lambda^2$$

$$A^2/\lambda^2 \sim R\kappa$$

$$\longrightarrow \quad \frac{U}{RL} \sim EtR\kappa\left(\frac{t^2}{\underline{\lambda^2}} + \frac{\underline{\lambda^2}}{R^2}\right)$$

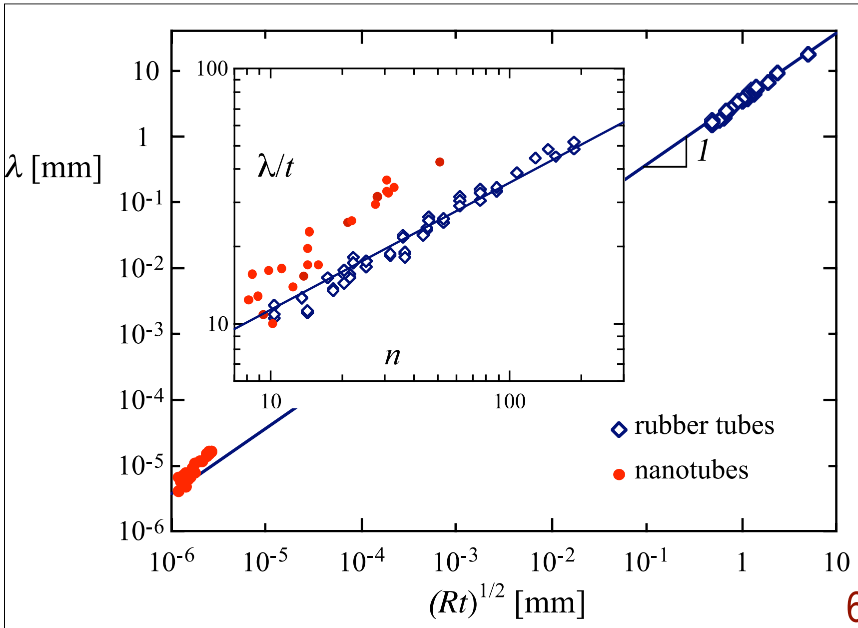
bending

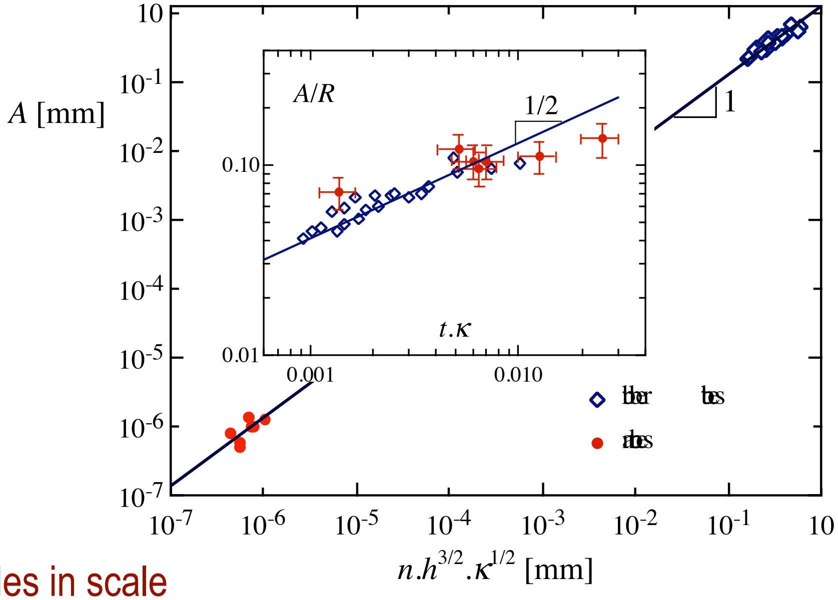
stretching (mattress)

$Min\ U$

$$\boxed{\begin{aligned}\lambda &\sim (Rt)^{1/2} \sim n^{1/2}t \\ A &\sim R(t\kappa)^{1/2} \sim n(t^3\kappa)^{1/2}\end{aligned}}$$

- No dependence on material
- Wavelength + Amplitude ... nonlinear
- Geometric theory





6 decades in scale

(rubber≠ carbon)

“Ganesha” instability

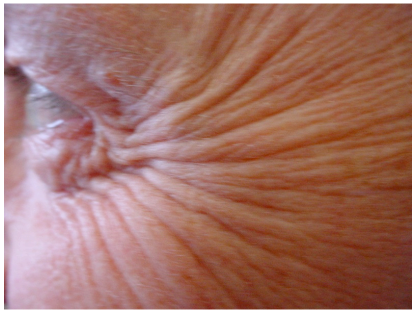
Ingredients for a theory of wrinkles:

- “Packing” constraint $\Delta \sim R\kappa$
- Bending energy penalty Et^3
- “Mattress” of springs $K \sim Et/R^2$

$$\lambda \sim (\frac{Et^3}{K})^{1/4} \qquad A \sim \Delta^{1/2}\lambda$$

Skin ?

Compression



Tension

Mattress stiffness $K = \frac{E_p}{L} f(\frac{E_f}{E_p}, \frac{\lambda}{l})$?

