Topological order & long-range entanglements: A totally new kind of quantum materials and an unification of light and electrons

Xiao-Gang Wen, Perimeter, Sept. 2012

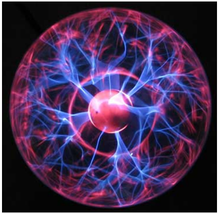
In primary school, we learned ...

there are four states of matter:

Solid

Liquid

Gas



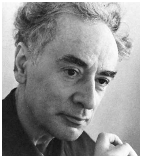
Plasma

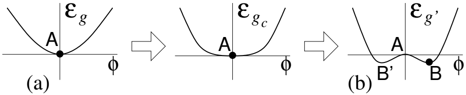
In university, we learned ...

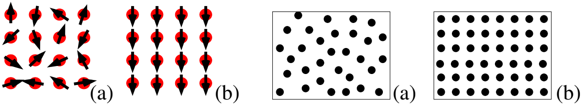
there are much more than four phases:

different phases = different symmetry breaking

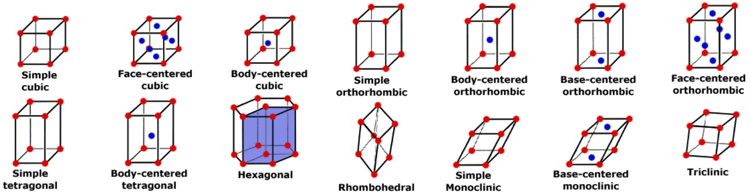
$\to$ Landau symmetry breaking theory





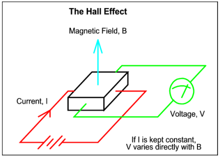


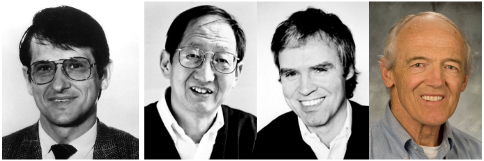
Group theory: From 230 ways of translation symmetry breaking, we obtain the 230 crystal orders in 3D



In graduate study after 1980's, we learned ...

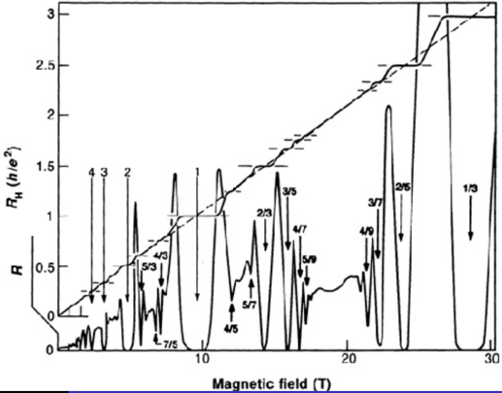
there are phases beyond symmetry-breaking:





$E_y = R_H j_x, \ \ R_H = \frac{p}{q}\frac{h}{e^2}$

- 2D electron gas in magnetic field has many **quantum Hall (QH) states**



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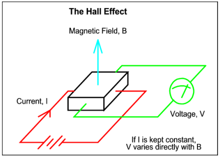
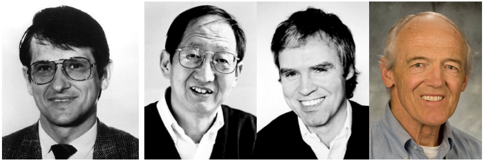
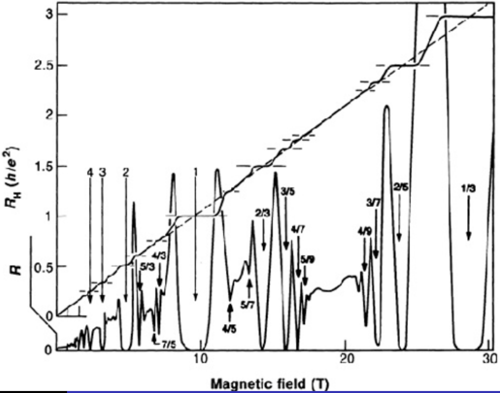




