Initial understanding of vector concepts among students in introductory physics courses

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We report the results of an investigation into physics students' understanding of vector addition, magnitude, and direction for problems presented in graphical form. A seven-item quiz, including free-response problems, was administered in all introductory general physics courses during the 2000/2001 academic year at Iowa State. Responses were obtained from 2031 students during the first week of class. We found that more than one quarter of students beginning their second semester of study in the calculus-based physics course, and more than half of those beginning the second semester of the algebra-based sequence, were unable to carry out two-dimensional vector addition. Although the total scores on the seven-item quiz were somewhat better for students in their second semester of physics in comparison to students in their first semester, many students retained significant conceptual difficulties regarding vector methods that are heavily employed throughout the physics curriculum.
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I. INTRODUCTION

Vector concepts and calculation methods lie at the heart of the physics curriculum, underlying most topics covered in introductory courses at the university level. As Knight[1] has emphasized, the vector nature of forces, fields, and kinematical quantities requires that students have a good grasp of basic vector concepts if they are to be successful in mastering even introductory-level physics. Knight has alluded to the surprising lack of published research regarding student learning of vector concepts, and his *Vector Knowledge Test* provided an invaluable first glimpse into the pre-instruction vector knowledge of students enrolled in the calculus-based physics course. Most of the problems on the *Vector Knowledge Test* focus on algebraic aspects of vectors. Another significant investigation has been reported by Kanim,[2] who explored students' understanding of vector concepts in the context of electric forces and fields. Aguirre,[3] and Aguirre and Rankin[4] have studied students' ideas about vector kinematics, but their inquiry focused on the interrelationships among velocity, acceleration, and force rather than properties of vectors *per se*. Recently, Ortiz *et al.*[5] have reported on student learning difficulties related to basic vector operations (such as dot and cross products) as employed in introductory physics courses.

Our instructional experience has led us to believe that students' poor understanding of vector ideas posed in graphical form presents a particularly troublesome obstacle to their success in mastering physics concepts. Graphical and geometrical interpretations of vector ideas pervade the entirety of the general physics curriculum. Despite most students' previous exposure to vector concepts in mathematics courses or in high-school physics (as indicated by various surveys), and the heavy emphasis we have placed on those concepts in our own instruction, students' persistent confusion about fundamental vector notions has bedeviled our instructional efforts. We decided therefore to carry out a systematic investigation of university physics students' knowledge of basic ideas of vector addition, magnitude, and direction during the initial weeks of their physics courses. To this end, we surveyed students in both the first- and second-semester courses of the two-semester general physics sequence, both in algebra-based and calculus-based courses.

II. METHODS

We constructed a quiz containing seven vector problems posed in graphical form (see the Appendix). The problems assess whether students can correctly identify vectors with identical magnitudes and directions, and whether they can carry out vector addition in one and two dimensions. On five of the problems, students are asked to give a free response or to select multiple options from a list. On the other two (#3 and #7), they are given possible choices. On four problems students are explicitly prompted to provide explanations of their work.

This diagnostic quiz was administered to students in all introductory general physics courses taught at Iowa State University (ISU) during the 2000/2001 academic year. (We did not include in our study one-semester elementary physics courses using little or no mathematics; these courses are intended as surveys for nontechnical students.) Very minor revisions were made to the quiz between fall and spring semesters.

ISU is a large public university with a focus on engineering and technical subjects. The average ACT Mathematics score of all freshmen entering ISU in fall 2000 was 24.5, compared to the national average of 21.8 for students who completed the core college-preparatory curriculum.[6] ISU ranks 16th nationally in number of undergraduate engineering degrees awarded. It therefore seems unlikely that our results will underestimate the average performance level of physics students nationwide.

The algebra-based general physics sequence consists of Physics 111 (mostly mechanics), and Physics 112 (mostly electricity and magnetism, and optics). The calculus-based sequence is comprised of Physics 221 (mechanics, electrostatics, and dc circuits), and Physics 222 (magnetism and electromagnetism, thermal physics, optics, and modern physics). In this paper, we will use the following designations for these courses: Physics 111: A-I; Physics 112: A-II; Physics

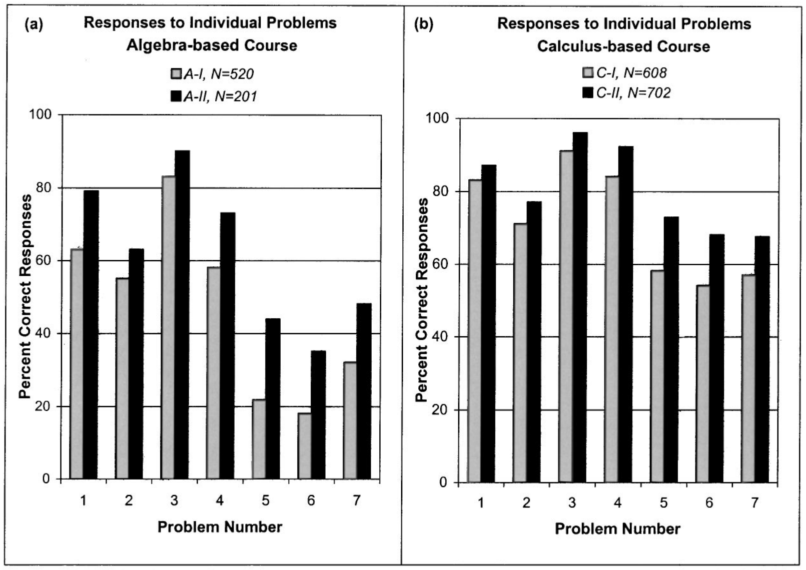

Fig. 1. Responses of students to individual problems on the vector concept diagnostic. Percent correct responses shown for students in (a) first- [A-I] and second-semester [A-II] algebra-based introductory physics; (b) first- [C-I] and second-semester [C-II] calculus-based introductory physics.