“Persistence” length L ?

deep “water”

$$\lambda \ll l, \quad L \sim \lambda, \quad \lambda \sim t(\frac{E_f}{E_p})^{1/3}$$

$$t/l \ll 1, \quad \lambda/t \sim 10$$

not very visible

shallow “water”

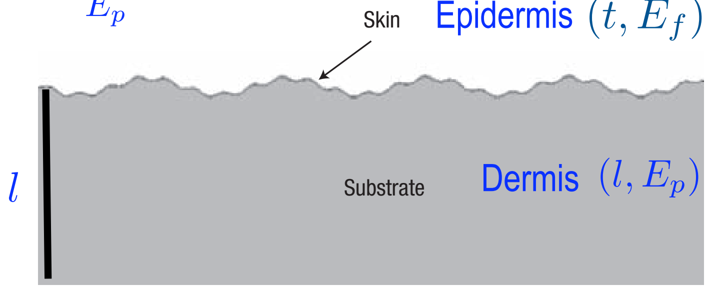
$$\lambda \gg l, \quad \lambda \sim (tl)^{1/2}(\frac{E_f}{E_p})^{1/6}$$

$$l/t \sim 10, \lambda/l \sim 1$$

visible - need Botox ?

Cerda, LM (PRL 2003)

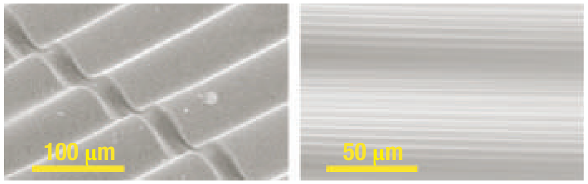
$$\frac{E_f}{E_p} \sim 10^3$$

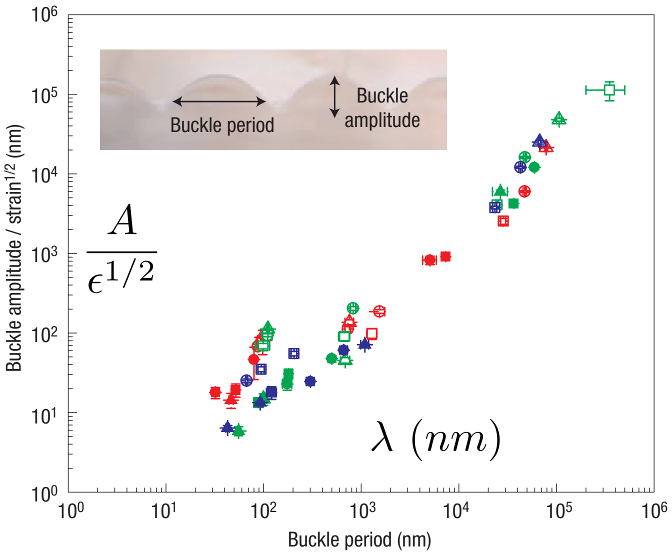


$$\lambda \sim (\frac{E_f t^3}{K})^{1/4}$$

J. Genzer, LM et al. (2005)

self-similar wrinkling





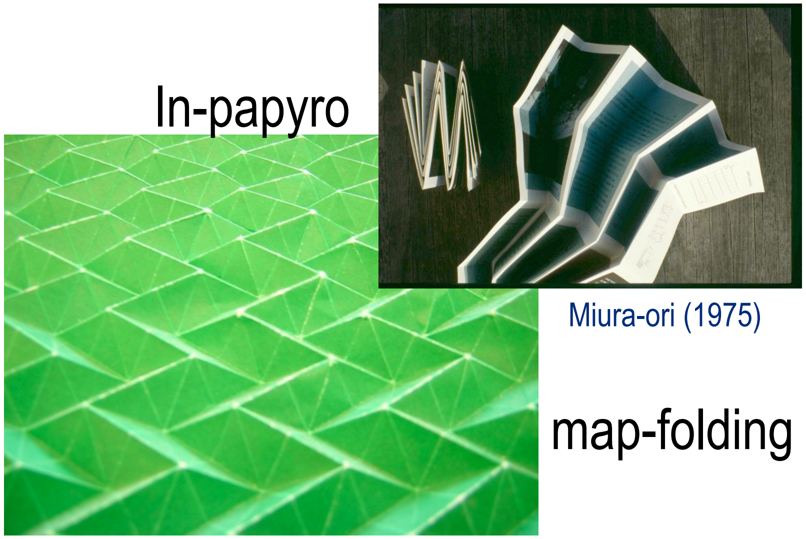
Controllable 1-d wrinkling patterns :

2-d : self-organized Origami

secondary instability of a periodic wrinkling pattern

LM, Rica (Science 2005)

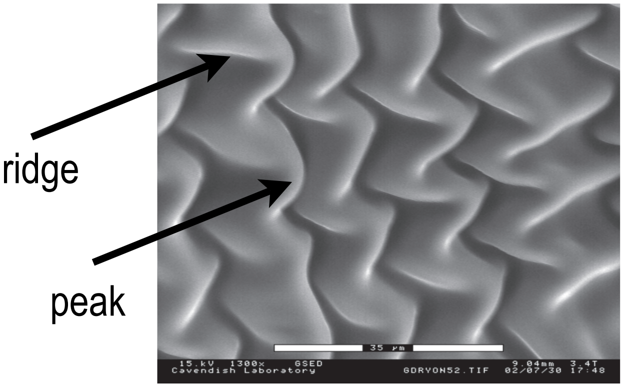
In-papyro



Miura-ori (1975)

