Entangled Granular Material

RESEARCH CORPORATION
for SCIENCE ADVANCEMENT
A foundation dedicated to science since 1912.

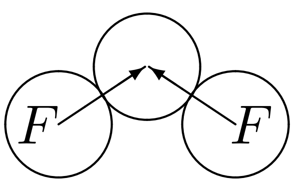
Key Facts about Granular Materials

- collections of macroscopic ($\sim$mm) particles interacting through contact forces (e.g. friction)

- Thermal energy irrelevant ($k_B T \sim 10^{-21}$ J $\ll PE \sim 10^{-5}$ J)

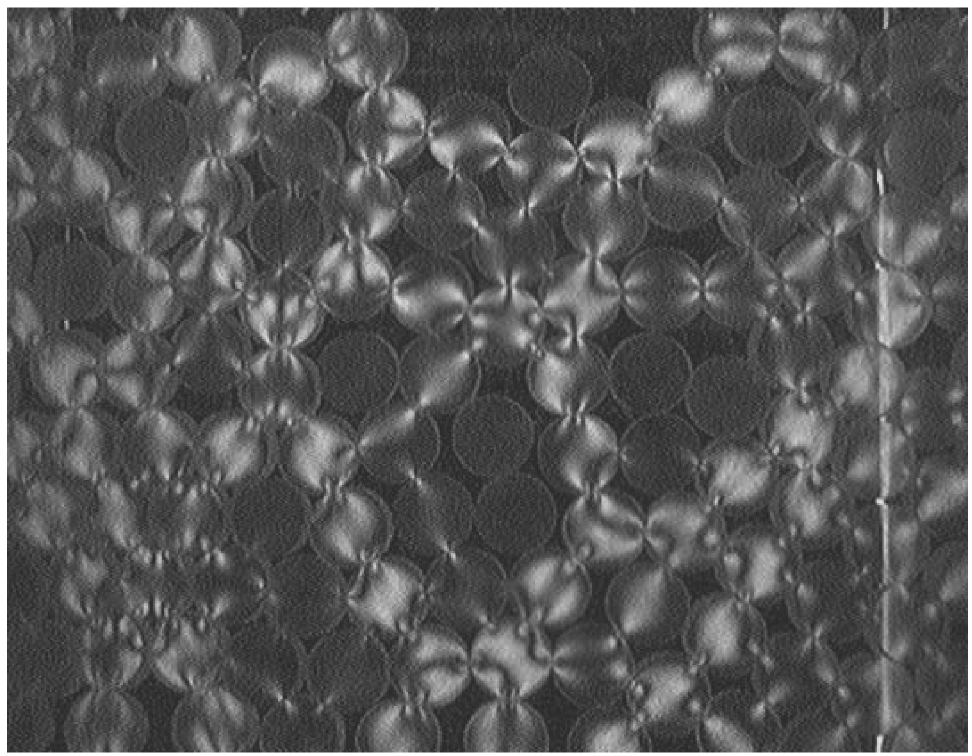
  * systems “frozen” in metastable state, unable to move to lower energy state without external help

- forces propagate in linear chains



$$2F \sin\theta = W$$

$$\Longrightarrow F = \frac{W}{2\sin\theta}$$



Ordinary granular materials are easily manipulated

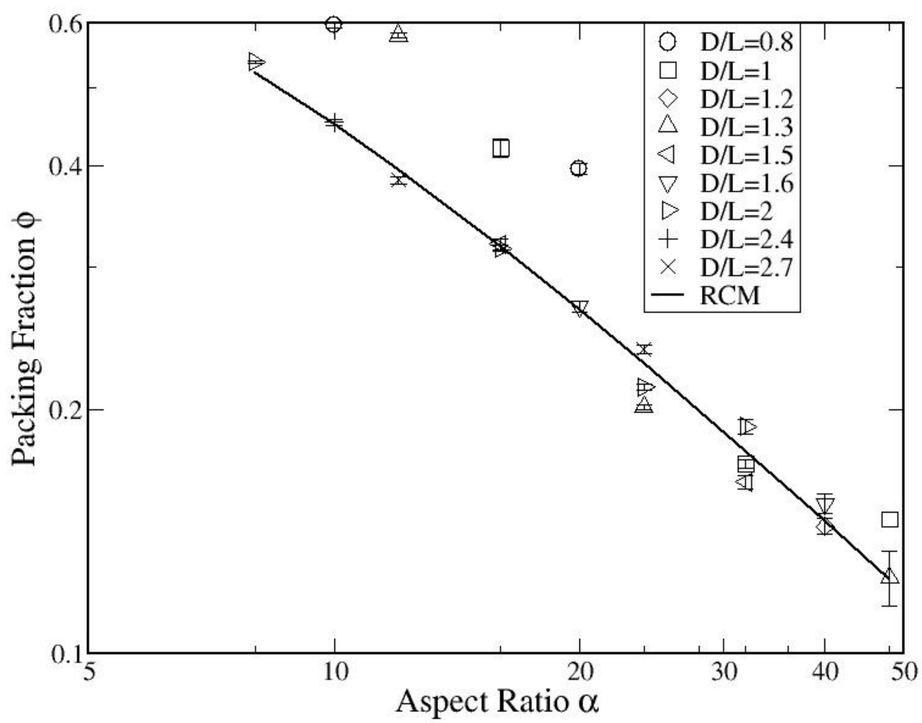
Geometrically cohesive particles are different

- Why are rodpiles so rigid?
- Is the rigid rodpile qualitatively different from the sandpile?
- What governs how granular materials rearrange?

