Reverse-Engineering the Solution of a "Simple" Physics Problem: Why Learning Physics Is Harder Than It Looks

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In this paper, we show an example of students working on a physics problem—an example that demonstrated to us that we had failed to understand the work they needed to do in order to solve a "simple" problem in electrostatics. Our critical misunderstanding was failing to realize the level of complexity that was built into our own knowledge about physics.

As physics teachers, we often stress the importance of problem solving. Unfortunately, many of our students appear to find it very difficult. Sometimes they generate ridiculous answers and seem satisfied. Sometimes they can do the calculations but not interpret the results. Sometimes, despite success in solving the problem, they seem to have a poor understanding of the physics.[1] We may give them explicit instructions on how to solve problems ("draw a picture," "find an equation," …) but it doesn't help.

We might respond that they need to take more math, but in the algebra-based physics class at the University of Maryland, almost all of our students have taken calculus and earned an A or a B. Many have been successful in organic chemistry, cellular biology, and genetics. Why do they have so much trouble with the math in an intro physics class?

As part of a research project to study learning in algebra-based physics,[2] the Physics Education Research Group at the University of Maryland videotaped students working together on physics problems. Analyzing these tapes gives us insights into the problems students have using math in the context of physics. First is that they have inappropriate expectations as to how to solve problems in physics.[3] The second problem seems to lie with us: As instructors, we may have misconceptions about how people think and learn, and these have implications about how we interpret what our students are doing.

"Packing" Knowledge Until You Don't See Its Parts: Compilation

Modern cognitive psychology and neuroscience have documented that much of our everyday functional knowledge is more complex than we give it credit for. One component of this is *automaticity.* Once we have learned to do something, like tie our shoes or ride a bicycle, it becomes easy and we can do it without thinking. However, we usually understand and remember the learning that goes into such tasks and we typically have patience in teaching them to our children.

We have other knowledge, though, that has invisible components. When our knowledge takes that form, it is hard to see why someone might not find the result obvious. Even an apparently simple thing like identifying an object is much more complicated than it appears.[4]

Some cognitive "illusions" dramatically demonstrate how much unnoticed processing the brain is doing for us. A nice example given by Ed Adelson is shown in Fig. 1. The squares of the checkerboard marked A and B are, in fact, exactly the same color. (If you don't believe this, make a copy of the page, cut out the squares, and place them next to each other, or check out Adelson's website.[5]) Your brain knows

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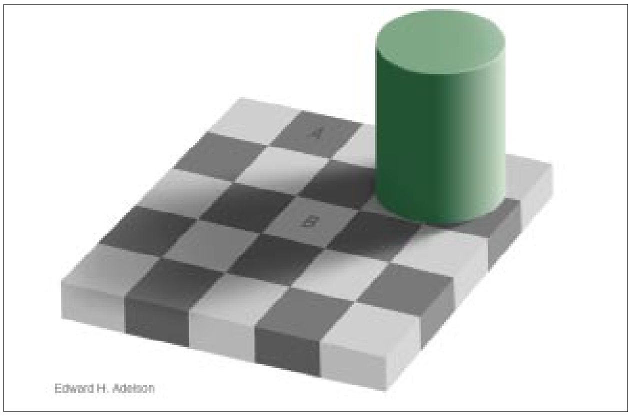


Fig. 1. An example of automatic processing that you cannot unpack. (Photo courtesy of E. Adelson)

enough to realize that if two objects appear to be the same color but one is in shadow, then the one in shadow must "really" be lighter—and that's how you see it. This particular example appears to be "wired up" very tightly and at a very young age. You can't see it any other way.

A second example is more obviously learned. You can no longer see the word "CAT" as only a series of lines and shapes. You probably not only get the meaning immediately, you have some visual image associated with the word. You have learned to interpret the shapes as letters, to see combinations of letters as words, and to associate the words with particular meanings. Although you can't undo this easily (looking at it upside down does it for some folks), you know that there was a time when all you could see were lines and shapes.