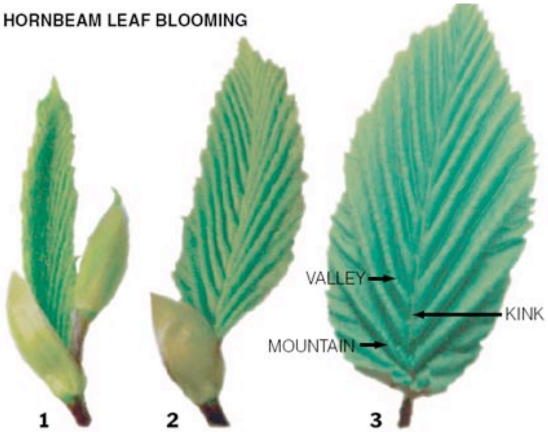
map-folding ?

In-vitro (drying gelatin)

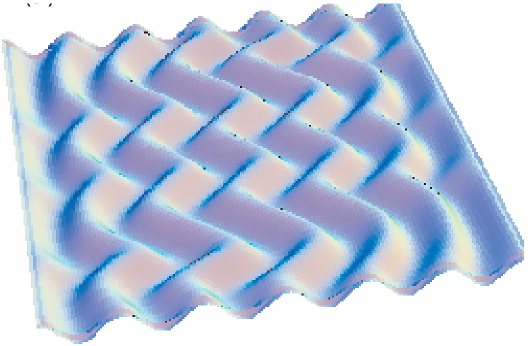


Rizzieri et al. (2005)

In-vivo



In-silico



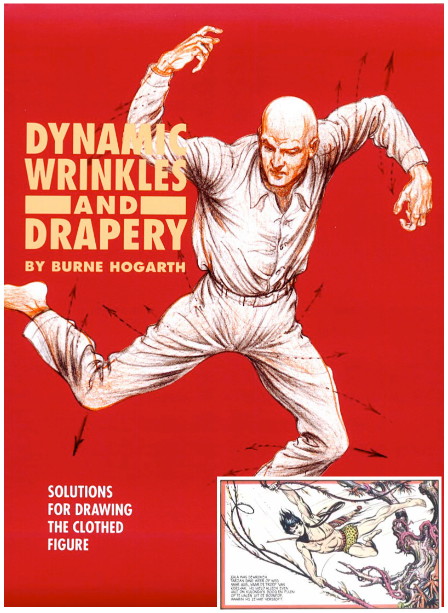
Newell-Whitehead-Segel eqn. (1969)

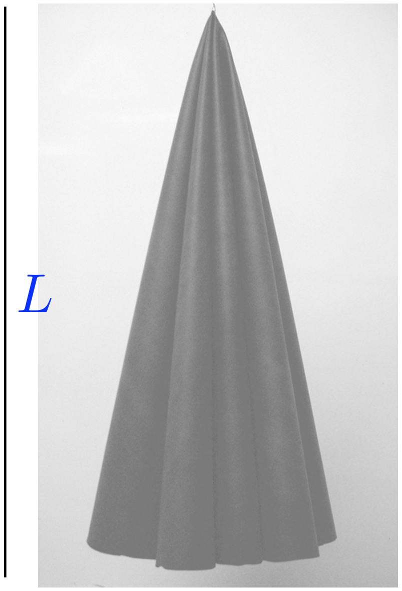
$$\epsilon A + h^2[(\partial_x - \frac{i}{\partial_{yy}})^2 - g|A|^2]A = 0$$

(differential) shrinkage - a model for morphogenesis ?

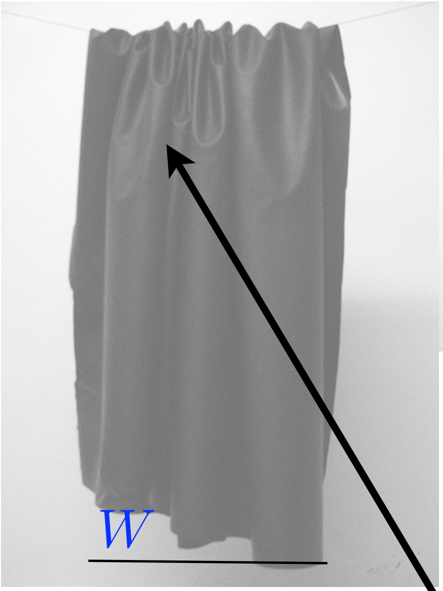
Elements of a drape ?