221: C-I; Physics 222: C-II, where I and II designate the first and second courses in each sequence, respectively. (That is, I is primarily a mechanics course, while II is primarily a course on electricity and magnetism. All four courses are taught during both the fall and spring semesters.) Results were obtained from a total of 2031 students, divided into the four courses as follows: Algebra-based physics: A-I, 520 total (fall: 287; spring: 233); A-II, 201 total (fall: 83; spring: 118). Calculus-based physics: C-I, 608 total (fall: 192; spring: 416); C-II, 702 total (fall: 313; spring: 389). (In the paper we refer to these courses as the ''four groups.'') Because the quiz was administered in both fall and spring offerings in all four courses during the academic year (that is, twice each in A-I, A-II, C-I, and C-II for a total of eight administrations), many students took the quiz twice, once in their fall-semester course and again in their spring-semester course. The number of repeat test-takers is not known.

We did not survey the students in this study sample with regard to their previous background in physics and mathematics. However, surveys carried out in ISU physics courses during the summer and in other years have indicated that nearly three quarters of students in the algebra-based courses, and more than 90% of those in the calculus-based courses, have studied physics in high school. In these surveys, a substantial majority of students report previous study of vectors including two-dimensional vector addition, either in their high-school physics classes or in high-school and/or college math courses. (This is the case for about two thirds of students in the algebra-based course, and about 90% of those in the calculus-based course.) These results are consistent with Knight's finding that 88% of students in the first quarter of the introductory calculus-based physics course at his institution had previous instruction on vectors. Of course, all students in the second-semester courses (that is, Physics 112 [A-II] and Physics 222 [C-II]) have had extensive exposure to vector representations and calculations in their first-semester university courses. They represent 44% of the total population sample in this study.

The quiz was administered in recitation sections (around 25 students each) during the first week of class in all four courses, before instruction on vectors took place. The quiz did not count toward a course grade and was not returned to the students. Students were asked to respond to the quiz so that instructors could get a better idea of their background knowledge in vectors. They were asked to fill in their names on the quiz to aid in record keeping. The same procedure was followed in both fall and spring semesters. Responses were obtained from the great majority of enrolled students. Responses on each problem were graded as correct or incorrect, and frequently appearing errors were noted and tabulated.

III. RESULTS

All statistical results we will cite in this paper (except for those in Sec. III C) reflect averages over the entire sample, that is, fall- and spring-semester offerings combined in the case of each of the four courses.

A. Responses to problems

Figure 1 shows the percentage of correct responses to each quiz item for students in all four courses. We now proceed to discuss the students' responses to each individual problem in more detail.

Problem #1: Vector magnitude. Performance on this problem was generally good, with a range of 63%–87% correct responses for the four different groups. However, more than one third of the students in A-I did *not* answer this question correctly, which indicates that student knowledge even on this basic vector property cannot be taken for granted. The most common error was to assume that vectors can only have equal magnitudes when they are parallel or antiparallel to each other (for example, choosing $|\vec{\mathbf{D}}|=|\vec{\mathbf{G}}|$, but not $|\vec{\mathbf{D}}| =|\vec{\mathbf{F}}|=|\vec{\mathbf{G}}|$).

Problem #2: Vector direction. A significant number of students in all classes made errors on this question (23%–45% incorrect responses). It is notable that there was very little difference in performance between students in the first- and second-semester courses, both in the algebra-based and calculus-based sequences. This small performance increment seems to suggest that, particularly on this problem, little increase in understanding occurs during the first-semester course (that is, in A-I and C-I).

The single most common incorrect response was to list both vectors $\vec{\mathbf{F}}$ *and* $\vec{\mathbf{G}}$, instead of $\vec{\mathbf{F}}$ *only*, thus reflecting confusion about the requirement that vectors with the same direction be parallel to each other. (Or, perhaps this response indicates confusion about how to *recognize* when two vectors are parallel.) This error represented 20% of all responses (almost half of all incorrect responses) in the algebra-based course, with no significant difference between A-I and A-II. However, there also were a significant number of students responding that the answer was "none"; this category comprised 11% of all responses in the algebra-based course (one quarter of all incorrect responses in both A-I and A-II). Remarkably, those students who answered "none" very often asserted explicitly that all of the angles—or the "slopes"—were different, despite the presence of the grid, which was intended to allow easy evaluation of the angles. The other option appearing with some frequency on students' responses was vector $\vec{\mathbf{C}}$, thus equating the direction of vector $\vec{\mathbf{A}}$ with that of $-\vec{\mathbf{A}}$. [It is worth noting that outside the U.S., the property we refer to as "direction" often is assumed to comprise two separate properties, that of "orientation" (line of action) and "sense" (loosely, "which way it points"), see, for example, Ref. 7.]

Problem #3: Qualitative vector addition. Performance on this problem was very good for students in all courses, with correct responses in the 83%–96% correct range. However, students were not asked to provide explanations of their answer, and evidence provided by student performance on problems #4 and #5 strongly suggests that many students arrived at the correct answer for problem #3 through use of a clearly incorrect algorithm (that is, the "split-the-difference" algorithm to be discussed after problem #5). Because use of this algorithm reflects substantial confusion regarding vector addition, it seems probable that problem #3 does not in itself provide valid assessment of students' understanding of this vector operation.