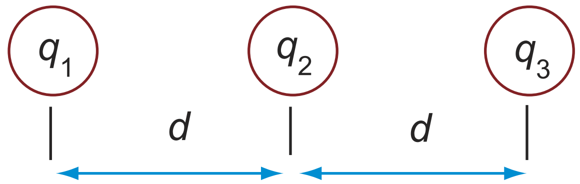
This same sort of process occurs as we learn throughout our lives. When professional physicists look at a graph, it is almost impossible not to see the y-intercept, the slope at each point, the maxima and minima, and so on. For many students in introductory physics, however, this process is not quick and automatic but takes explicit recall and reasoning.

We refer to this process of binding knowledge tightly so that its parts are inaccessible to the user as *compilation*. (This is referred to in the neuroscience literature as *binding*.) The metaphor here is computer code. Once a program written in a high-level computer language has been debugged and is stable, it is convenient to convert it into machine language so it doesn't need to be translated each time it runs. This "executable" is fast, but if you are only given a machine-language executable, it is immensely difficult to back-interpret it to understand what it is actually doing. To understand what is going on in such a program, a computer programmer who wants to recreate it may have to *reverse-engineer* it.

Once we learn how to do something, it can be difficult to empathize with someone who does not know how to do that thing. This lack of empathy may lead physics teachers to forget what it is like to actually learn physics—and, therefore, prevent them from understanding how their students are unable to solve "simple" physics problems. Some physics teachers may think, "If it takes a student an hour to solve a problem to which I can just write down the answer, then that student does not know enough physics—and she is wasting her time spending that long on such a 'simple' problem." In this paper we want to demonstrate that this is not the case. To do this, we reverse-engineer what solving a "simple" physics problem really entails. We analyze a group of students' solution to this simple problem and show that, while the students take much longer than the typical teacher would to solve this problem, their solution involves activities that can be seen as a process of compiling their physics knowledge and that are appropriate for students at their stage of learning.

An Example: The Three-Charge Problem

In what follows we observe students working together on an electrostatics problem that we refer to as the "Three-Charge Problem" (Fig. 2).



In the figure above three charged particles lie on a straight line and are separated by distances d. Charges q_1 and q_2 are held fixed.

Charge q_3 is free to move but happens to be in equilibrium (no net electrostatic force acts on it).

If charge q_2 has the value Q, what value must the charge q_1 have?

Fig. 2. The "Three-Charge Problem." An example of physics knowledge that instructors need to unpack.

How instructors solve this problem

We have asked this problem of many physics instructors. It rarely takes anyone more than a few seconds to work his/her way through to a correct answer and some give it immediately. A typical instructor's solution might be: "Well, charge 3 is twice as far away from charge 1 as it is from charge 2. So if the forces balance, charge 1 has to be $-4Q$, opposite sign to balance Q and four times as big because Coulomb's law says the force falls like the square of the distance."

What instructors want students to do

Although most instructors can do this kind of a calculation in their heads, we, as instructors, intended for the students to go through a bit more math. The problem states there is "no net electrostatic force" acting on charge q_3. This implies that the sum of all the forces acting on q_3 is equal to zero, which is written formally in symbols as

$$\mathbf{F}_{q_2 \rightarrow q_3} + \mathbf{F}_{q_1 \rightarrow q_3} = 0. \quad (1)$$

(We see below that this form of the equation is deceptively simple and hides a lot of conceptual information.) Using Coulomb's law yields

$$\frac{kQq_3}{d^2}\hat{i} + \frac{kq_1q_3}{(2d)^2}\hat{i} = 0, \quad (2)$$

where we have set $q_2 = Q$ and written (for simplicity as we did in the class) $k = 1/4\pi\varepsilon_0$. Finally, we bring the second term to the right side of the equation and cancel similar terms, resulting in $q_1 = -4Q$.

What the students did

We examine a videotaped episode in which a group of students solves the "Three-Charge Problem." This episode occurred in the second week of the second semester of the two-semester sequence and has three female students (pseudonyms Alisa, Bonnie, and Darlene) working together. All of the students had taken the transformed course in the first semester and were familiar with the ideas that a single problem might take a long time to solve and that qualitative and quantitative considerations might both be needed.