Cerda, LM, Passini (PNAS 2004)





point



line

curve

"gravity" length

$$l_g \sim (\frac{Et^2}{\rho g})^{1/3} \sim O(cm)$$

gravity = "spring"

tension $T \sim \rho g t L$

$$K \sim T/L^2 \sim \frac{\rho g t}{L}$$

# of folds

$$n \sim \frac{L}{\lambda} \sim (\frac{\underline{L}}{l_g})^{3/4}$$

$$\lambda \sim (\frac{Et^3}{K})^{1/4} \sim L^{1/4} l_g^{3/4}$$

# of folds

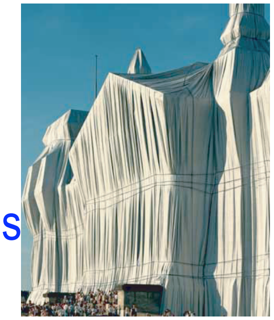
$$n \sim \frac{W}{\lambda} \sim \frac{W}{\underline{L}^{1/4} l_g^{3/4}}$$

Inverse cascade

$$L_p^i \sim \lambda_p^{i^2}$$

"persistence" of a wrinkle

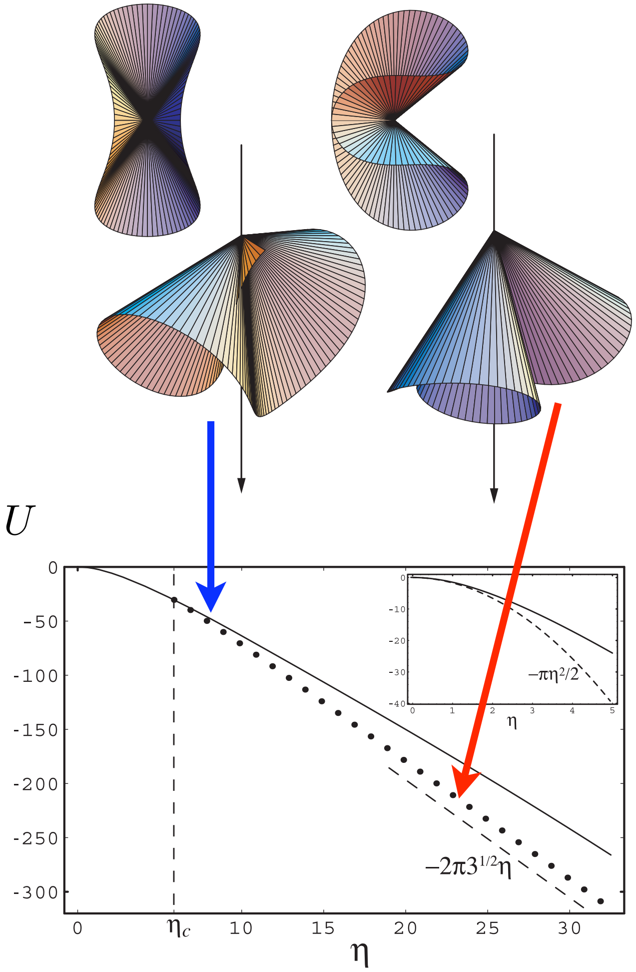
- Spacing of guy ropes
Christo's drapes !



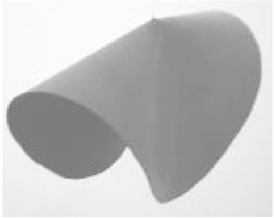
Configurational multi-stability ?

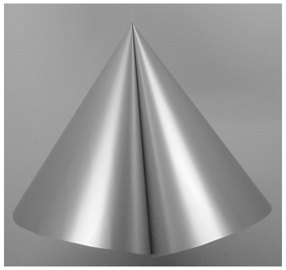
Developable cones ... $\eta = (\frac{R}{l_g})^3/ln(R/R*)$

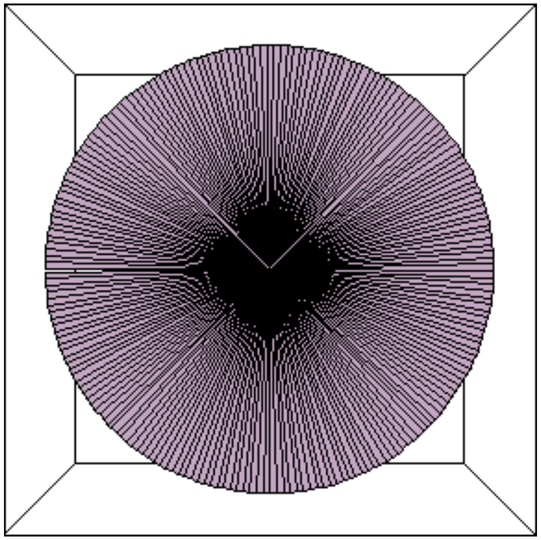
$$\ddot{\kappa} + (a^2 + \kappa^2/2)\kappa = -\eta\kappa \quad \text{f(geometry)}$$