$E_y = R_H j_x, \quad R_H = \frac{p}{q}\frac{h}{e^2}$

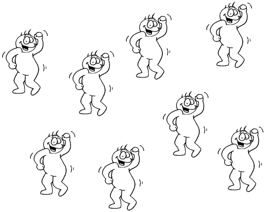
- 2D electron gas in magnetic field has many **quantum Hall (QH) states** that all have the **same symmetry**.
- Different QH states cannot be described by symmetry breaking theory.
- We call the new order **topological order** Wen 89



Symmetry breaking orders through pictures

Ferromagnet

Anti-ferromagnet



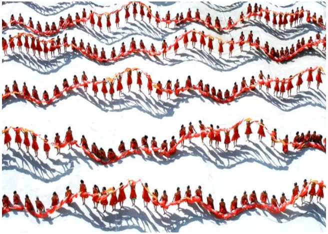
Superfluid of bosons

Superconductor of fermions

- **Long-range order**: every spin/particle is doing the same thing.

Topological orders through pictures

FQH state



String liquid (spin liquid)

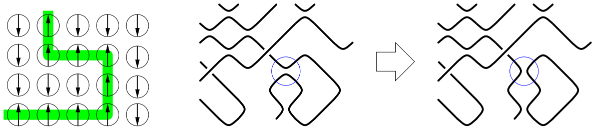
- **Global dance**:
  All spins/particles dance following a local dancing "rules"
  → The spins/particles dance collectively
  → a global dancing pattern.

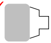
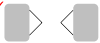
Local dancing rule → global dancing pattern

- Local dancing rules of a FQH liquid:
  (1) every electron dances around clock-wise
  (Φ_{FQH} only depends on $z = x + iy$)
  (2) takes exactly three steps to go around any others
  (Φ_{FQH}'s phase change 6π)
  → Global dancing pattern $\Phi_{\text{FQH}}(\{z_1, ..., z_N\}) = \prod(z_i - z_j)^3$

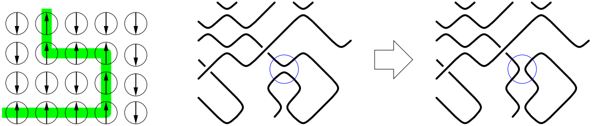
- Local dancing rules are enforce by the Hamiltonian
  to lower the ground state energy.

Local dancing rule $\to$ global dancing pattern



- Local dancing rules of a string liquid:
  (1) Dance while holding hands (no open ends)
  (2) Φ_{str} () $=$ Φ_{str} () , Φ_{str} () $=$ Φ_{str} ()
  $\to$ Global dancing pattern Φ_{str} () $= 1$

Local dancing rule → global dancing pattern



- Local dancing rules of a string liquid:
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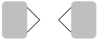
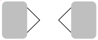
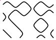
  (2) Φ_{str} () $=$ Φ_{str} () , Φ_{str} () $=$ Φ_{str} ()
  → Global dancing pattern Φ_{str} () $= 1$

- Local dancing rules of another string liquid:
  (1) Dance while holding hands (no open ends)
  (2) Φ_{str} () $=$ Φ_{str} () , Φ_{str} () $= -\Phi_{\text{str}}$ ()
  → Global dancing pattern Φ_{str} () $= (-)^{\#\text{ of loops}}$

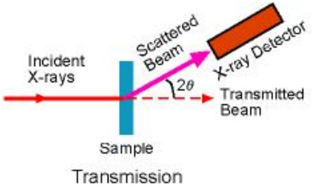
- Two string-net condensations → two topological orders Levin-Wen 05

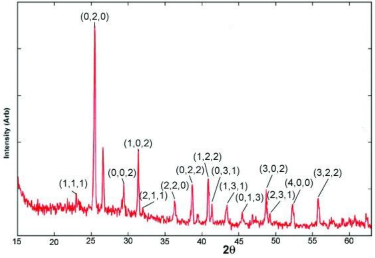
What really is topological order (through experiments)

To define a physical concept, such as symmetry-breaking order or topological order, is to design a probe to measure it

For example,

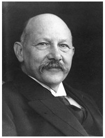
- crystal order is defined/probed by X-ray diffraction:

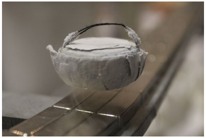




Symmetry-breaking orders through experiments

Order	Experiment
Crystal order	X-ray diffraction
Ferromagnetic order	Magnetization
Anti-ferromagnetic order	Neutron scattering
Superconducting order	Zero-resistance & Meissner effect
Topological order (Global dancing pattern)	???