Problem #4: One-dimensional vector addition. The students in the calculus-based courses performed very well on problem #4: C-I, 84% correct; C-II, 92% correct. However, a substantial fraction of the students in the algebra-based courses were not able to solve this problem: A-I, 58% correct; A-II, 73% correct.

In A-I, 19% of all incorrect responses consisted of a two-headed arrow as shown in Fig. 2(a); in A-II, this response was only 11% of the incorrect responses. Often this arrow was eight boxes long, but other lengths were common. Representative explanations for this response were, "$\vec{\mathbf{R}}$ is made by connecting the end of $\vec{\mathbf{A}}$ to the end of $\vec{\mathbf{B}}$," and "It is just the two vectors put together." Another common error in the algebra-based course (23% of all incorrect responses in A-I and A-II combined) was to show a horizontal resultant with incorrect magnitude and/or direction.

Many students produced a sloping resultant; in A-I these represented 20% of the incorrect responses, which rose to 36% in A-II. Most of these students did not show their work, but those who did typically had a diagram similar to one of those in Figs. 2(b)–2(d). Sometimes these students would explain that they were using the "tip-to-tail" method, or words to that effect.

Performance on problem #4 was not as good as it was on problem #3, particularly in the algebra-based courses. We suspect that, in comparison to problem #3, it may be more

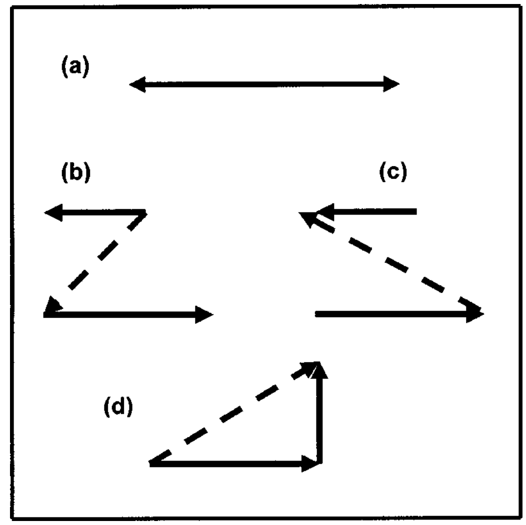

Fig. 2. Common student errors on problem #4 (addition of collinear vectors): (a) two-headed arrow; (b) tail-to-tail; (c) tip-to-tip; (d) re-orientation of top vector.

difficult to obtain a correct solution for problem #4 by using an incorrect algorithm. We will return to this issue in the discussion of problem #5.

Problem #5: Two-dimensional vector addition. The vast majority of problems in the general physics curriculum that involve vector quantities require an understanding of this basic operation. We found that most students in the calculus-

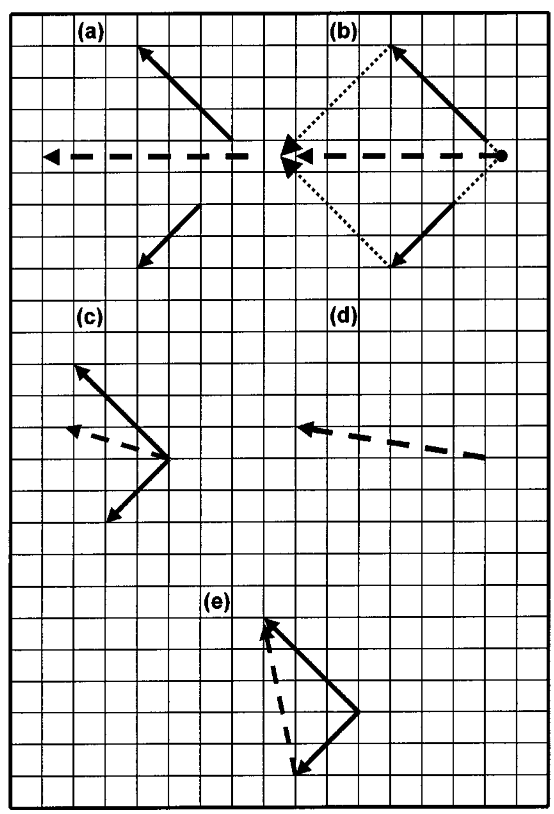

Fig. 3. Common student errors on problem #5 (addition of noncollinear vectors): (a) zero vertical component; (b) split-the-difference algorithm; (c) incorrect parallelogram addition; (d) incorrect horizontal component; (e) tip-to-tip error.

based course solved this problem correctly (58% in C-I, 73% in C-II), but only a minority of students in the algebra-based course could do so (22% in A-I, 44% in A-II).

The most common error for all four groups was to draw the resultant vector aligned along the horizontal axis (or nearly so), pointing toward the left [Fig. 3(a)]. The magnitudes of the horizontal components in this class of responses varied widely. Although some of the students who made this error were successful in determining the net horizontal component (that is, five boxes, leftward), all failed to realize that the net vertical component would be one box upward. Many students' diagrams explicitly showed the algorithm they used to obtain this result: *Join vectors* $\vec{\mathbf{A}}$ *and* $\vec{\mathbf{B}}$ *at a common vertex, and form the resultant by "splitting the difference" to obtain a net vertical component of zero* [see Fig. 3(b)]. This response was usually a clear attempt to implement a parallelogram addition rule. Some students explicitly used a very similar algorithm [see Fig. 3(c)] to obtain an apparently related error, that is, a resultant vector with the correct vertical component and pointing toward the left, but with an incorrect horizontal component. Although a particular example of this response is shown in Fig. 3(d), the magnitudes of the horizontal components represented in students' responses covered a wide range. It was not clear to us how they were able to arrive at the correct vertical component while still having an incorrect horizontal component. It seems possible that the positioning of the $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ vectors on the page—that is, one on top of the other—contributed to this outcome. It is noteworthy that in a large proportion of cases where students drew diagrams suggestive of the parallelogram addition rule, they were unsuccessful in arriving at a correct answer to this problem. Instead they produced variants of Figs. 3(b) or 3(c), or made some other error due to imprecise drawing of the parallelogram.