A striking feature of the students' problem solving is that it takes so long compared to a typical instructor's solution. The students work for nearly 60 minutes before arriving at a solution—almost two orders of magnitude longer than the typical teacher! Is this a cause for concern? In our analysis of our students' approach in solving this and in other problems, two factors seemed critical in understanding what the students were doing.

First, we note that much of the knowledge that students are using is not integrated; results that would be considered trivially identical by an instructor are treated as distinct and unrelated. The students have not yet compiled these distinct knowledge elements the way experts have.

Second, we observe that students tend to solve problems by working in locally coherent activities in which they use only a limited set of the knowledge that they could in principle bring to bear on the problem. We refer to each of these activities as an *epistemic* or *knowledge-building game*. Each game has allowed moves, a starting point, a goal or ending point, and a *form* or visible result. Most important, while students are playing one game, they ignore moves that they consider as not pertinent, thereby excluding much relevant knowledge. We see an example of this in our episode. (These games are discussed in detail in Ref. 3.)

An example of a knowledge-building game many of us have seen when interacting with students is "Recursive Plug-and-Chug." The student's goal is to calculate a numerical answer. The opening move is to identify the target variable to be calculated. The next move is to find an equation containing that variable. Then, check if the rest of the variables in the equation are known. If so, calculate the resulting quantity. If not, find another target variable in the equation and repeat the process. The game does *not* include developing a story about the problem, evaluating the relevance of the equation, or making sense of the answer. Without these sense-making moves, inappropriate and even bizarre results go unnoticed. The output form of this game is a string of equations—one that might look identical to that produced by a student playing a more productive game, such as "Making Meaning with Mathematics."

Knowledge-building games the students use

In our example, we can identify five different knowledge-building games that this group of students played to solve the "Three-Charge Problem." We describe each in turn and give an example from the transcript.

"Physical Mechanism": Understanding the physical situation. The students start by attempting to understand the physical situation articulated in the problem statement. Their reasoning is based on intuitive knowledge about and experience with physical phenomena rather than on formal physics principles.

Darlene: *I'm thinking that the charge q_1 must have its... negative Q.*
Alisa: *We thought [q_1] would be twice as much, because it can't repel q_2, because they're fixed. But, it's repelling in such a way that it's keeping q_3 there.*
Bonnie: *Yeah. It has to—*
Darlene: *Wait, say that.*
Alisa: *Like—q_2 is— q_2 is pushing this way, or attracting—. There's a certain force between two Q, or q_2 that's attracting.*
Darlene: *q_3.*
Alisa: *But at the same time you have q_1 repelling.*

Darlene initiates this exchange with a possible solution to this problem: the charge on q_1 is "negative *Q*." Although this is wrong, it has a good piece of physics: q_1 must have the opposite sign to q_2 if the forces they exert on q_3 are to balance. Rather than simply accept or reject this suggestion, the students discuss the physical mechanism that acts to keep q_3 from moving: q_2 is attracting and q_1 is repelling q_3, or vice versa. If the students were only attempting to find a solution to this problem, then a discussion about the physical mechanism seems unnecessary—they would only need to assess the correctness of Darlene's assertion. That the students discuss a possible physical mechanism involved in the physical situation is an indication that the students are attempting to develop a conceptual understanding of this problem. Research on quantitative problem solving in physics discusses the importance of conceptual understanding.[6] The instructor's solution outlined above does not explicitly contain a description of the physical mechanism underlying the physical situation, but it is nonetheless there—compiled into the way instructors think about and approach the problem. They not only "know the right formula," they know what the formula means and how to use it.

The next exchange indicates, however, that the students' intuitions about the physical situation are not always consistent with expert physics principles. The students are still struggling with reconciling the principle of superposition with their everyday ideas.