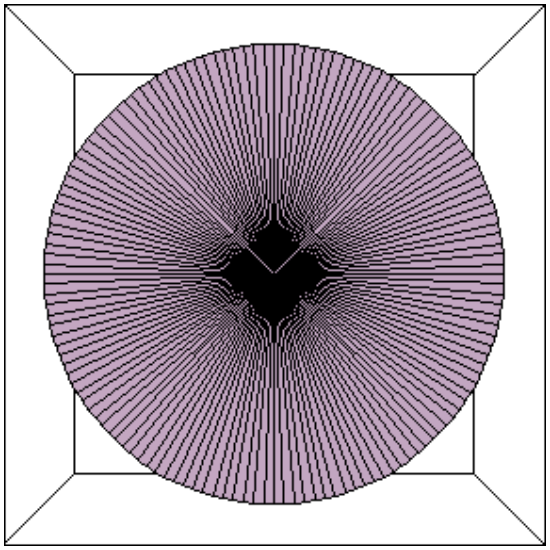


observations









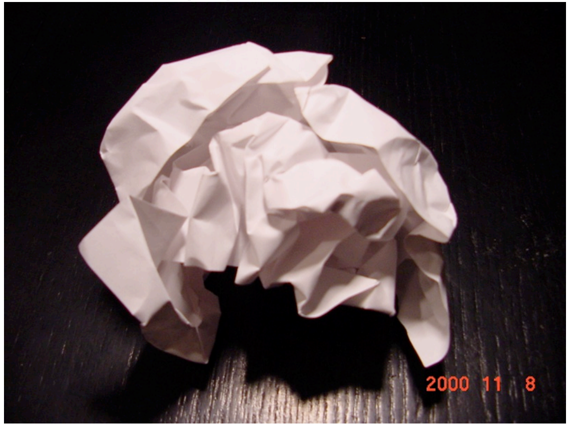
Symmetry breaking bifurcations and “catwalk transitions” ?

Crumpling

Cerda, LM, (PRS (Lond) 2005)

foil

paper



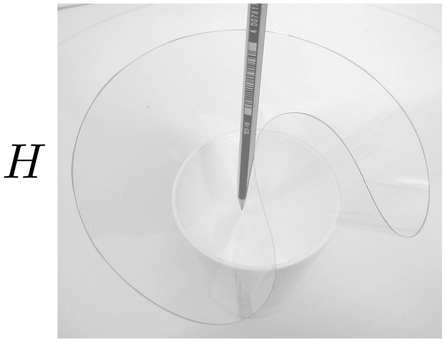
Q.
1. Description ?
2. Statics ? Dynamics ?

A.
Minimize energy + constraints...

flat — isometric bending (cylinders, cones, tangent developables) $\kappa_G = 0$ — bending+stretching (inhomogeneous) $\kappa_G \neq 0$ → Elastic energy

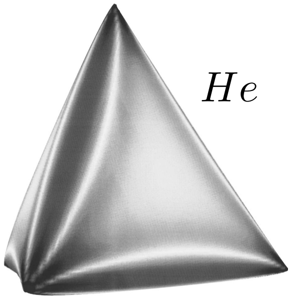
Crumpled surfaces: $\kappa_G = 0$ except along peaks and ridges.

Peaks



$U \sim Et^3 \ln(\frac{R}{R*})$ conical "dislocation"

Ridges



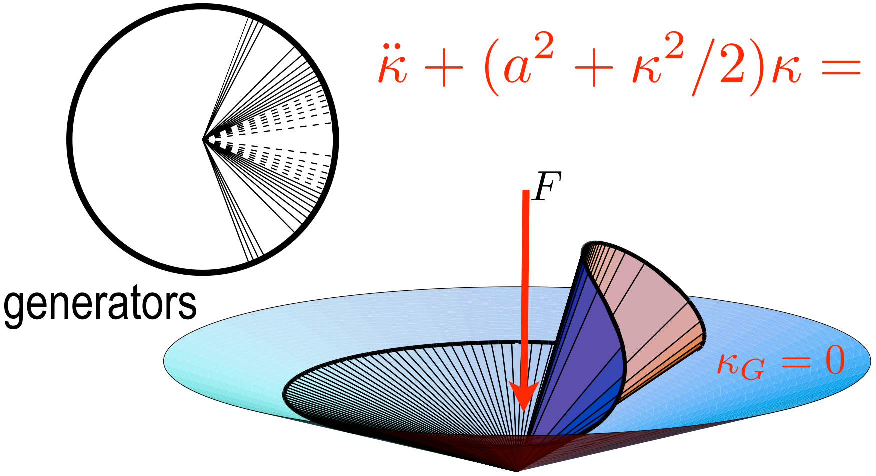
Witten et al. (1995)

Q.
1. Structure (shape) of "defects" ?
2. Response ?
3. Interaction ? *H, He, ...*
4. Dynamics ?

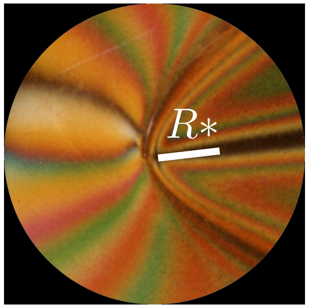
Cerda, LM (PRL 1998)
Cerda, Chaieb, Melo, LM (Nature, 1999)