Rods: Static Packings and Solid Plugs

- Philipse (*Langmuir*, 1996)
  - ∗ above aspect ratio ~ 35 pile emerges as solid plug
  - ∗ mean-field *Random Contact Model* (packing fraction $\propto v_{excl}^{-1}$)
  - ∗ assumes no orientational correlation, constant coordination number

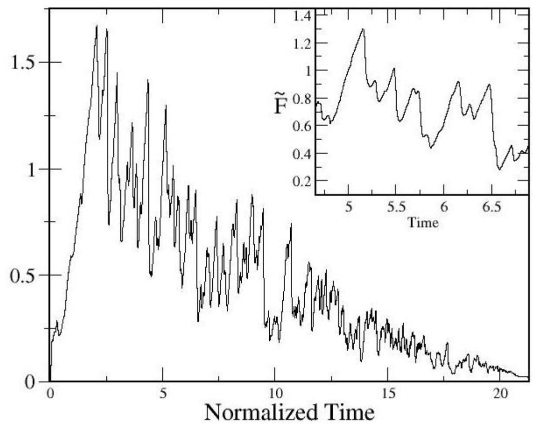
- Blouwolff and Fraden (*Europhysics Letters*, 2006)
  - ∗ small variation in coordination number ($6 \leq \langle z \rangle \leq 10$), essentially validates RCM



Kenneth Desmond and SVF, PRE (2006)

GCGM display solid & granular behavior Desmond & SVF (2006)

Stick-slip



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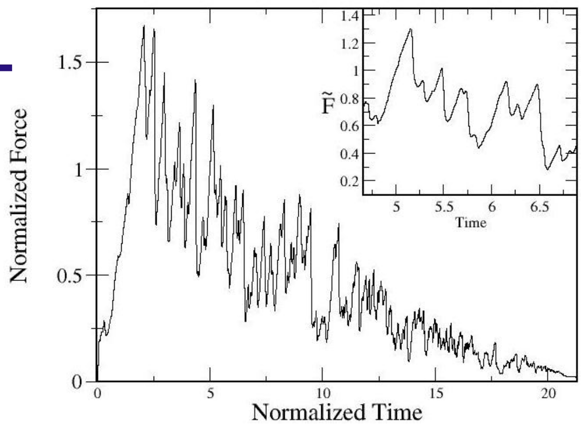
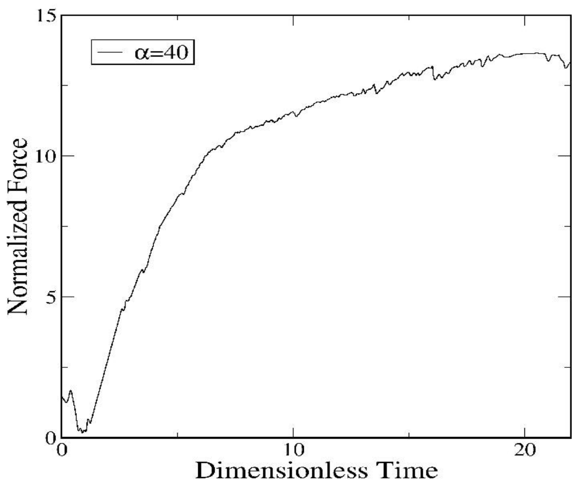