Most students who drew resultant vectors similar to those in Figs. 3(a) and 3(d) did not show a diagram to explain how they obtained their result. Therefore, we cannot be certain that they used the same algorithm to obtain this split-difference resultant. The proportion of the entire class that gave incorrect responses corresponding to either Fig. 3(a) or Fig. 3(d) (regardless of the horizontal component) was A-I, 42%; A-II, 29%; C-I, 21%; and C-II, 13%.

The next most common error on this problem originated from mistaken employment of a "tip-to-tip" algorithm in which the resultant vector begins at the tip of vector $\vec{\mathbf{A}}$ and ends at the tip of vector $\vec{\mathbf{B}}$ or, less often, points from the tip of $\vec{\mathbf{B}}$ to that of $\vec{\mathbf{A}}$. (This error also has been described by Knight.[1]) In this case the interpretation of students' responses was unambiguous because their diagrams explicitly showed the algorithm they had employed. There are two versions of this error: either the vectors are first brought together to a common vertex (see Fig. 3(e); this procedure actually produces the difference vector [$\vec{\mathbf{B}}-\vec{\mathbf{A}}$]), or they are left in place and the "resultant" arrow is drawn directly on the original diagram. This type of response (either version) was given by 9% of students in the algebra-based course and 6% of those in the calculus-based course, with very little difference between the I and II courses.

As was noted in connection with problems #3 and #4, the number of correct responses on problem #3 was well above that on problem #4. We now see that it was also far higher than the correct response rate on problem #5. In view of the obvious route for obtaining a correct answer to problem #3 by using the incorrect "split-the-difference" algorithm, we now believe that problem #3 is not a valid indicator of students' knowledge of vector addition.

Problem #6: Two-dimensional vector subtraction. In principle this problem could be solved with the same algorithms used for problem #5, combined with some algebraic manipulation and knowledge of how to form $-\vec{\mathbf{A}}$ from $\vec{\mathbf{A}}$. However, students probably have less practice with a specific algorithm for carrying out vector subtraction, compared with vector addition. That is, students may have memorized "place the tail of one to the tip of the other" as an addition algorithm without gaining enough understanding to extend this idea to a similar problem posed as a subtraction. One might therefore expect that performance on problem #6 would be inferior to that on problem #5, and indeed it was. However the difference was generally rather small: only 4–5% fewer correct in the calculus-based course, and 4% and 9% fewer, respectively, in A-I and A-II. Overall, error rates on problem #6 ranged from 32% incorrect in C-II, up to 82% incorrect in A-I.

In the calculus-based course (both C-I and C-II combined), 83% of the students who answered problem #5 correctly also answered problem #6 correctly. Similarly, 89% of those who answered problem #6 correctly also answered problem #5 correctly. (There was no significant difference between C-I and C-II students regarding this pattern.) This response pattern suggests that for students in the calculus-based course, problem #5 and problem #6 provide a roughly equivalent indication of students' understanding of two-dimensional vector addition.

By contrast, in the algebra-based course, only 67% of students who answered problem #5 correctly also answered problem #6 correctly. Of the students who answered problem #6 correctly, 83% also solved problem #5. (Again, there was no significant difference between A-I and A-II.) So, for students in the algebra-based course, problem #6 was indeed significantly more difficult than problem #5 ($p<0.01$ according to a z test for difference between correlated proportions[8]). In this case the two problems did not provide equivalent indications of students' knowledge, because a correct solution to problem #6 was correlated with superior performance on this two-problem subset.

There were a wide variety of incorrect responses to problem #6. Many students' explanations made it clear that they were trying to find a $\vec{\mathbf{B}}$ such that $\vec{\mathbf{R}}$ would be the "average," in some sense, of $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$. However, lacking an algorithm for this purpose, students often resorted to guessing or estimating the direction of vector $\vec{\mathbf{B}}$. A common response was to draw $\vec{\mathbf{B}}$ as a horizontal vector (vertical component$=0$) pointing to the right; one-quarter of all incorrect responses were of this type in both algebra-based and calculus-based courses (algebra based, 26%; calculus based, 25%). These vectors were drawn either with their tails in contact with the tail of $\vec{\mathbf{A}}$ or, more often, as isolated vectors in the blank grid space to the right of $\vec{\mathbf{A}}$ and $\vec{\mathbf{R}}$. Most students did not explain their reasoning, but some offered clear descriptions of their thinking such as "$\vec{\mathbf{R}}$ should be a combination of $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ so I tried to put it between $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$"; "The magnitude of $\vec{\mathbf{B}}$ and $\vec{\mathbf{A}}$ are equal, so the direction of the resultant is directly between the two." Overall, a large majority of students with incorrect