Analytical (outer) solution *(purely geometric)*

$$\ddot{\kappa} + (a^2 + \kappa^2/2)\kappa = 0$$



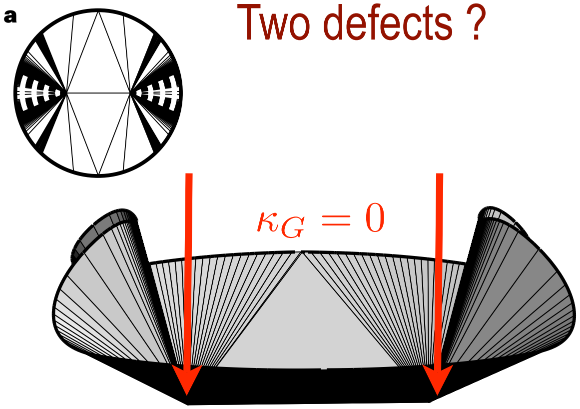
Core shape/scaling



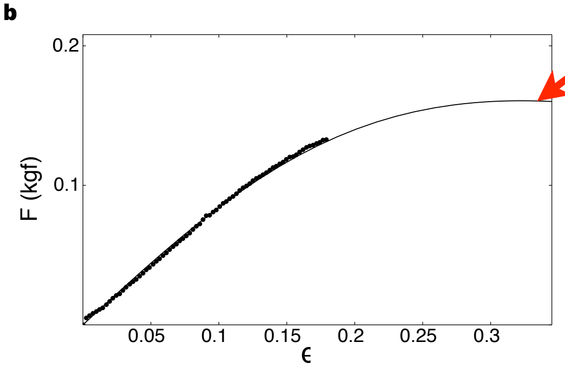
$$R* \sim t^{1/3} R^{2/3} \epsilon^{-1/3}$$

non-isometric
$\kappa_G \neq 0$

Two defects ?

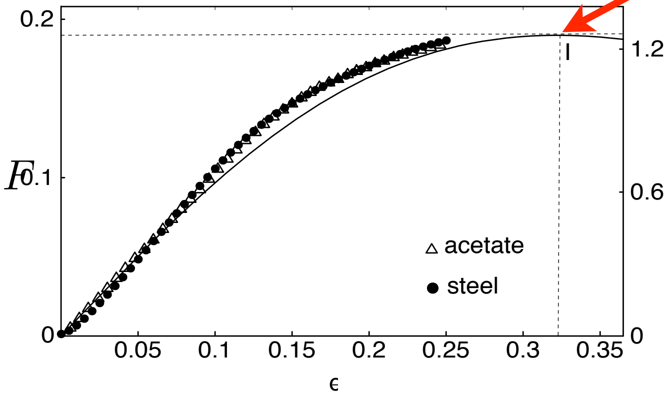


Instability !



Instability !

Single defect response ?



$1, \quad 2, \quad .....\infty \quad ?$

“Crumpulence” ?

Big crumples fold into little crumples
That store energy in bending
And little crumples have lesser crumples
And so on to stretching

with apologies to Swift/Richardson

Fluids do it too !

LM et al, (Nature, 1998), Skorobogaity, LM (EPL; 2000); Silveira et al (Science, 2000)

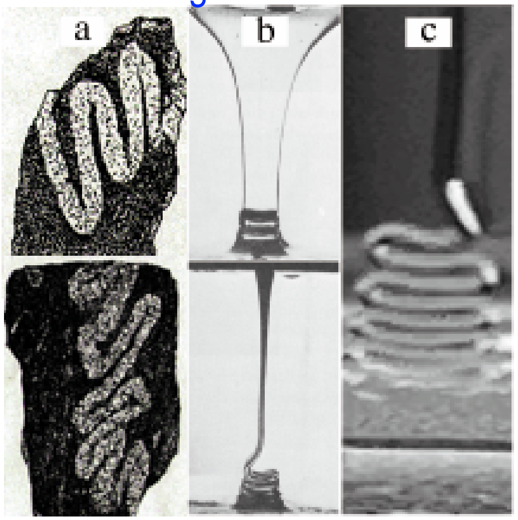
Stokes-Rayleigh analogy

Hookean solid	Newtonian fluid
Displacement	Velocity
Strain	Strain rate
Shear modulus	Shear viscosity

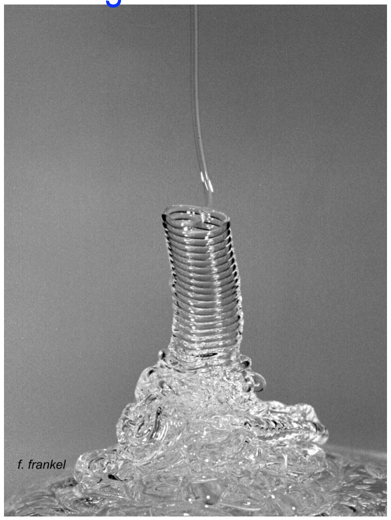
$$+ \quad Ca = \mu U/\sigma \gg 1$$

i.e., free-surfaces are free !

Folding sheets



Coiling filaments

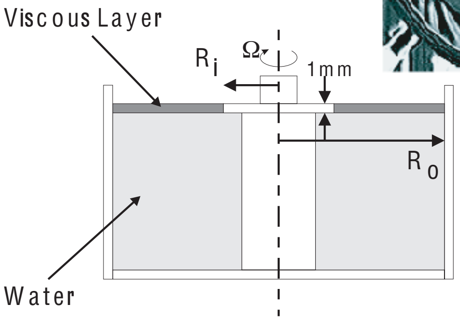


$$l_g \omega \sim U$$

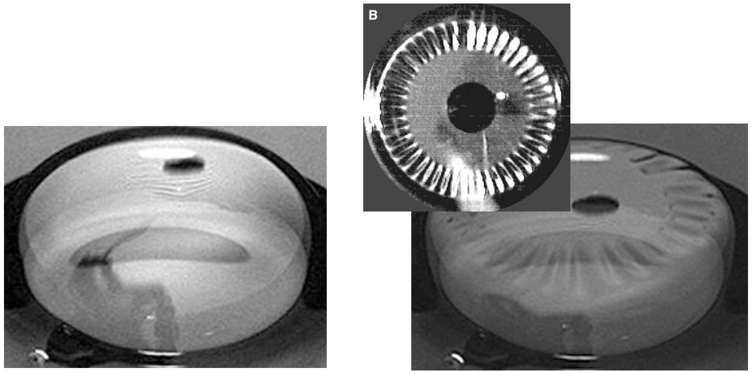
$$l_g \sim (\frac{\mu U t^2}{\rho g})^{1/4}$$