Solid



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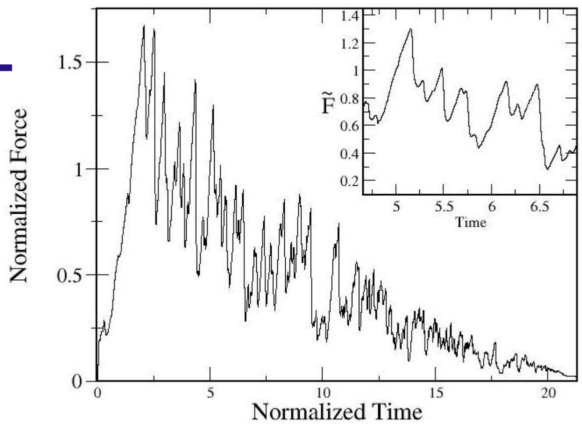
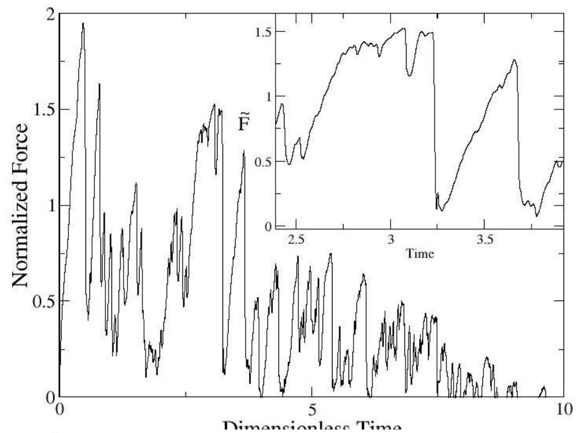
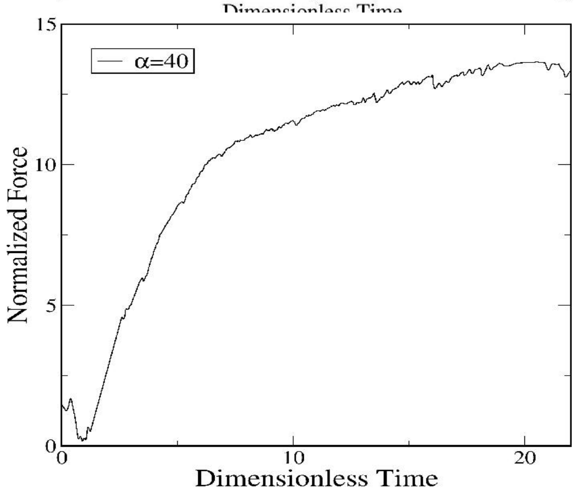
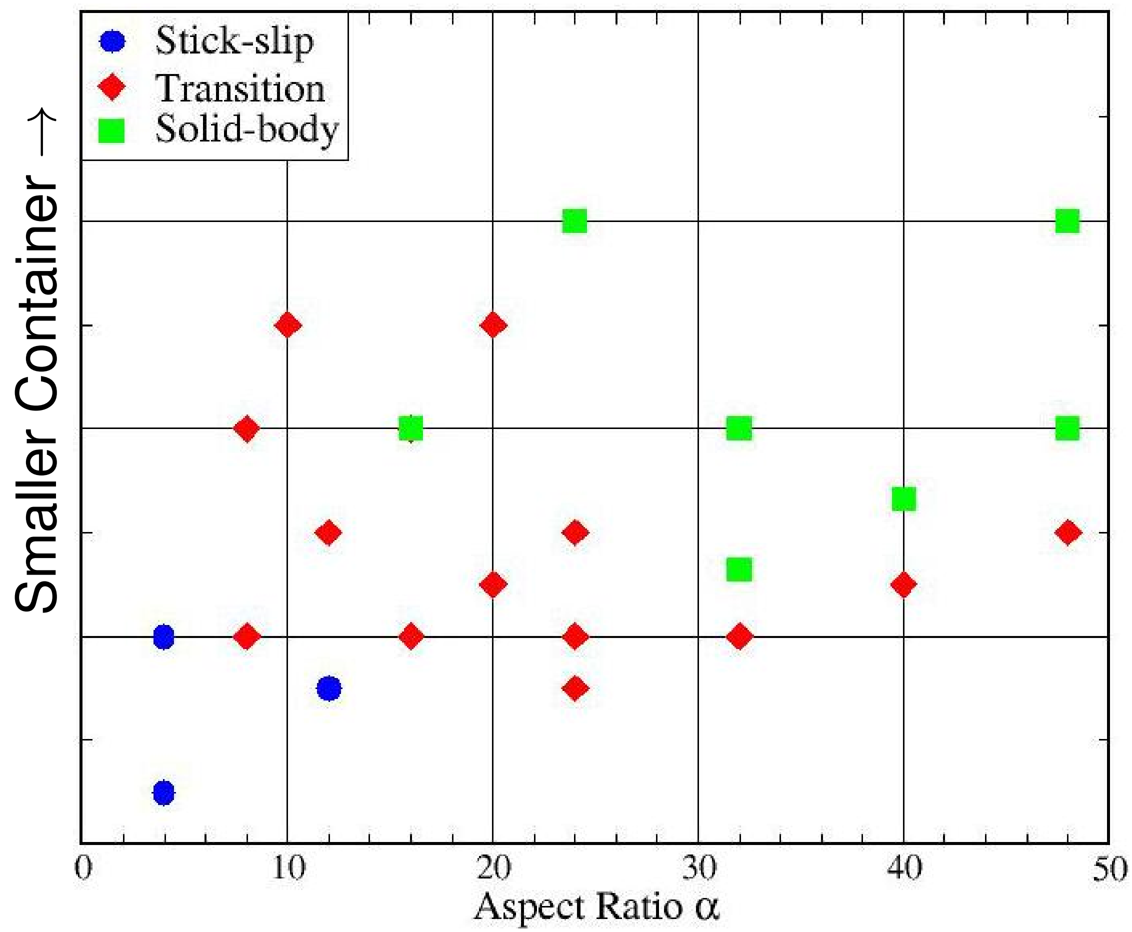