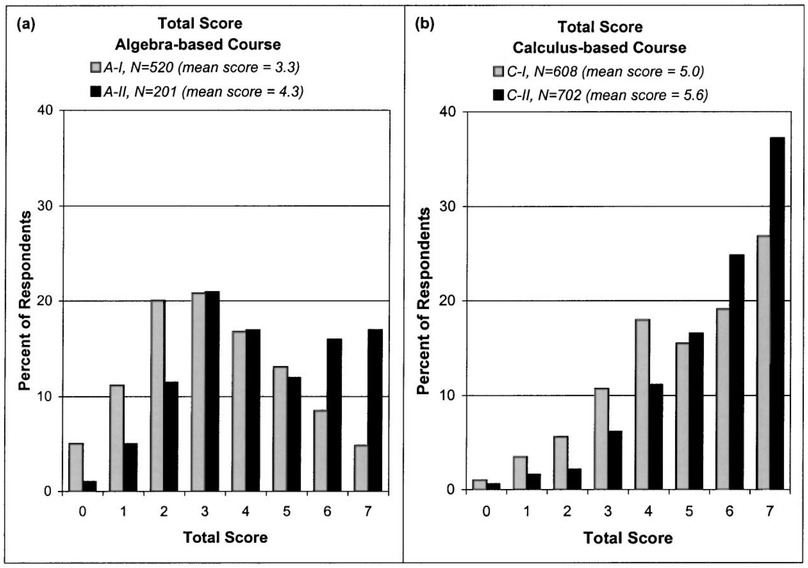

Fig. 4. Distribution of total scores on vector concept diagnostic in percent of class obtaining a particular score (score range: 0–7): (a) first- [A-I] and second-semester [A-II] algebra-based introductory physics; (b) first- [C-I] and second-semester [C-II] calculus-based introductory physics.

responses to this problem realized that $\vec{\mathbf{B}}$ would have a positive horizontal component, but were unable to determine its precise value.

Problem #7: Comparison of resultant magnitude. This problem is another application of vector addition for which students are unlikely to have memorized a specific algorithm. With no grid available, students do not have at hand as straightforward a calculation procedure as might be employed in problems #5 and #6. However, only a qualitative response is required on problem #7, while a precise quantitative answer is needed for problem #5; moreover, there are only three possible choices. This smaller selection of options may mitigate the additional challenge posed by problem #7 (if there is any). In any case, the only group for whom performance on problems #5 and #7 differed by more than 5% was students in A-I; they achieved 32% correct on problem #7 compared to only 22% correct on problem #5. However, it is interesting to note that 23% of the C-II students who successfully solved problem #5 also gave incorrect responses to problem #7. It seems that the apparently superior algorithmic skill of the C-II students did not always translate to a situation in which a grid was lacking.

Many students who chose the correct ("smaller than") response in problem #7 gave a satisfactory explanation of their answer, often accompanied by a diagram that reflected use of the parallelogram or tip-to-tail addition rules to demonstrate that $|\vec{\mathbf{R}}_{\mathbf{A}}|<|\vec{\mathbf{R}}_{\mathbf{B}}|$. Among those students who gave incorrect answers, there was a preference for the "equal to" response (that is, magnitude of resultant of pair **A** is equal to that of pair **B**), very often justified by an explanation such as "the vectors in **A** and **B** are equal magnitude," and sometimes accompanied by an invalid application of the Pythagorean formula to pair **B**. The ratio of "equal to" responses in comparison to "larger than" responses was almost exactly 1:1 in A-I, but in A-II the "equal to" response jumped in popularity to nearly a 2:1 ratio compared to "larger than." In both C-I and C-II, the "equal to" response was the more common incorrect response by nearly a 3:2 ratio. The "larger than" response was justified by the larger vertex angle or the "larger area covered" in diagram A. Explanations such as these were typical: "**A** is larger because arrows are further apart"; "**A** larger, the angle is greater between the vectors"; "larger than because both vectors are farther apart than the ones in **B**."

B. Total score comparisons

The distribution of students' total score on the diagnostic (maximum=7.0) is shown in Fig. 4. The scores in A-I are fairly normally distributed around a mean value of 3.3, while the A-II distribution (mean score=4.3) is somewhat bimodal. The distributions in the calculus-based course are very strongly skewed toward higher scores (although that in C-I is also somewhat bimodal). Mean scores for the calculus-based course are C-I: 5.0; C-II: 5.6. These distributions suggest that the diagnostic is a good reflection of the mean level of knowledge of students in the algebra-based courses, whereas the average level of vector knowledge of students in the calculus-based courses goes beyond that characterized by this diagnostic.

C. Differences in performance between fall- and spring-semester courses

We were surprised to find that on many of the quiz items, there appeared to be a significant difference in performance between students in the fall and spring offerings of the *very* same course (for example, the fall and spring offerings of A-I). Students enrolled in C-I during the spring semester of 2001 had higher scores on all seven quiz items than students in the fall 2000 semester of the same course. The mean scores (percent correct out of seven problems; s.d.=standard deviation) were: spring, 2001 ($N=416$): 74% correct (s.d. =25%); fall, 2000 ($N=192$): 65% correct (s.d.=27%). The difference in mean scores is statistically significant at the $p=0.0003$ level according to a two-sample t-test. A very similar fall–spring discrepancy was found for students in A-I (spring, 51%; fall, 44%; $p<0.001$). For C-II there was a smaller but still statistically significant superiority, this time however in the *fall* semester mean scores (fall, 83%; spring, 78%, $p<0.01$) while in A-II, the fall–spring difference in mean scores was very small and not statistically significant.