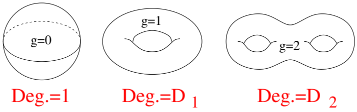


- All the above probes are linear responses. But topological order cannot be probed/defined through linear responses.

Topological orders through experiments

Topological order can be defined "experimentally" through two unusual topological probes (at least in 2D)

(1) **Topology-dependent ground state degeneracy** D_g Wen 89



(2) **Non-Abelian geometric's phases** of the degenerate ground state from deforming the torus: Wen 90

- Shear deformation T: $|\Psi_\alpha\rangle \to |\Psi'_\alpha\rangle = T_{\alpha\beta}|\Psi_\beta\rangle$

- 90° rotation S: $|\Psi_\alpha\rangle \to |\Psi''_\alpha\rangle = S_{\alpha\beta}|\Psi_\beta\rangle$

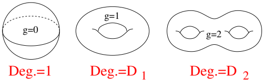
- Topological degeneracies and non-Abelian geometric phases, T and S, define topological order "experimentally".

Symmetry-breaking/topological orders through experiments

Order	Experiment
Crystal order	X-ray diffraction
Ferromagnetic order	Magnetization
Anti-ferromagnetic order	Neutron scattering
Superconducting order	Zero-resistance & Meissner effect
Topological order (Global dancing pattern)	Topological degeneracy, non-Abelian geometric phase

- The linear-response probe **Zero-resistance** and **Meissner effect** define **superconducting order**. Treating the EM fields as non-dynamical fields
- The topological probe **Topological degeneracy** and **non-Abelian geometric phase** define a completely new class of order – **topologically order**.

What is the significance of topological order?



represent experimental probes of topological order which can hardly be realized in experiments.

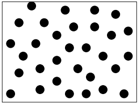
How to measure/study topological order in practice?

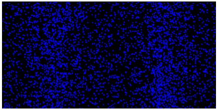
- Topological orders produce **new kind of waves**
  (collective excitations above the topo. ordered ground states).
  → *change our view of universe*
- The defects of topological order carry
  fractional statistics (including non-Abelian statistics) and
  fractional charges (if there is symmetry).
  → *a medium for topological quantum memory and computations.*
- Some topological orders have topologically protected **gapless boundary excitations**
  → *perfect conducting surfaces despite the insulating bulk.*

Principle of emergence: from order to physical properties

Different orders → different wave equations for the deformations of order → different physical properties.

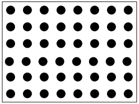
- Atoms in superfluid have a random distribution
  → cannot resist shear deformations (which do nothing)
  → liquids do not have shapes



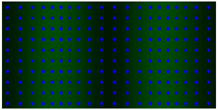


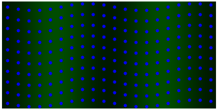
Wave Eq. → Euler Eq.
$\partial_t^2 \rho - \partial_x^2 \rho = 0$ One longitudinal mode

- Atoms in solid have a ordered lattice distribution
  → can resist shear deformations
  → solids have shapes



Wave Eq. → elastic Eq.
$\partial_t^2 u^i - C^{ijkl} \partial_{x^j} \partial_{x^k} u^l = 0$





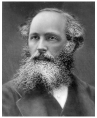
One longitudinal mode and two transverse modes

Origin of photons, gluons, electrons, quarks, etc

- Do all waves and wave equations emerge from some orders?

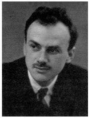
Wave equations for elementary particles

- Maxwell equation → Photons
  $\partial \times \mathbf{E} + \partial_t \mathbf{B} = \partial \times \mathbf{B} - \partial_t \mathbf{E} = \partial \cdot \mathbf{E} = \partial \cdot \mathbf{B} = 0$



- Yang-Mills equation → Gluons
  $\partial^\mu F^a_{\mu\nu} + f^{abc} A^{\mu b} F^c_{\mu\nu} = 0$

- Dirac equation → Electrons/quarks
  $[\partial_\mu \gamma^\mu + m]\psi = 0$



What orders produce the above waves? What are the origins of light (gauge bosons) and electrons (fermions)?

Elementary or emergent?

- But none of the symmetry breaking orders can produce:
  - electromagnetic wave satisfying the Maxwell equation
  - gluon wave satisfying the Yang-Mills equation
  - electron wave satisfying the Dirac equation.