Transition



Solid





Column Collapse of Granular Rods

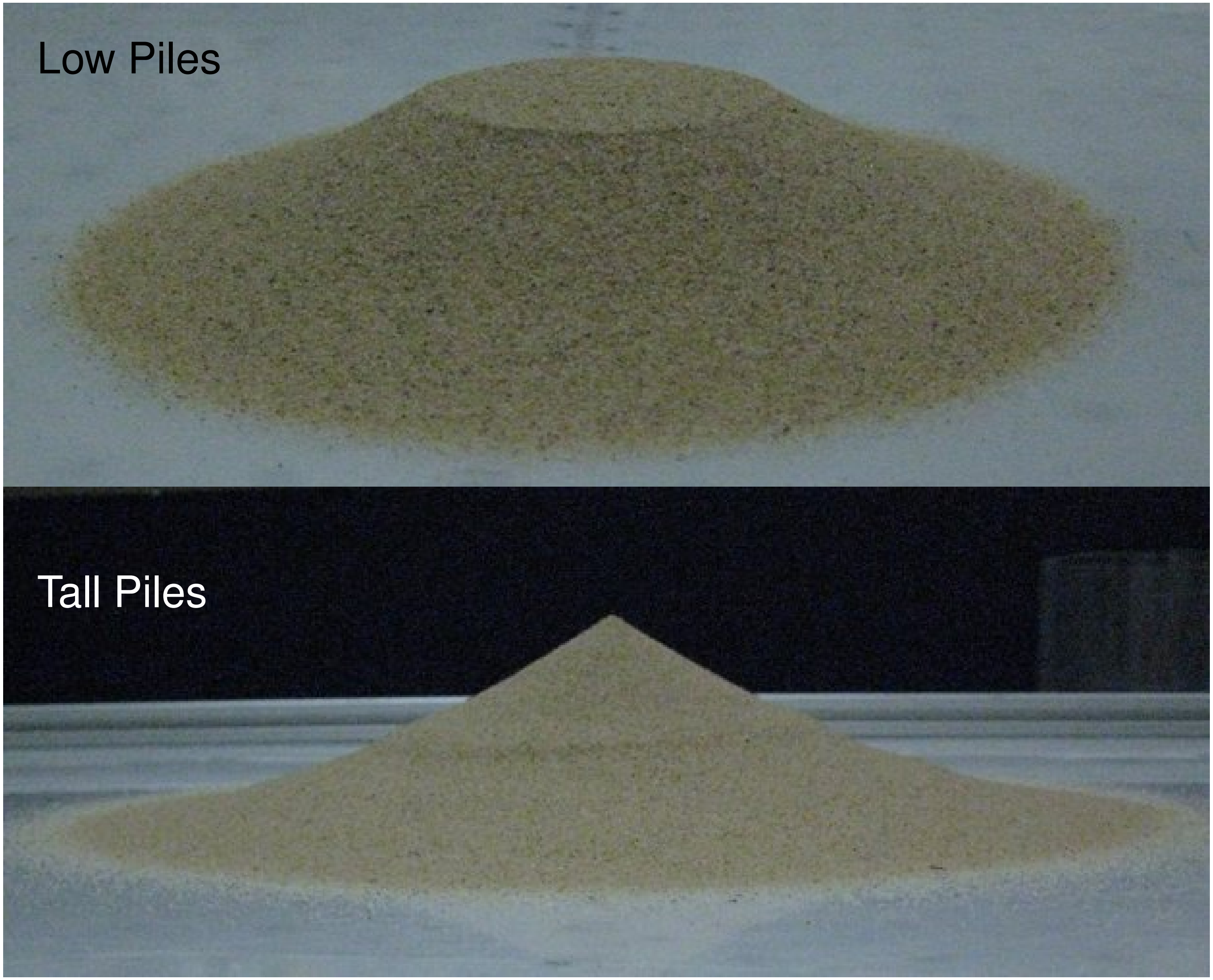
Melissa Trepanier

DMR #0706353

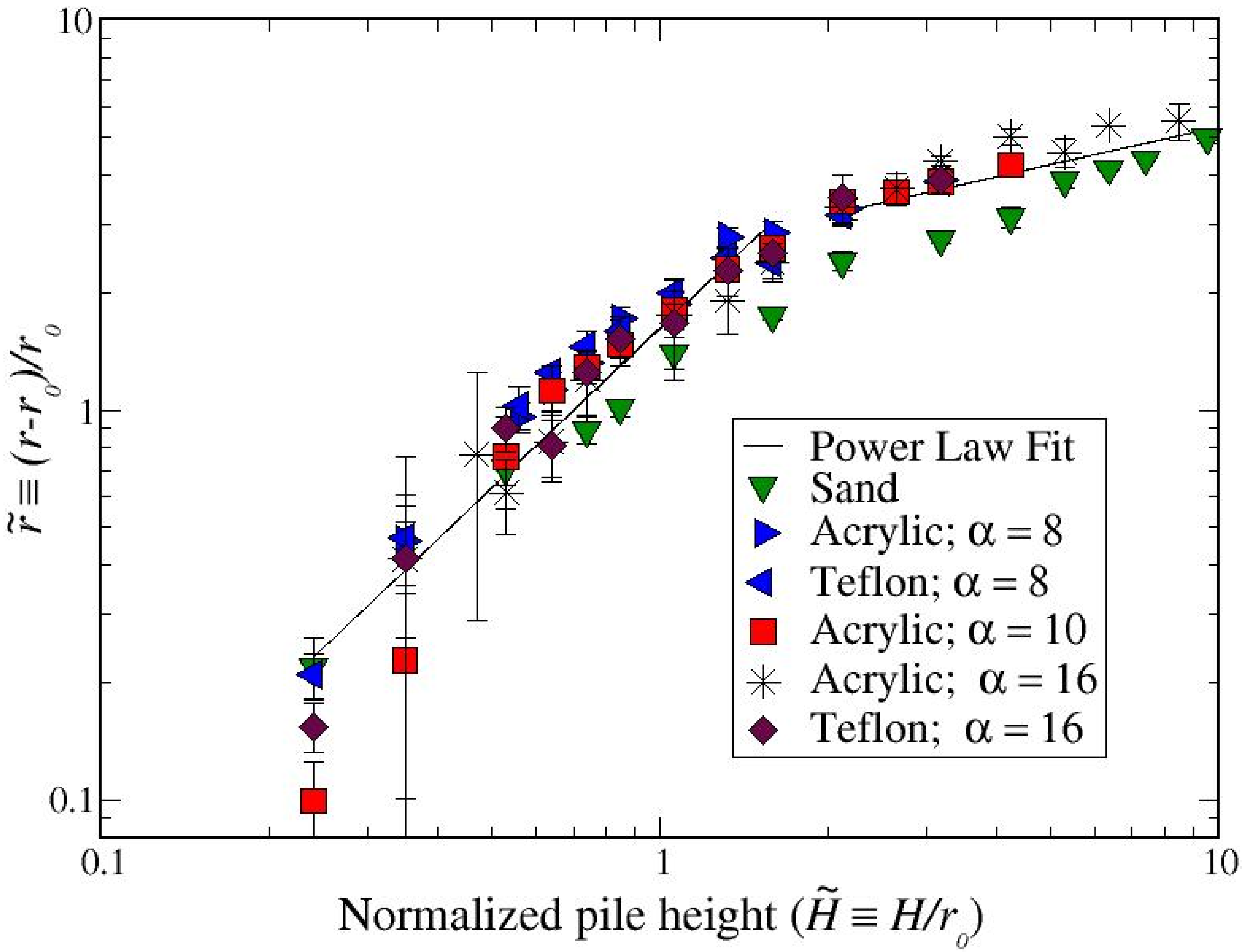
Experimental Setup

- $D = 11.43, 15.24$ cm cylinders
- Sand, Acrylic & Teflon Rods
  - $L : 2.5 - 7.5$ cm
  - $w : 0.16 - 0.6$ cm
  - aspect ratio 4-48
- independent of cylinder velocity
- average runoff in 4 directions

Geometric Transition

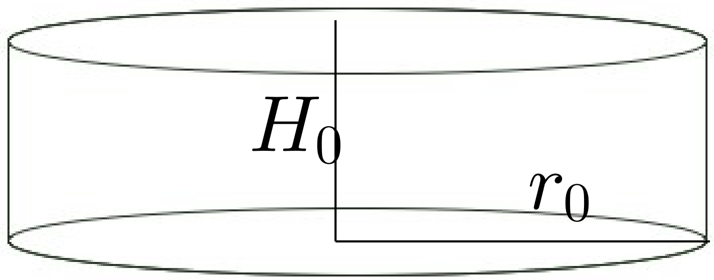


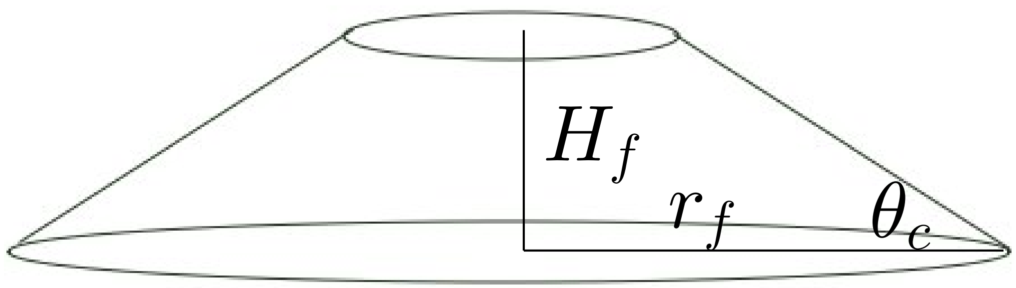
Transition Power Laws