elastic analog $l_g \sim (\frac{Et^2}{\rho g})^{1/3}$

Wrinkling of a sheared annular film



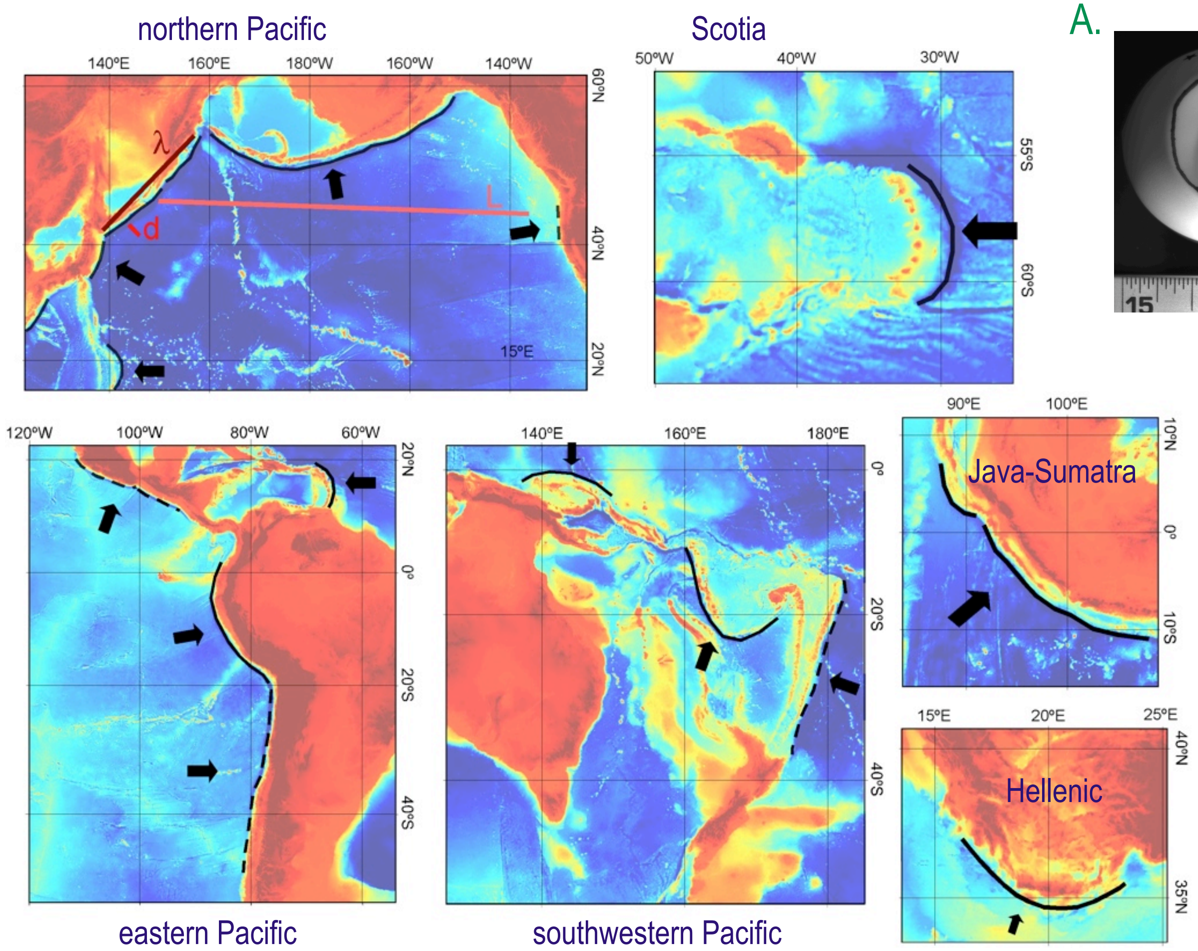
Rippling of a collapsing bubble



Island arcs ? Shellular subduction

$$R/h \sim 6000/60 \sim 100$$

LM, Bendick, Liang (2008)


northern Pacific

Scotia

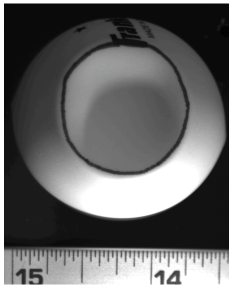
Java-Sumatra

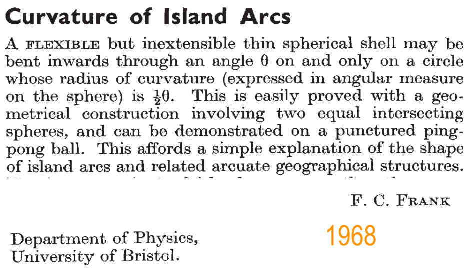
Hellenic

eastern Pacific

southwestern Pacific

A.