For C-II, on a three-item group of closely related problems (problems #5, #6, and #7), fall performance was significantly better (fall, 76%; spring, 64%; $p<0.001$ according to a chi-square test). In A-II, a fall–spring difference on the same three-item group was again present (fall, 49%; spring, 38%), but did not quite rise to the level of statistical significance ($p=0.12$), perhaps because of the relatively small sample size.

Although it seems clear that the discrepancy in performance between students in the fall- and spring-semester offerings of A-I and C-I is not due to chance—and the same may be true for the inverse effect observed in A-II and C-II—we do not have data that would allow us to determine the cause. Many factors might contribute (for example, students repeating courses, advanced students preferring ''off-sequence'' offerings, etc.), but at this point we can only speculate on this matter.

IV. DISCUSSION

The concepts probed in this diagnostic are among the most basic of all vector ideas. Students are assumed to have a good understanding of them throughout all but the first week or two of the introductory physics curriculum. Although a very brief (less than one lecture) discussion related to these concepts is usually provided near the beginning of the first-semester course, students often are assumed to have been exposed to vector ideas either in their mathematics courses or in high-school physics, with the further assumption that very little review is needed. The emphasis of the discussion and use of vector concepts in the college-level physics course is decidedly on the algebraic aspects and is directed toward calculational competence. As a consequence, graphical and geometrical interpretations of vector operations may be somewhat neglected.

(As a point of reference on this issue, we note that of the seven high-school physics textbooks surveyed in a recent study,[9] all but one[10] cover vector concepts to some extent, including one- and two-dimensional vector addition presented in graphical form. Most of these texts go into considerable detail. No doubt the actual extent of vector coverage in high-school physics courses varies very widely throughout the nation.)

We found that a significant proportion of students in our sample had serious conceptual confusion related to basic vector concepts represented in graphical form, even though surveys suggest that most of them had previous instruction in vectors. (More than 44% of students in our sample had taken at least one full semester of university-level mechanics.) Even in the second semester of the calculus-based physics course (that is, C-II)—in which students are assumed from the very first day to have considerable expertise with vector methods—more than one-quarter of the class could not carry out a two-dimensional vector addition. Our data from the second semester of the algebra-based course (that is, A-II) suggest that the majority of students in the first semester of this course (A-I) never successfully mastered this operation. This finding should have rather sobering implications for instructors who assume that, for example, students beginning study of electric field superposition are competent with vector addition.

On many of our quiz items, improvements in student performance from first to second semester were small or practically nonexistent, indicating that little learning of the ideas had taken place during the first-semester course. This small performance improvement was observed for both algebra-based and calculus-based courses. It seems that the bulk of students' basic geometrical understanding of vectors was brought with them to the beginning of their university physics course and was little changed by their experiences in that course, at least during the first semester.

It seemed clear that, although most students were unable to solve two or more of the problems, they did have some degree of basic knowledge which they attempted to apply to the problems they missed. For instance, there often were efforts to apply a tip-to-tail rule or a parallelogram addition rule which were unsuccessful due to imprecise execution. Frequently, students did not accurately copy the magnitude and/or the direction of the vectors they were attempting to add. Often, they were uncertain as to which ''tail'' was supposed to be in contact with which ''tip.''

Many students had an intuitive feel for how vectors should add which, it was clear, was based on their experience with forces. Although the word ''force'' is not used in the quiz, many students referred to the vectors as ''forces'' and used dynamical language to describe their thinking, such as how one vector was ''pulling'' the other in a certain direction, or how the ''pulls'' of two vectors would balance out. In many cases students were able to estimate the approximate direction of a resultant without being able to give a correct quantitative answer.

It seemed to us that many of the students' errors could perhaps be traced to a single general misunderstanding, that is, of the concept that vectors may be moved in space in order to combine them as long as their magnitudes and directions are exactly preserved. We suspect that, to some extent, this misunderstanding results in part from lack of a clear concept of how to determine operationally a vector's direction (through slope, angle, etc.)

As mentioned in Sec. I, very few reports of students' vector understanding have been published. We may make direct comparison, however, with the results reported by Knight[1] for problem 5 of his *Vector Knowledge Test*. This problem is very similar to problem #5 on our own quiz. Knight found that 43% of students in the first-quarter calculus-based course at California Polytechnic State University, San Luis Obispo, were able to answer that problem correctly. This statistic may be compared to the 58% correct response rate we observed on problem #5 in the first-semester calculus-based course (C-I) at ISU. Although the difference is statistically significant it is not particularly large, and might be accounted for by slight differences both in the test problems and in the student populations.

Another comparison we may make is to the results reported by Kanim on a problem involving net electrical force on a charge;[11] this problem is similar to our problem #7. He reports that 70% of students in a second-semester calculus-based course at the University of Illinois gave a correct response to that question, nearly identical to the 68% correct response rate to problem #7 in our second-semester calculus-based course (C-II). Kanim reports similar results on related problems among students at the University of Washington and elsewhere.

V. CONCLUSION

In previous investigations, Knight[1] and Kanim[2] have documented a variety of serious student difficulties with both algebraic and graphical aspects of vector concepts among students in introductory physics courses at several institutions similar to our own. Their results and ours consistently support a conclusion that significant additional instruction on vectors may be needed if introductory physics students are to master those concepts. We suspect that most instructors would be unsatisfied with a situation in which more than half of the students are still unable, after a full semester of study, to carry out two-dimensional vector addition (as we found to be the case in the algebra-based course).