$$\tilde{r} = \begin{cases} \tilde{H}^{1.2\pm0.1} & \tilde{H} < 1.1 \pm 0.3 \\ \tilde{H}^{0.6\pm0.1} & \tilde{H} > 1.1 \pm 0.3 \end{cases}$$

Conservation of Volume I: Mesas





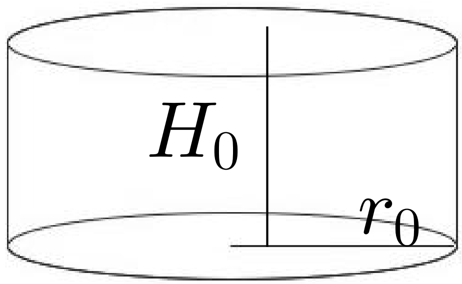
$$\pi r_0^2 H_0 = \pi \left(\frac{H_f}{r_0}\right)^3 \left[\frac{1}{3\tan^2\theta_c} - \frac{1}{\tan\theta_c}\frac{r_f}{H_f} + \left(\frac{r_f}{H_f}\right)^2\right]$$

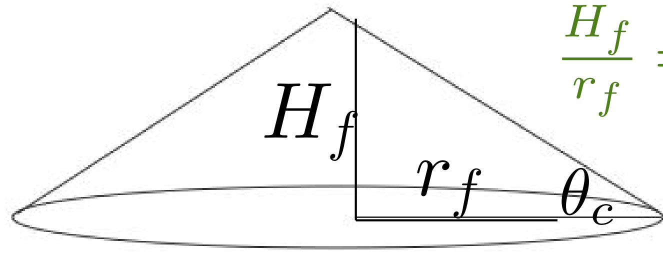
- $H_f = H_0, \tan\theta_c = \frac{H_f}{r_f}$

$$\frac{r_f}{r_0} = \frac{1}{2\tan\theta_c}\left(\frac{H_f}{r_0} + \sqrt{4\tan^2\theta_c - \frac{H_f^2}{3r_0^2}}\right) \approx \frac{1}{2\tan\theta_c r_0} H_f$$