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- We have seen that topological order corresponds to to new states of quantum matter, such as FQH states and spin-liquid states.

- Can topological order produces the Maxwell equation, Yang-Mills equation, and the Dirac equation?

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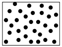
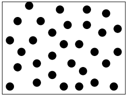
- Can topological order produces the Maxwell equation, Yang-Mills equation, and the Dirac equation?

Yes

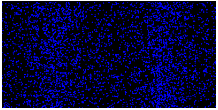
- A particular topologically ordered state, string-net liquid, provide a unified origin of light, electrons, quarks, gluons, ...

Topological order (closed strings) → emergence of electromagnetic waves (photons)

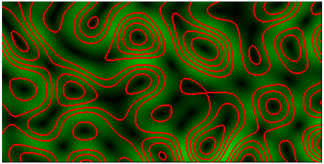
- Wave in superfluid state $|\Phi_{SF}\rangle = \sum_{\text{all position conf.}} \Big|$  $\Big\rangle$:

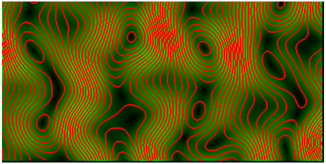


density fluctuations:
Euler eq.: $\partial_t^2 \rho - \partial_x^2 \rho = 0$
→ Longitudinal wave



- Wave in closed-string liquid $|\Phi_{\text{string}}\rangle = \sum_{\text{closed strings}} \Big|$  $\Big\rangle$:

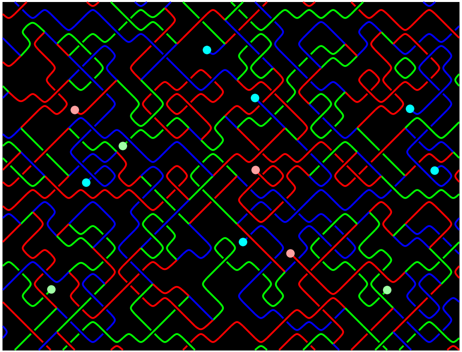




String density $\mathbf{E}(\mathbf{x})$ fluctuations → waves in string condensed state.
Strings have no ends → $\partial \cdot \mathbf{E} = 0$ → **only two transverse modes**.
Equation of motion for string density → Maxwell equation:
$\dot{\mathbf{E}} - \partial \times \mathbf{B} = \dot{\mathbf{B}} + \partial \times \mathbf{E} = \partial \cdot \mathbf{B} = \partial \cdot \mathbf{E} = 0$. ($\mathbf{E}$ electric field)

Topological order (string nets)→ Emergence of Yang-Mills theory (gluons)

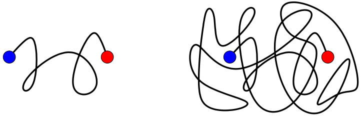
- If string has different types and can branch
  → string-net liquid → Yang-Mills theory
- Different ways that strings join → different gauge groups



A picture of our vacuum A string–net theory of light and electrons

Closed strings → Maxwell gauge theory
String-nets → Yang-Mills gauge theory

Topological order →
Emergence of electrons



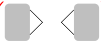
- In string condensed states, the ends of string be have like point particles
  - with quantized (gauge) charges
  - with Fermi statistics

  Levin-Wen 2003

- **String-net/topological-order provides a way to unify gauge interactions and Fermi statistics in 3D**

Emergence of fractional spin/statistics

- Why electron carry spin-1/2 and Fermi statistics?

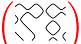
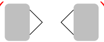
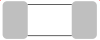
- Φ_{str} () $= 1$ string liquid Φ_{str} () $= \Phi_{\text{str}}$ ()

  360° rotation:  $\to$  and  $=$  $\to$ 

   $+$  has a spin 0 mod 1.  $-$  has a spin $1/2$ mod 1.

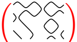
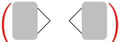
Emergence of fractional spin/statistics

- Why electron carry spin-1/2 and Fermi statistics?