It is clear from our findings that many students have substantial intuitive knowledge of vectors and vector superposition, obtained to some extent by study of mechanics, and yet are unable to apply their knowledge in a precise and therefore fruitful manner. They seem to lack a clear understanding of what is meant by vector direction, of how a vector may be ''moved'' so long as its magnitude and direction are strictly preserved, and of exactly how to carry out such moves by parallel transport. Many students are confused about the tip-to-tail and parallelogram addition rules.

One way in which vector addition may be introduced is through the use of displacement vectors, because students all have experiences that could allow understanding of how a 50-m walk to the east and subsequent 50-m walk to the north is equivalent to a 71-m walk to the northeast. Students could be guided to determine similar equivalent displacements—perhaps initially by using a grid—when the component displacements are at arbitrary angles. In order to solidify the notion of vector addition, it also would be important for students to practice applying these methods when no grid or other means for quantitative measurement is available. Many of the responses by students in our study (in particular, to problem #7) suggest that an ability to solve vector problems when a grid is available do not always translate to a similar ability in the absence of a grid. Recent interviews carried out by our group lend support to this observation.[12] We believe that curricular materials that guide students through a series of exercises in which they perform vector additions and subtractions (both with and without use of a grid) may be useful in improving their understanding of these ideas.

Further research will be needed to determine whether curricular materials based on such a strategy are effective in improving both students' performance on assessments such as the quiz used in our study, and students' ability to provide explanations of their work with precision (describing a clearly delineated calculational procedure) and accuracy (describing a *correct* calculational procedure). Additional research (such as that initiated by Oritz *et al.*[5]) is necessary to probe students' understanding of more advanced vector concepts such as scalar and vector products.

As a consequence of our findings, we have increased the amount of instructional time we devote specifically to vector concepts. We have developed some instructional materials[13] in a format similar to the problems on our diagnostic quiz, and continue development and assessment of additional materials. Our group has carried out a preliminary series of student interviews to shed additional light on student understanding of vector concepts. We are also extending our research to assess students' understanding of more advanced concepts, such as scalar and vector products, coordinate systems and rotations, etc. In addition, we are examining student understanding of vector ideas, specifically in the context of physics concepts such as superposition of forces and fields.

ACKNOWLEDGMENTS

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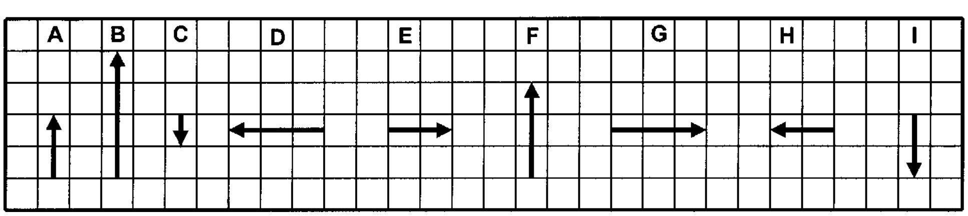
APPENDIX: VECTOR CONCEPT QUIZ

Name: ____________________

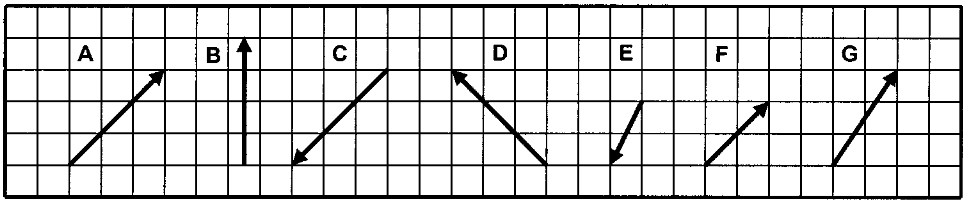
Class: ____________________ Section: ____________________

1. Consider the list below and write down **all** vectors that have the same magnitudes as each other. For instance if vectors $\vec{\mathbf{W}}$ and $\vec{\mathbf{X}}$ had the same magnitude, and the vectors $\vec{\mathbf{Y}}$, $\vec{\mathbf{Z}}$, and $\vec{\mathbf{A}}$ had the same magnitudes as each other (but different from $\vec{\mathbf{W}}$ and $\vec{\mathbf{X}}$) then you should write the following: $|W|=|X|$, $|Y|=|Z|=|A|$.



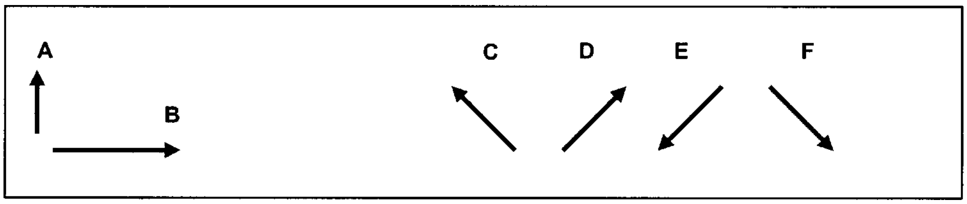
Answer ______________________________

2. List all the vectors that have the same **direction** as the first vector listed, $\vec{\mathbf{A}}$. If there are none, please explain why.