Darlene: *How is [q_1] repelling when it's got this charge in the middle?*
Alisa: *Because it's still acting. Like if it's bigger than q_2 it can still, because they're fixed. This isn't going to move to its equilibrium point. So, it could be being pushed this way.*
Darlene: *Oh, I see what you're saying.*
Alisa: *Or, pulled. You know, it could be being pulled more, but it's not moving.*
Darlene: *Uh-huh.*

The arrangement of the charges cues Darlene to think that the presence of q_2 somehow hinders or blocks the effect of q_1 on q_3, which does not agree with the superposition principle. Exploring the possibility that q_2 blocks the effect of q_1 on q_3 is not a step in the instructor's solution outlined above—it's something the instructor knows and takes for granted. The exploration is not a dead end for the students; it is a step in helping them to develop an intuitive sense of superposition.

Alisa's argument is particularly interesting. She uses an incorrect qualitative argument (overcoming—"if it's bigger…it could be being pulled more") rather than the correct quantitative one (Coulomb's law), even though she later shows she knows the quantitative approach. Formal arguments are not part of the "Physical Mechanism" game. Here, as in many other episodes we've seen, students who are in the middle of one game tend not to use other knowledge that they have.

"Pictorial Analysis": Drawing a picture. The students make progress on this problem by attempting to develop a conceptual understanding of the physical situation in terms of their intuitive ideas; however, this game does not lead to a stable solution within the group. After their apparent initial agreement in developing a conceptual understanding of the physical situation, and in particular on determining that charges q_1 and q_2 had to have opposite signs, Darlene decides she is not convinced.

Darlene: *I think they all have the same charge.*

Bonnie: *You think they all have the same charge? Then they don't repel each other.*
Darlene: *Huh?*
Bonnie: *Then they would all repel each other.*
Darlene: *That's what I think is happening.*
Bonnie: *Yeah, but q_3 is fixed. If it was being repelled—*
Alisa: *No, it's not. q_3 is free to move.*
Bonnie: *I mean, q_3 is not fixed. That's what I meant….*
Darlene: *So, the force of q_2 is pushing away with is only equal to d.*
Bonnie: *Yeah, but then...*
Darlene: *These two aren't moving.*
Bonnie: *Wouldn't this push it somewhat?*
Alisa: *Just because they're not moving doesn't mean they're not exerting forces.*
Darlene: *I know.*

The TA (Tuminaro) notices the students' failing to communicate clearly and lock down their apparent gains, and suggests that they draw a picture. The students do not use an algorithmic pictorial analysis technique (e.g., free-body diagrams) but rely on their intuitive ideas to generate a picture. The picture helps the students organize their thoughts and agree on the relative sign on each of the charges.

Alisa: *So, maybe this is pushing...*
Darlene: *That's [q_2] repelling and q_1's attracting?*
Bonnie: *Yeah, it's just that whatever q_2 is, q_1 has to be the opposite. Right?*
Alisa: *Not necessarily.*
Darlene: *Yeah.*
Bonnie: *OK, like what if they were both positive?*
Alisa: *Well, I guess you're right, they do have to be different, because if they were both positive...*
Bonnie: *Then, they'd both push the same way.*
Alisa: *And, if this were positive it would go zooming that way.*
Darlene: *They would both push.*
Alisa: *And, if this were negative it would go there.*
Bonnie: *It would go zooming that way.*
Alisa: *And, if they were negative...*
Darlene: *It would still—they'd all go that way.*
Alisa: *It would be the same thing.*

The picture enhances the students' ability to reason about this problem, enabling them to agree on a clear intuitive understanding of the physical situation: "Whatever [the sign of] q_2, q_1 has to be the opposite."

"Mapping Mathematics to Meaning": Identifying the relevant physics. At this point, the students have not made use of Coulomb's law—they have relied solely on their intuitive ideas. Yet, their intuitive reasoning helps them understand the physical situation, and, ultimately, to realize that Coulomb's law is essential for this problem.