- $\Phi_{\text{str}}\big($  $\big) = 1$ string liquid $\Phi_{\text{str}}\big($  $\big) = \Phi_{\text{str}}\big($  $\big)$

  360° rotation:  $\to$  and  $=$  $\to$ 

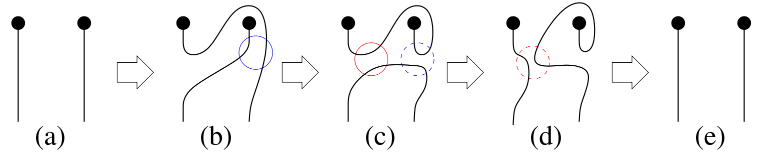
   $+$  has a spin 0 mod 1.  $-$  has a spin $1/2$ mod 1.

- $\Phi_{\text{str}}\big($  $\big) = (-)^{\#\text{ of loops}}$ string liquid $\Phi_{\text{str}}\big($  $\big) = -\Phi_{\text{str}}\big($  $\big)$

  360° rotation:  $\to$  and  $= -$  $\to -$ 

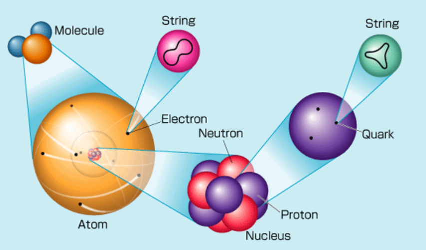
   $+\ \mathrm{i}$  has a spin $-1/4$ mod 1.  $-\ \mathrm{i}$  has a spin $1/4$ mod 1.

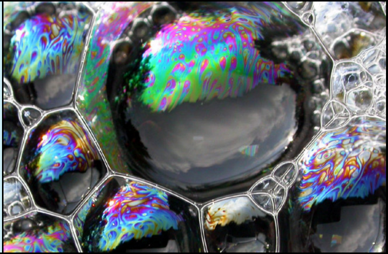
Spin-statistics theorem



- (a) → (b) = exchange two string-ends.
- (d) → (e) = 360° rotation of a string-end.
- Amplitude (a) = Amplitude (e)
- Exchange two string-ends plus a 360° rotation of one of the string-end generate no phase.

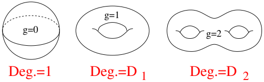
→ **Spin-statistics theorem**





Topological order
is the source of many wonders

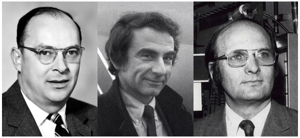
What is the microscopic picture of topological order?



represent an experimental definition of topological order.

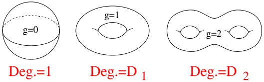
- But what is the microscopic understanding of topological order?

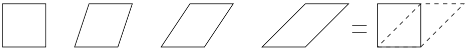
- Zero-resistance and Meissner effect $\rightarrow$ experimental definition of superconducting order.
- It took 40 years to gain a microscopic picture of superconducting order: **electron-pair condensation**



Bardeen-Cooper-Schrieffer 57

What is the microscopic picture of topological order?





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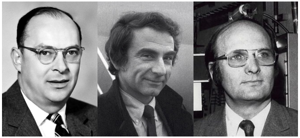

Bardeen-Cooper-Schrieffer 57

- It only took 20 years to gain a microscopic picture of topological order: **long-range entanglements** (global dancing patterns) Chen-Gu-Wen 10

Quantum entanglements through examples

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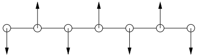
-  $= |\downarrow\rangle \otimes |\uparrow\rangle \otimes |\downarrow\rangle \otimes |\uparrow\rangle \otimes |\downarrow\rangle \ldots$ $\to$ unentangled