Conservation of Volume II: Cones

"Classic" angle of repose theory





$$\frac{H_f}{r_f} = \tan\theta_c$$

$$\pi r_0^2 H_0 = \frac{1}{3}\pi H_f r_f^2 \Longrightarrow \frac{r_f}{r_0} \sim (H_0)^{1/3}$$

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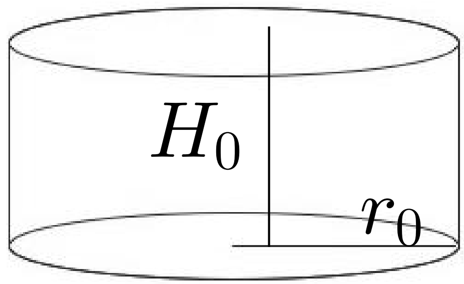

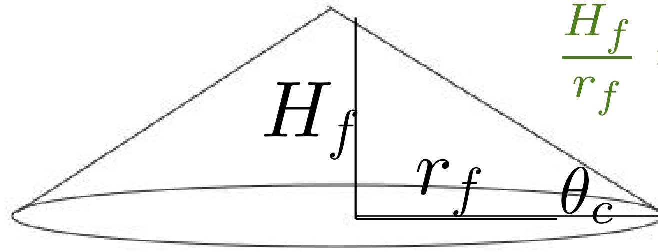

$$\frac{H_f}{r_f} = \tan\theta_c$$

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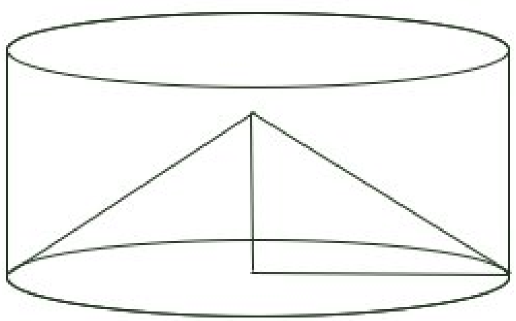
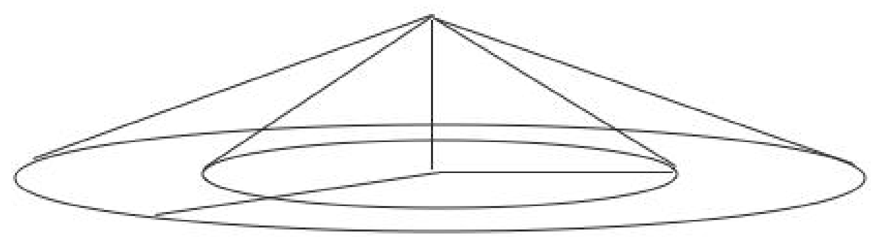
Internal stable cone