Bonnie: *Yeah. Negative two Q, since it's twice as far away.*
Alisa: *And, this is negative Q.*
Bonnie: *Negative two Q.*
Darlene: *Negative two Q.*
Alisa: *Are we going to go with that?*
Bonnie: *I think it makes sense.*
Darlene: *That makes...*
Alisa: *Well, I don't know, because when you're covering a distance you're using it in the denominator as the square.*
Bonnie: *Oh! Is that how it works?*
Alisa: *And [...inaudible...] makes a difference.*
Bonnie: *Yeah, you're right.*
Tuminaro: *So, how do you know that?*
All: *From the Coulomb's law.*
Bonnie: *So, it should actually be negative four q? Or what? Since it has…*
Alisa: *Cause we were getting into problems in the beginning of the problem with [a three-charge problem in which shifting a middle charge destroys a balance of forces because of Coulomb's law] because I thought that, like, if you move this a little bit to the right, the decrease for this would make up for the increase for this. But, then we decided it didn't. So, that's how I know that I don't think it would just increase it by a factor of two.*

The students relied on their intuitive ideas to generate a conceptual understanding of the situation—two mutually exclusive influences acting on q_3 that exactly cancel each other. Yet Alisa, recalling her experience of working on an earlier problem, realizes that their intuitive ideas are not enough; they need Coulomb's law.

"Mapping Meaning to Mathematics": Translating conceptual understanding into mathematical formalism. After some false starts and nearly 60 minutes, the students finally solve this problem, integrating their conceptual understanding, developed in terms of their common sense ideas, with formal application of Coulomb's law.

Tuminaro: *What did you do there?*
Alisa: *What did I do there?*
Tuminaro: *Yeah, can I ask?*
Alisa: *All right, so because this isn't moving, the two forces that are acting on it are equal: the push and the pull.*

Alisa reiterates the group's conceptual understanding of the physical situation: two mutually exclusive influences exactly canceling yielding no result. Next, she writes down the two forces in terms of Coulomb's law:

Alisa: *So, the F—I don't know if this is the right F symbol—but, the F q_2 on q_3 is equal to this [see Eq.(3)]. And, then the F q_1 on q_3 is equal to this [see Eq. (4)], because the distance is twice as much, so it would be four d squared instead of d squared.*

$$F_{q_2 \rightarrow q_3} = \frac{kQq_3}{d^2} \tag{3}$$

$$F_{q_1 \rightarrow q_3} = \frac{kxQq_3}{4d^2} \tag{4}$$

Alisa: *And then I used x Q like or you can even do—yeah—x Q for the charge on q_1, because we know in some way it's going to be related to Q like the big Q we just got to find the factor that relates to that…Then, I set them equal to each other…*

Alisa uses Coulomb's law to write the form of the two forces, but she does not formally invoke the other physics principle outlined in the ideal solution: Newton's second law. Rather, Alisa relies on her conceptual understanding of the physical situation to write that the forces must be equal—not on a formal application of Newton's second law. This feature of her solution lends additional evidence that Alisa is making sense of this problem rather than following a problem-solving algorithm.

After setting up the equation, Alisa is left with an algebra problem, which she has little trouble solving:

Alisa: *… and I crossed out, like, the q_2 and the k and the d squared, and that gave me Q equals x Q over four. And then x Q equals four Q, so x would have to be equal to four. That's how you know it's four Q.*

The other students then evaluate the plausibility of Alisa's recited solution, another indication that these students are making sense of this problem:

Bonnie: *Well, shouldn't it be—well, equal and opposite, but...*
Alisa: *Yeah, you could stick the negative.*
Bonnie: *Yeah.*
Darlene: *I didn't use Coulomb's equation, I just—but it was similar to that.*
Bonnie: *That's a good way of proving it.*
Darlene: *Uh-huh.*
Bonnie: *Good explanation.*
Alisa: *Can I have my A now?*

Alisa's final question is meant in jest, but shows that she realizes that she has understood and solved this problem successfully.

What Takes So Long?

The fact that students can take so long to solve problems like this one is a matter of concern to many instructors. Some instructors—and many students—assume that if students can't do the problem in a short amount of time, they can't do it at all. Our example shows this not to be the case. Given the time, students can recall and construct the background knowledge needed to make sense of the problem a step at a time. This is a long but likely necessary part of the process of creating their own knowledge compilations.