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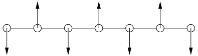
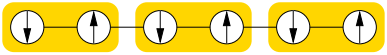
-  $= |\downarrow\rangle \otimes |\uparrow\rangle \otimes |\downarrow\rangle \otimes |\uparrow\rangle \otimes |\downarrow\rangle ...$ $\to$ unentangled
-  $= (|\downarrow\uparrow\rangle - |\uparrow\downarrow\rangle) \otimes (|\downarrow\uparrow\rangle - |\uparrow\downarrow\rangle) \otimes ...$ $\to$
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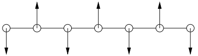
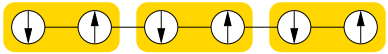
-  $= |\downarrow\rangle \otimes |\uparrow\rangle \otimes |\downarrow\rangle \otimes |\uparrow\rangle \otimes |\downarrow\rangle ...$ → unentangled
-  $= (|\downarrow\uparrow\rangle - |\uparrow\downarrow\rangle) \otimes (|\downarrow\uparrow\rangle - |\uparrow\downarrow\rangle) \otimes ...$ →
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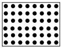
- Crystal order: $|\Phi_{\text{crystal}}\rangle = \Big|$  $\Big\rangle = |0\rangle_{x_1} \otimes |1\rangle_{x_2} \otimes |0\rangle_{x_3} \cdots$
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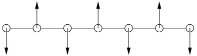
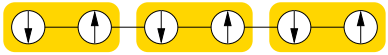
-  $= |\downarrow\rangle \otimes |\uparrow\rangle \otimes |\downarrow\rangle \otimes |\uparrow\rangle \otimes |\downarrow\rangle \ldots$ → unentangled
-  $= (|\downarrow\uparrow\rangle - |\uparrow\downarrow\rangle) \otimes (|\downarrow\uparrow\rangle - |\uparrow\downarrow\rangle) \otimes \ldots$ →
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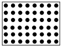
- Crystal order: $|\Phi_{\text{crystal}}\rangle = \Big|$  $\Big\rangle = |0\rangle_{x_1} \otimes |1\rangle_{x_2} \otimes |0\rangle_{x_3} \cdots$
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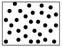
  $|\Phi_{\text{SF}}\rangle = \sum_{\text{all conf.}} \Big|$  $\Big\rangle$

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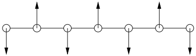
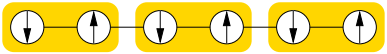
-  $= |\downarrow\rangle \otimes |\uparrow\rangle \otimes |\downarrow\rangle \otimes |\uparrow\rangle \otimes |\downarrow\rangle ...$ $\to$ unentangled
-  $= (|\downarrow\uparrow\rangle - |\uparrow\downarrow\rangle) \otimes (|\downarrow\uparrow\rangle - |\uparrow\downarrow\rangle) \otimes ...$ $\to$
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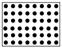
- Crystal order: $|\Phi_{\text{crystal}}\rangle = \Big|$  $\Big\rangle = |0\rangle_{x_1} \otimes |1\rangle_{x_2} \otimes |0\rangle_{x_3} \cdots$
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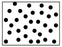
  $|\Phi_{SF}\rangle = \sum_{\text{all conf.}} \Big|$  $\Big\rangle = (|0\rangle_{x_1} + |1\rangle_{x_1} + ..) \otimes (|0\rangle_{x_2} + |1\rangle_{x_2} + ..) ...$
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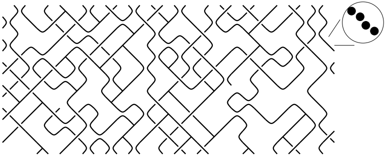
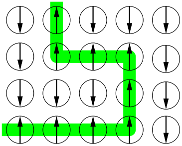
How to make long range entanglements (topo. orders)

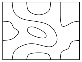
To make topological order, we need to sum over different product states, but we should not sum over everything.

- Sum over a subset of the particle configurations, by first join the particles into strings, then sum over the loop states





$\rightarrow$ string-net condensation (string liquid): Levin-Wen 05

$|\Phi_{loop}\rangle = \sum_{\text{all loop conf.}} \Big|$  $\Big\rangle$ which is not a direct-product state and not a local deformation of direct-product states

$\rightarrow$ non-trivial **topological orders** (long-range entanglements)

Long range entanglements
$\rightarrow$ A new and deeper understanding of quantum phases

For gapped systems with no symmetry:

- According to Landau theory, no symmetry to break
  $\rightarrow$ all systems belong to one trivial phase

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  - There are **long range entangled (LRE) states**
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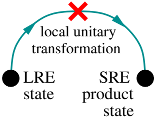

Long range entanglements
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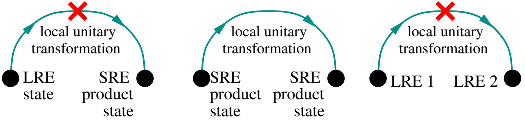
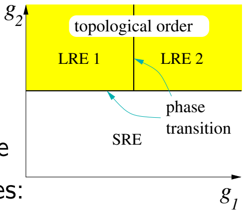
- According to entanglement picture:
  - There are **long range entangled (LRE) states** → many phases
  - There are **short range entangled (SRE) states** → one phase