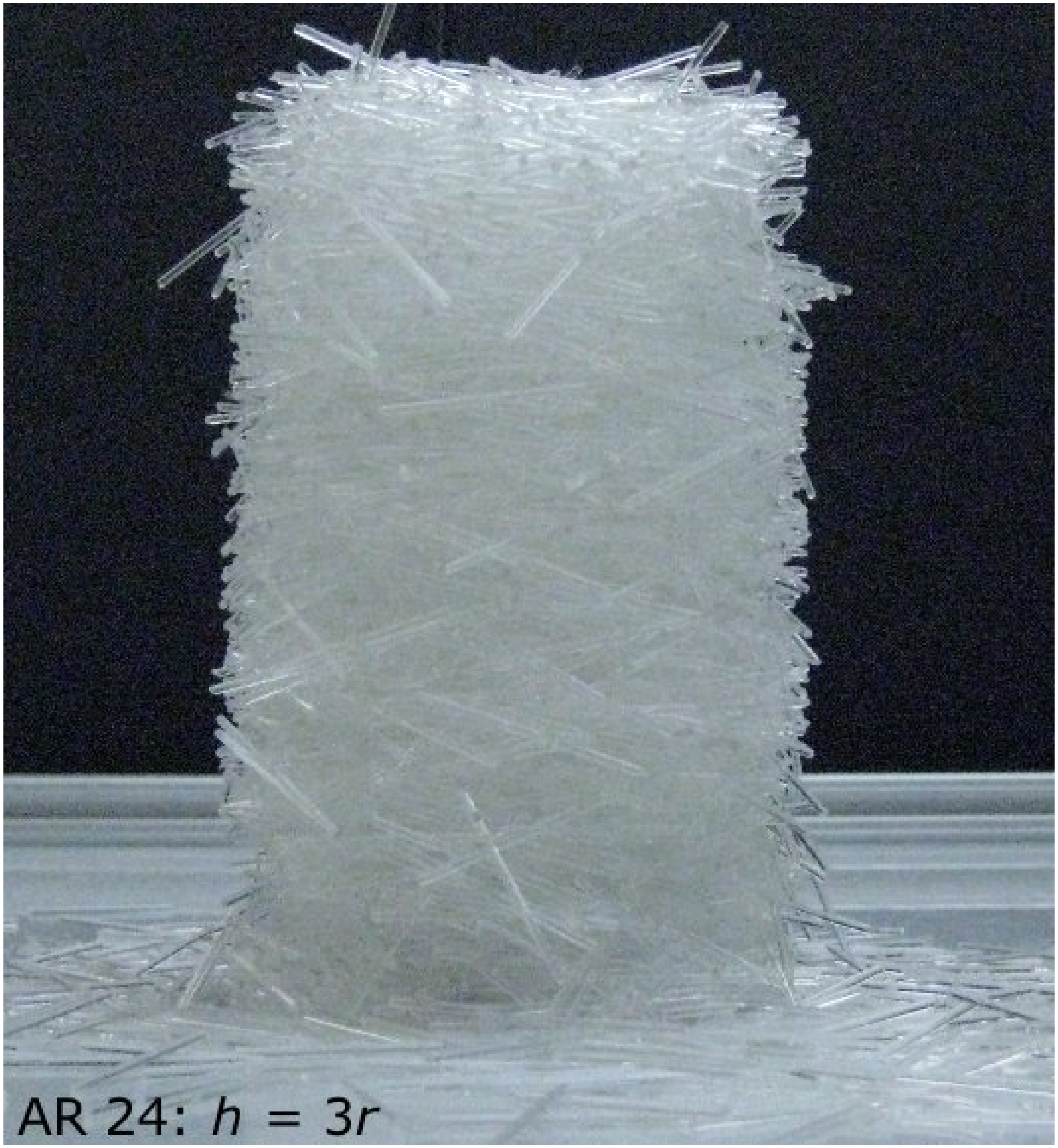
$$\frac{H_f}{r_0} = \tan\theta_c$$



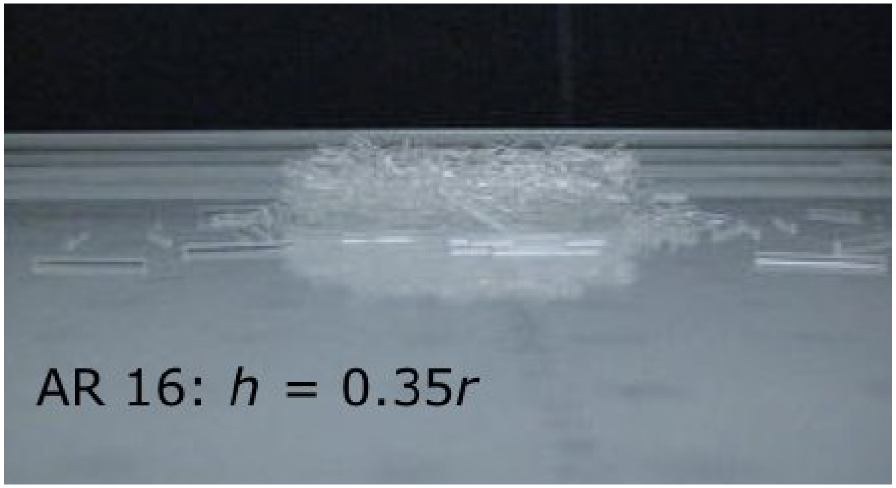
$$\pi r_0^2 H_0 = \frac{1}{3}\pi H_f r_f^2 \Longrightarrow \pi r_0^2 H_0 = \frac{1}{3}\pi r_0 \tan\theta_c r_f^2$$