Curvature of Island Arcs

A FLEXIBLE but inextensible thin spherical shell may be bent inwards through an angle θ on and only on a circle whose radius of curvature (expressed in angular measure on the sphere) is $\frac{1}{2}\theta$. This is easily proved with a geometrical construction involving two equal intersecting spheres, and can be demonstrated on a punctured ping-pong ball. This affords a simple explanation of the shape of island arcs and related arcuate geographical structures.

F. C. FRANK

Department of Physics,
University of Bristol.

1968

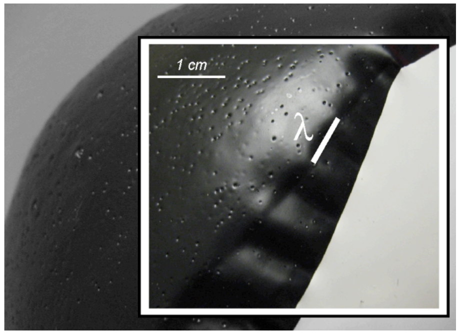
No !

Q.
- Localized deformations ?
- Arc-like ? Straight ?
- Polarity ?

- Incomplete spherical cap
- Finite thickness, complex rheology
- Mantle resists deformation
- Variable buoyancy

Stability of (partially) negatively buoyant lithosphere ?

(Stokes-Rayleigh analogy)



Bendick

- Free edge ... geometrically soft, physically dense.
- Subduction onset - reduction in effective perimeter
- i.e. edge buckling !

Elastic model

$$U_e \sim E_l h_l^3 R \frac{w^2}{\lambda^3} + E_l h_l \frac{w^2}{R}\lambda + \frac{E_m}{H_m} w^2 R \lambda$$

lithosphere bending

lithosphere stretching

mantle deformation

Viscous model

μ_l

$$\lambda \sim \left(\frac{E_l h_l^3}{\frac{E_m}{H_m} + \frac{E_l h_l}{R^2}}\right)^{1/4}$$

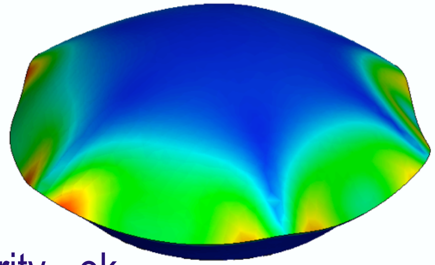
μ_m

μ_l

$$\lambda \sim \left(\frac{\mu_l h_l^3 H_m}{\mu_m}\right)^{1/4}$$

soft / viscous mantle model ...

Liang

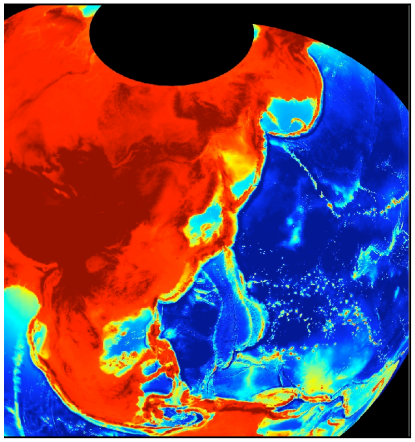


- Polarity - ok
- Localized deformations - ok
- Arc-like ? Straight ? - ok

$$\lambda \sim 500km$$

$$n \sim W/\lambda \sim 1-5$$

but subcritical instability -
i.e. heterogeneity dominated ...



Lessons: geometry + simple physics = rich field to mine

Questions:

- Non-Euclidean geometries ?

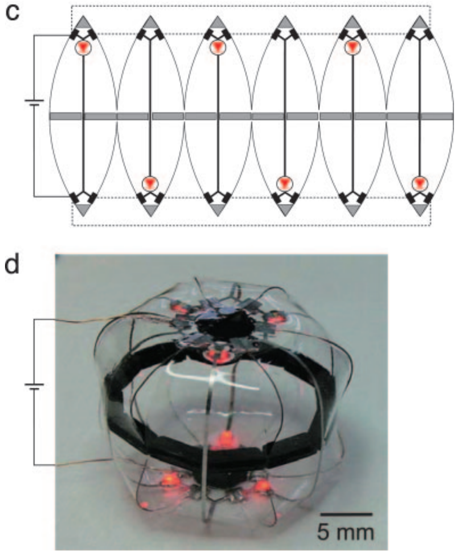
- Cuts, tears and stitches ?

- Dynamics and wave turbulence ?

Crocheted hyperbolic plane

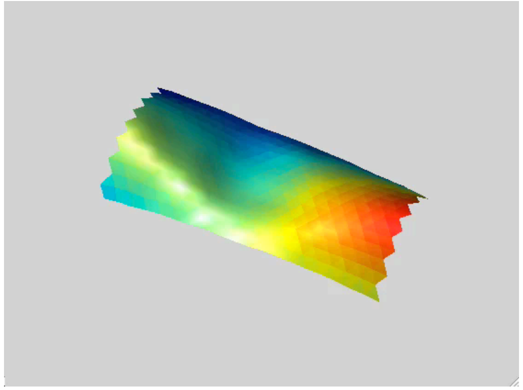
D. Taimina, S. Rowel (2000)

Self assembly in 3D



M. Boncheva et al. (PNAS 2005)

Defect mediated Transitions



Experiment (10000 frames/s) !