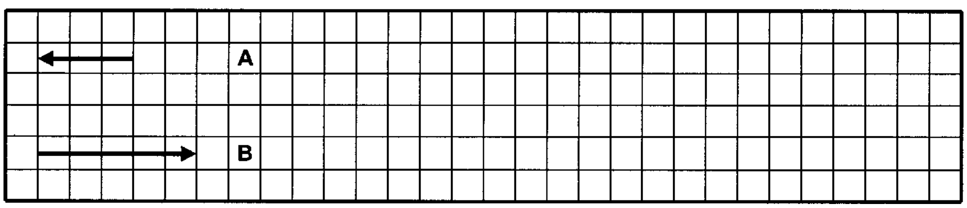
Explain ____

3. Below are shown vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$. Consider $\vec{\mathbf{R}}$, the vector sum (the "resultant") of $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$, where $\vec{\mathbf{R}}=\vec{\mathbf{A}}+\vec{\mathbf{B}}$. Which of the four other vectors shown ($\vec{\mathbf{C}}$,$\vec{\mathbf{D}}$,$\vec{\mathbf{E}}$,$\vec{\mathbf{F}}$) has most nearly the ***same direction*** as $\vec{\mathbf{R}}$?



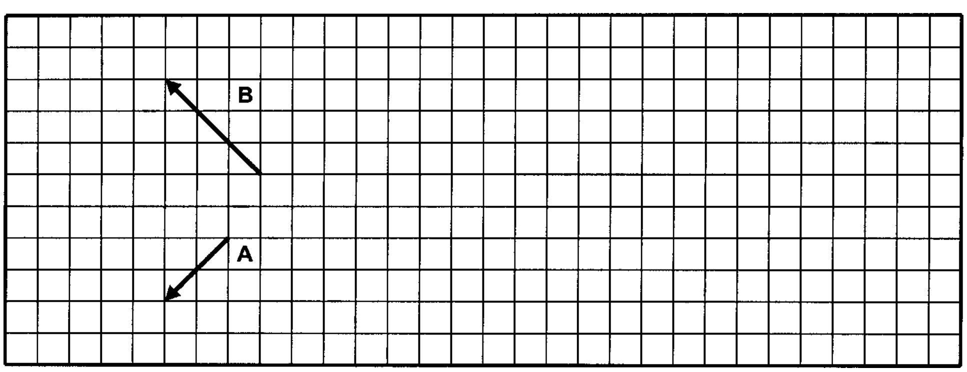
Answer ____

4. In the space to the right, draw $\vec{\mathbf{R}}$ where $\vec{\mathbf{R}}=\vec{\mathbf{A}}+\vec{\mathbf{B}}$. Clearly label it as the vector $\vec{\mathbf{R}}$. Explain your work.

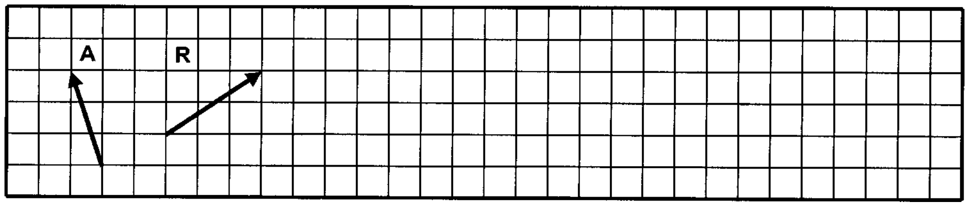


Explain ____

5. In the figure below there are two vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$. Draw a vector $\vec{\mathbf{R}}$ that is the sum of the two, (i.e., $\vec{\mathbf{R}}=\vec{\mathbf{A}}+\vec{\mathbf{B}}$). Clearly label the resultant vector as $\vec{\mathbf{R}}$.



6. In the figure below, a vector $\vec{\mathbf{R}}$ is shown that is the ***net resultant*** of two other vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ (i.e., $\vec{\mathbf{R}}=\vec{\mathbf{A}}+\vec{\mathbf{B}}$). Vector $\vec{\mathbf{A}}$ is given. Find the vector $\vec{\mathbf{B}}$ that when added to $\vec{\mathbf{A}}$ produces $\vec{\mathbf{R}}$; clearly label it $\vec{\mathbf{B}}$. **DO NOT** try to combine or add $\vec{\mathbf{A}}$ and $\vec{\mathbf{R}}$ directly together! Briefly explain your answer.



Explain ________________________________

____

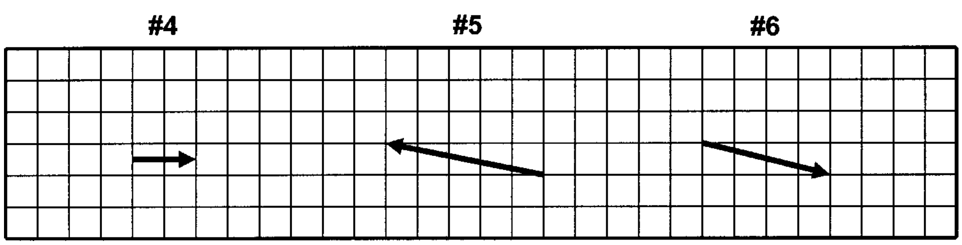
7. In the boxes below are two pairs of vectors, pair **A** and pair **B**. (All arrows have the same length.) Consider the magnitude of the ***resultant*** (the vector sum) of each pair of vectors. Is the magnitude of the resultant of pair **A** *larger than*, *smaller than*, or *equal to* the ***magnitude*** of the resultant of pair **B**? Write an explanation justifying this conclusion.

Explain ________________________________

____

Problem solutions:

1. $|A|=|E|=|H|=|I|$, $|D|=|F|=|G|$
2. **F**
3. **D**



7. smaller than.

[a]Electronic mail: nguyenn@iastate.edu
[b]Electronic mail: dem@iastate.edu

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