Prof. Aephraim Steinberg

Physics 485/1860: Problem Set #1
assigned 26 September, 2002
(due 8 October, 2002)

1. Milonni and Eberly, Problem 2.10

2. Milonni and Eberly, Problem 2.12

3. By calculating the work done on an electron by an applied electromagnetic field, it is possible to derive the absorption cross-section.

   (a) Consider a Lorentz atom of natural frequency ω_0 driven exactly on resonance by a coherent drive $E(t) = E_0 \cos \omega_0 t$. Derive the expression for the position of the electron versus time.

   (b) Since the force applied to the electron is $-eE$, the instantaneous work done on the atom is $-eE(t)\dot{x}(t)$. Calculate the energy deposited per optical period, and deduce the average power absorption.

   (c) By comparing this absorbed power with the incident intensity corresponding to a field E_0, find the expression for the resonant absorption cross-section.
   Using the decay rate calculated in 2.10, rewrite this expression and simplify– give a numerical estimate for a typical atom.

   (d) Consider how the above calculations would be modified by inclusion of an oscillator strength (Jackson sec. 7.5) in the Lorentz model. What effect would this have on the decay rate? ...on the resonant absorption cross-section? ...on the frequency-integrated absorption cross-section?

4. For an atom with two levels $|g\rangle$ and $|e\rangle$ with energies E_g and E_e respectively, we define the dipole matrix element by $\langle g|x|g\rangle = \langle e|x|e\rangle = 0$ and $\langle e|x|g\rangle \equiv x_0 = [\langle g|x|e\rangle]^*$.

   (a) For an atom in a state $|\Psi\rangle = c_g|g\rangle + c_e|e\rangle$ at $t = 0$, what is the expectation value of the dipole moment $-ex$ as a function of time? (Recall that c_g and c_e may be complex.)

   (b) Sketch the amplitude of $\langle x\rangle$ versus the expectation value of the atom's energy.
   Make the comparable sketch for the amplitude and energy of a Lorentz-model oscillator.
   Compare and contrast.

(c) Suppose the Lorentz model were "true," *i.e.*, that atoms *were* in fact little harmonic oscillators. Of course, the correct description of these oscillators would still be quantum mechanical, so the atom could be in the ground state $|0\rangle$, or excited states $|1\rangle$,$|2\rangle$,...; these states are related by raising and lowering operators $a^\dagger_{HO}$ and a_{HO} with the well-known matrix elements $a^\dagger_{HO}|n\rangle = \sqrt{n+1}|n+1\rangle$ and $a_{HO}|n\rangle = \sqrt{n}|n-1\rangle$. Recall that $x = \sqrt{\hbar/2m\omega}(a_{HO} + a^\dagger_{HO})$.
A "quasiclassical," or coherent, state of the harmonic oscillator can be represented as an eigenstate $|\alpha\rangle$ of the lowering operator:

$$a_{HO}|\alpha\rangle = \alpha\ |\alpha\rangle\ . \tag{1}$$

(Note that a_{HO} is non-Hermitian, and that $|\alpha\rangle$ is *not* an eigenstate of $a^{\dagger_{HO}}$.)
What are the expectation value of the dipole moment $-ex$ and of the energy $\left(a^\dagger_{HO}a_{HO} + 1/2\right)\hbar\omega_0$ for an atom in a state $|\alpha\rangle$? Does this more closely resemble the classical Lorentz-model atom or the quantum two-level atom?

(d) Consider the 1s-2p transition in Hydrogen. Calculate the transition frequency $\omega/2\pi$ and the wavelength $\lambda = 2\pi c/\omega$. For a harmonic oscillator at this frequency, what would $\langle -ex\rangle$ be in an equal superposition of the ground and first excited states? Is this a reasonable quantity to expect for a real (quantum-mechanical) Hydrogen atom, and if so, why?

(e) The spontaneous decay rate of Rubidium in its first excited state is $\gamma/2\pi = 6$ MHz. It decays by emitting a photon with a wavelength of 780 nm. Given that the cross-section for absorption is $3\lambda^2/2\pi$, how many photons per second would a single ground-state atom absorb from a 780 nm beam with an intensity of I mW/cm^2? (Recall problem 3.)
How many photons per second will a single excited-state atom be stimulated to emit by the same beam (remember the discussion in class)?
Combining the spontaneous emission, absorption, and emission, write down the rate equation for n_e, the population in the excited state. Write down the steady state solution for n_e as a function of intensity I. How does it behave for low intensities? Does this fit the expectations we have from the Lorentz model? At what intensity does this begin to change?
This is known as the saturation intensity. Note that it is easily achievable with lasers; since this calculation assumed the light to be on resonance, the same effect would be exceedingly difficult to achieve with broadband light (in effect, it would require a light source with a temperature on the order of the order of the excitation energy, around an electron Volt, or $12,000$ K).