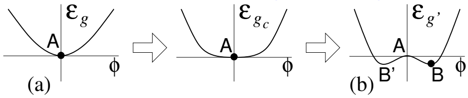


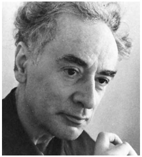


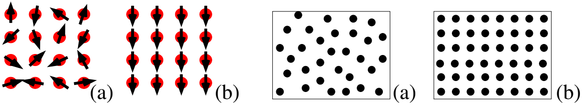
- All SRE states belong to the same trivial phase
- LRE states can belong to many different phases:
  patterns of long range entanglements = topological orders

Short-range entanglements that break symmetry → Landau symmetry breaking phases

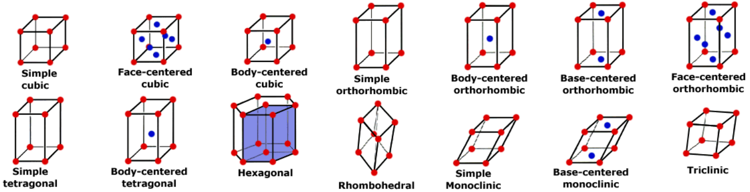
different phases = different symmetry breaking







From 230 ways of translation symmetry breaking, we obtain the 230 crystal orders in 3D



Short-range entanglements w/ symmetry → SPT phases

For gapped systems with a symmetry G (no symmetry breaking):

- there are **LRE symmetric states** → many different phases
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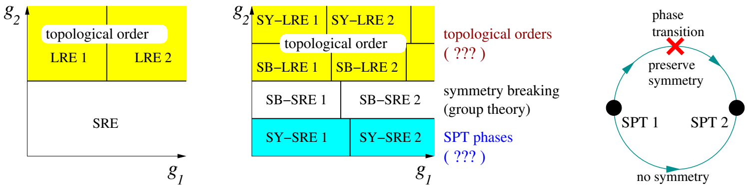

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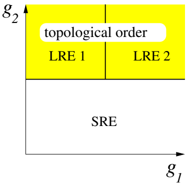
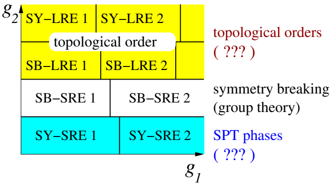
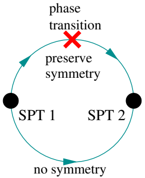
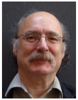






- Haldane phase of 1D spin-1 chain w/ $SO(3)$ symm. Haldane 83

1D 

Short-range entanglements w/ symmetry → SPT phases

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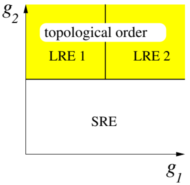
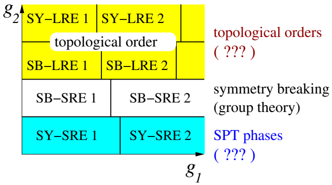
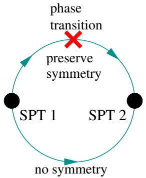
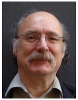
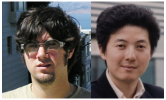
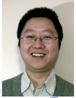
or **symmetry protected topological (SPT)** phase







- Haldane phase of 1D spin-1 chain w/ $SO(3)$ symm. Haldane 83
- Topo. insulators w/ $U(1) \times T$ symm.: 2D Kane-Mele 05; Bernevig-Zhang 06 and 3D Moore-Balents 07; Fu-Kane-Mele 07

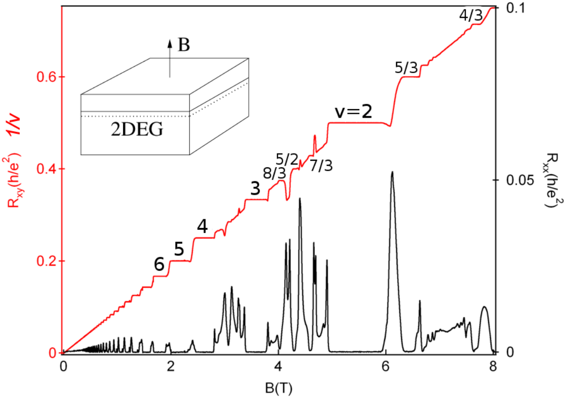
1D  2D   3D  

Compare topological order and topological insulator

- **Topological order** describes states with *long-range entanglements*
  - The essence: long-range entanglements
- **Topological insulator** is a state with *short-range entanglements, particle number conservation, and time reversal symmetry*, which is an example of SPT phases.
  - The essence: symmetry entangled with short-range entanglements