The details of the transcript[7] reveal some of the physics knowledge the students call on explicitly in working through this problem (Table I). Many of these items are new to them (only learned last semester) and are still being reconciled with their intuitive knowledge.

Time Well Spent

The students' problem-solving approach, while it takes much longer than an average teacher's approach, has expert-like features. Two of these are (1) the students rely heavily on their conceptual understanding, and (2) they themselves choose their solution path.

The students solve the problem by thinking of two mutually exclusive influences exactly canceling each other (and then applying Coulomb's law). At no point during the entire problem-solving episode do they use or make explicit reference to Newton's second law, even though it is the relevant physics principle for why q_3 remains in equilibrium. This is not a negative. It shows that the students are using their own understanding of the physical situation to generate a solution (an expert-like characteristic) rather than doggedly applying a formal physical principle (a novice-like characteristic).

Further, the students generate their own problem-solving path. They do not defer to the TA or let him direct what they should do next. Even as they follow the TA's suggestion to draw a picture, they use phrases like "we thought" or "I decided" (not "the TA said" or "the book says")—evidence that they feel they are in control of their approach.

Recognition Versus Formal Manipulation

In physics, especially at the college level, we tend to focus our attention on formal manipulation. We often don't realize how much of an expert's success is based on a more fundamental cognitive ability: recognition. The ability to handle language and formal reasoning is analogous to serial processing in computer technology—sequential and reasonably slow. The ability to recognize faces and places is analogous to parallel processing—all happening at once and very quickly.[8] When we identify a coffee cup or recognize a friend, we don't go through a formal check-list of properties; we just recognize them. Many physicists have had the experience of going through a tedious calculation, making mathematical manipulation after manipulation and then reaching a particular point and saying: "Oh! Now I get it." Either the rest of the calculation now becomes trivial or the previous steps now are obvious rather than formal.

In this paper, we reverse-engineered a simple physics problem, comparing the way instructors who already have lots of compiled knowledge solve it to the way a group of novice students who are still working on compiling their knowledge solve it. An immediately obvious difference between the two is that the students' solution took much more time—nearly two orders of magnitude more. Our analysis shows two things. First, even simple physics problems include conceptual and technical subtleties that experts tend to forget about. Experts are so familiar with these subtleties that they don't notice them, but they present significant difficulties for novices. Second, what we may at first judge to be poor student problem-solving behavior may actually be very desirable behavior. Careful analysis of the students' solution in the three-charge problem shows that their solution shares many features with an expert's solution. In addition, the

Table I. Some of the knowledge required by the student in each game.

E-Game	Physics Knowledge Needed
Physical Mechanism	• Like charges repel, unlike attract. • Attractions and repulsions are forces. • Forces can add and cancel. (one does not "win"; one is not "blocked") • "Equilibrium" corresponds to balanced, opposing forces (not a single strong "holding" force). • Electric force both increases with charge and decreases with distance from charge. • Objects respond to the forces they feel (not those they exert). • "Fixed" objects don't give visible indication of forces acting on them; "free" ones do.
Pictorial Analysis	• Only forces on the test charge require analysis. • Each other charge exerts one force on test charge. • Each force may be represented by a vector. • "Equilibrium" corresponds to opposing vectors. • Vertical and horizontal dimensions are separable. • 1D is sufficient for analysis.
Mapping Mathematics to Meaning	• Electric force both increases with charge and decreases with distance from charge. • Electric force decreases with the square of the distance.
Mapping Meaning to Mathematics	• Charges of indeterminate sign are appropriately represented by symbols of indeterminate sign. • Coefficients may relate similar quantities. • Balanced forces correspond to algebraically equal Coulomb's-law expressions.

students work toward the consolidation and reconciliation of new knowledge, which is just what they need to do at their stage of learning. To better assist our students, we need to better understand both what they know and the hidden components of our own knowledge. Only then can we effectively "reverse-engineer" what we know to help our students build expert problem-solving skills.

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