QUICK NOTE ON ADDING COMPLEX NUMBERS & ADDING VECTORS

You have already begun to see how much of QM amounts to calculating amplitudes and absolute-squaring them to find probabilities.

Interference, then, is all about what probabilities $P_T = |A_1 + A_2|^2$ do as a function of phase.

Take $A_1 = \sqrt{P_1}\, e^{i\varphi_1}$, $A_2 = \sqrt{P_2}\, e^{i\varphi_2}$.

Brute force:
$$|A_1 + A_2|^2 = \left(\sqrt{P_1}\, e^{-i\varphi_1} + \sqrt{P_2}\, e^{-i\varphi_2}\right)\left(\sqrt{P_1}\, e^{i\varphi_1} + \sqrt{P_2}\, e^{i\varphi_2}\right)$$
$$= P_1 + P_2 + \sqrt{P_1 P_2}\left(e^{i(\varphi_1 - \varphi_2)} + e^{i(\varphi_2 - \varphi_1)}\right)$$
$$= \boxed{P_1 + P_2 + 2\sqrt{P_1 P_2}\cos(\varphi_2 - \varphi_1)}$$

Note: $\text{Min} = P_1 - P_2 - 2\sqrt{P_1 P_2} = \left(\sqrt{P_1} - \sqrt{P_2}\right)^2$

$\text{Max} = P_1 + P_2 + 2\sqrt{P_1 P_2} = \left(\sqrt{P_1} + \sqrt{P_2}\right)^2$

Observation: only relative phases ever matter.

$$\left|\sqrt{P_1}\, e^{i\varphi_1} + \sqrt{P_2}\, e^{i\varphi_2}\right|^2 = \left|e^{i\varphi_1}\left(\sqrt{P_1} + \sqrt{P_2}\, e^{i\Delta\varphi}\right)\right|^2 \quad (\text{where } \Delta\varphi \equiv \varphi_2 - \varphi_1)$$
$$= \underbrace{\left|e^{i\varphi_1}\right|^2}_{1} \cdot \left|\sqrt{P_1} + \sqrt{P_2}\, e^{i\Delta\varphi}\right|^2$$
$$= \left|\sqrt{P_1} + \sqrt{P_2}\, e^{i\Delta\varphi}\right|^2$$

Elegant math tricks:

① $|A+B|^2 = (A^*+B^*)(A+B) = |A|^2+|B|^2+ A^*B + \underbrace{AB^*}_{=(A^*B)^*}$

$\underbrace{A^*B+(A^*B)^*}_{}$

$A^*B+(A^*B)^* = 2\mathrm{Re}(A^*B)$

$\therefore |A+B|^2 = |A|^2+|B|^2+2\underbrace{\mathrm{Re}(A^*B)}_{|A|\cdot|B|\cdot\cos(\varphi_B-\varphi_A)}$

② $|e^{i\varphi_1}+e^{i\varphi_2}|^2$ may alternatively be simplified

$$\left|\left(e^{i\frac{\varphi_1+\varphi_2}{2}}\right)\left(e^{i\frac{\varphi_1-\varphi_2}{2}}+e^{-i\frac{\varphi_1-\varphi_2}{2}}\right)\right|^2$$

$$= \underbrace{|e^{i\varphi_{avg}}|^2}_{1}\cdot\Big|\underbrace{e^{i\Delta\varphi/2}+e^{-i\Delta\varphi/2}}_{2\cos(\Delta\varphi/2)}\Big|^2$$

$$= 4\cos^2\frac{\Delta\varphi}{2}$$

(which of course $= 2+2\cos\Delta\varphi$, the form we derived earlier; but often such trig identities are easier to see with complex exponentials.)

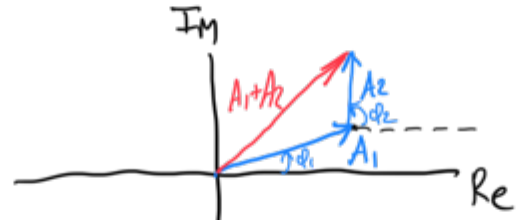
Geometrical picture...

Recall that $|A|^2$ is just the square of the length of the vector representing A in the complex plane.

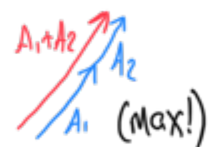
Complex numbers add exactly like vectors:

$$(R_1 + i I_1) + (R_2 + i I_2) = (R_1 + R_2) + i(I_1 + I_2)$$

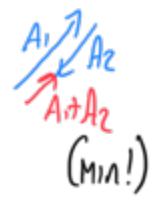
$$(R_1, I_1) + (R_2, I_2) = (R_1 + R_2,\ I_1 + I_2)$$



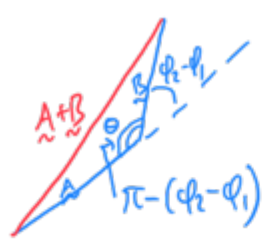
If $\varphi_1 = \varphi_2$, $|A_1 + A_2| = |A_1| + |A_2|$:

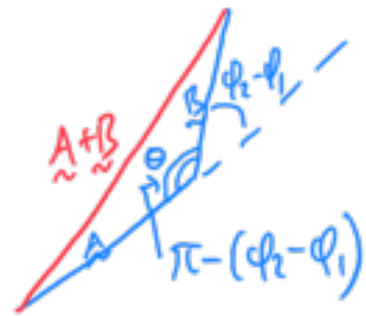


If $\varphi_1 = \varphi_2 + \pi$, $|A_1 + A_2| = \big||A_1| - |A_2|\big|$:



In general, the length of the resultant depends on the angle between the two vectors, but not their individual orientations.





Law of cosines: $|\underset{\sim}{A}+\underset{\sim}{B}|^2 = |\underset{\sim}{A}|^2 + |\underset{\sim}{B}|^2 - 2|A|\cdot|B|\cos\theta$

$$= |A|^2 + |B|^2 + 2|A|\cdot|B|\cos(\varphi_2-\varphi_1)$$

Note that $\Delta\varphi = 0 \Rightarrow |A+B|^2 = |A|^2+|B|^2+2|A|\cdot|B| = (|A|+|B|)^2$

$\Delta\varphi = \pi \Rightarrow |A+B|^2 = |A|^2+|B|^2-2|A|\cdot|B| = (|A|-|B|)^2$

$\Delta\varphi = \pi/2 \Rightarrow |A+B|^2 = |A|^2+|B|^2$

(right triangle)

amplitudes which are 90° out of phase add "in quadrature," like sides of a right triangle — the same result you'd get if you incoherently added probabilities (neither constructive nor destructive interference).