	Topo. order	Topo. Ins.	Band Ins.
Entanglements	long range	short range	short range
Fractional charges of finite-energy defects	Yes	No	No
Fractional statistics of finite-energy defects	Yes	No	No
Proj. non-Abelian stat. of infinite-energy defects	Yes $>$ Majorana	Yes only Majorana	Yes only Majorana
Gapless boundary	topo. protected	symm. protected	not protected

Where are long-range entanglements: FQH states

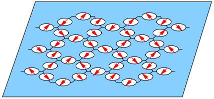


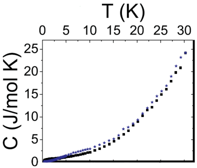
Filling fractions $\nu = 1/3, 2/3, 4/3, 5/3, ...$ → Abelian FQH states
→ $U(1)$ or $U(1) \times U(1)$ Chern-Simons theories

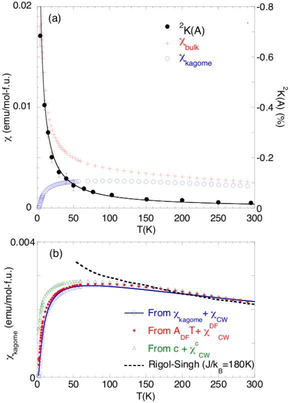
Filling fractions $\nu = 5/2, ...$ → non-Abelian FQH states
→ $U(1) \times SO(5)$ Chern-Simons theory

Where are long-range entanglements: spin liquid states

Herbertsmithsite: spin-1/2 on Kagome lattice $H = J \sum \mathbf{S}_i \cdot \mathbf{S}_j$.



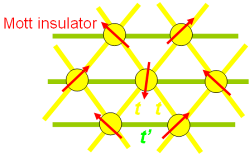


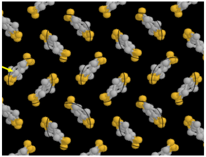


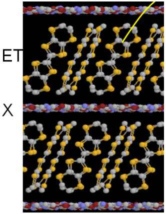
- $J \sim 200K$, no phase trans. down to $50mK$ → spin liquid Helton etal 06
- Numerical calculations → Z_2 topological order (emergence of Z_2 gauge theory) Misguich-Bernu-Lhuillier-Waldtmann 98; Jiang-Weng-Sheng 08;Yan-Huse-White 10

Where are long-range entanglements: Organics κ-(ET)$_2$X

Hubbard model on triangular lattice:

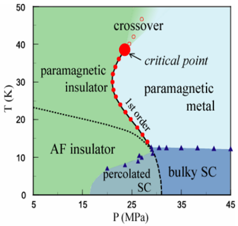


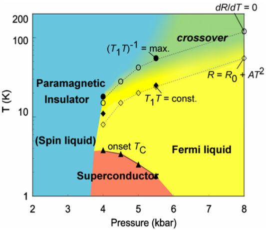




$t'/t = 0.5 \sim 1.1$

X=$Cu[N(CN)_2]Cl$, $Cu_2(CN)_3$,...





Spin interaction
$J = 250K$
But no AF order down to $35mK$

$Cu[N(CN)_2]Cl$ $t'/t = .75$ $Cu_2(CN)_3$ $t'/t = 1.06$

- Emergence of $U(1)$ gauge theory int. with spinon Fermi surface.

Highly entangled quantum matter: A new chapter of condensed matter physics

Quantum States of Matter (gapped)

Tensor category theory

- Long–Rang Entangled = Topological order
  - FQH State
  - Super–conductor
  - Spin Liquid

Gauge theory Fractional statistics
Fermi statistics Boundary excitations
Topological degeneracy

Group Theory

- Short–Range Entangled
  - Symm. Breaking
    - Crystal
    - Liquid Crystal
    - Magnet
  - With symmetry (Group cohomology theory)
    - Haldane Phase
    - Topological Insulator

Shape Insulator
Information storage
Gapless Goldstone mode

Boundary excitations
Anti–localization