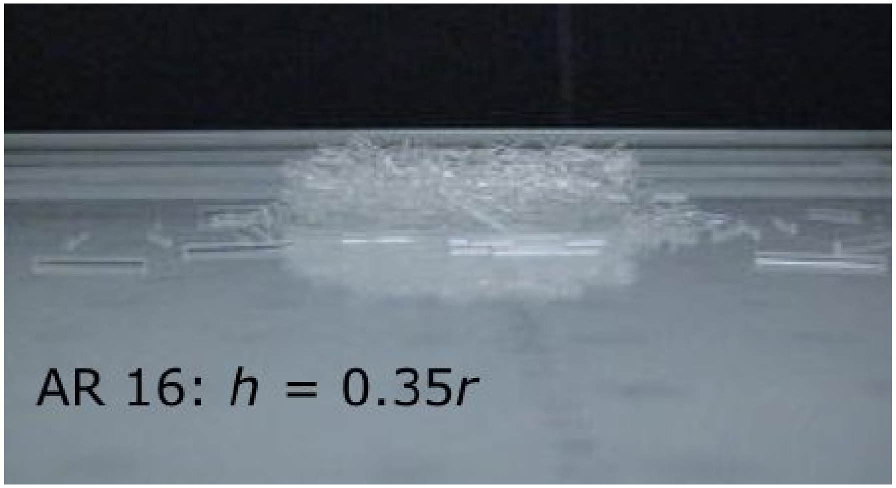
$$\Longrightarrow r_f \sim H_0^{1/2}$$

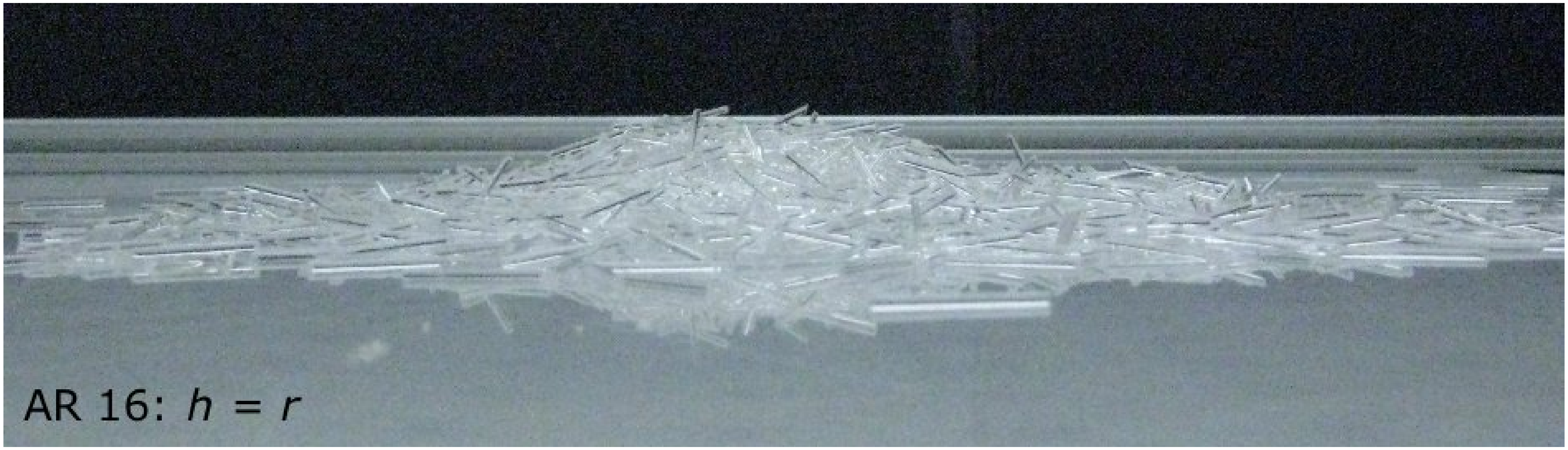
Long thin rods form stable piles



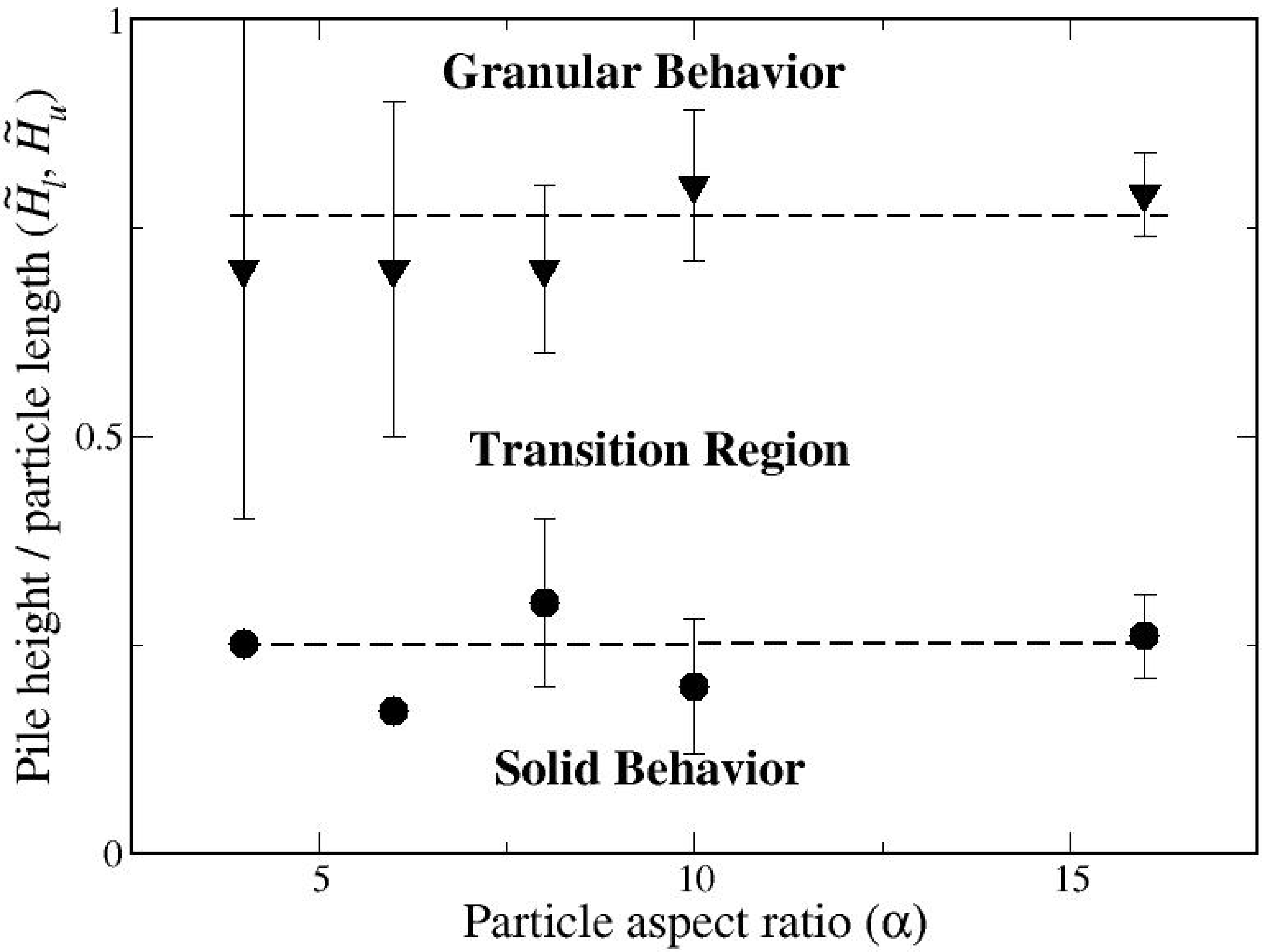
Moderately long rods also don't collapse!



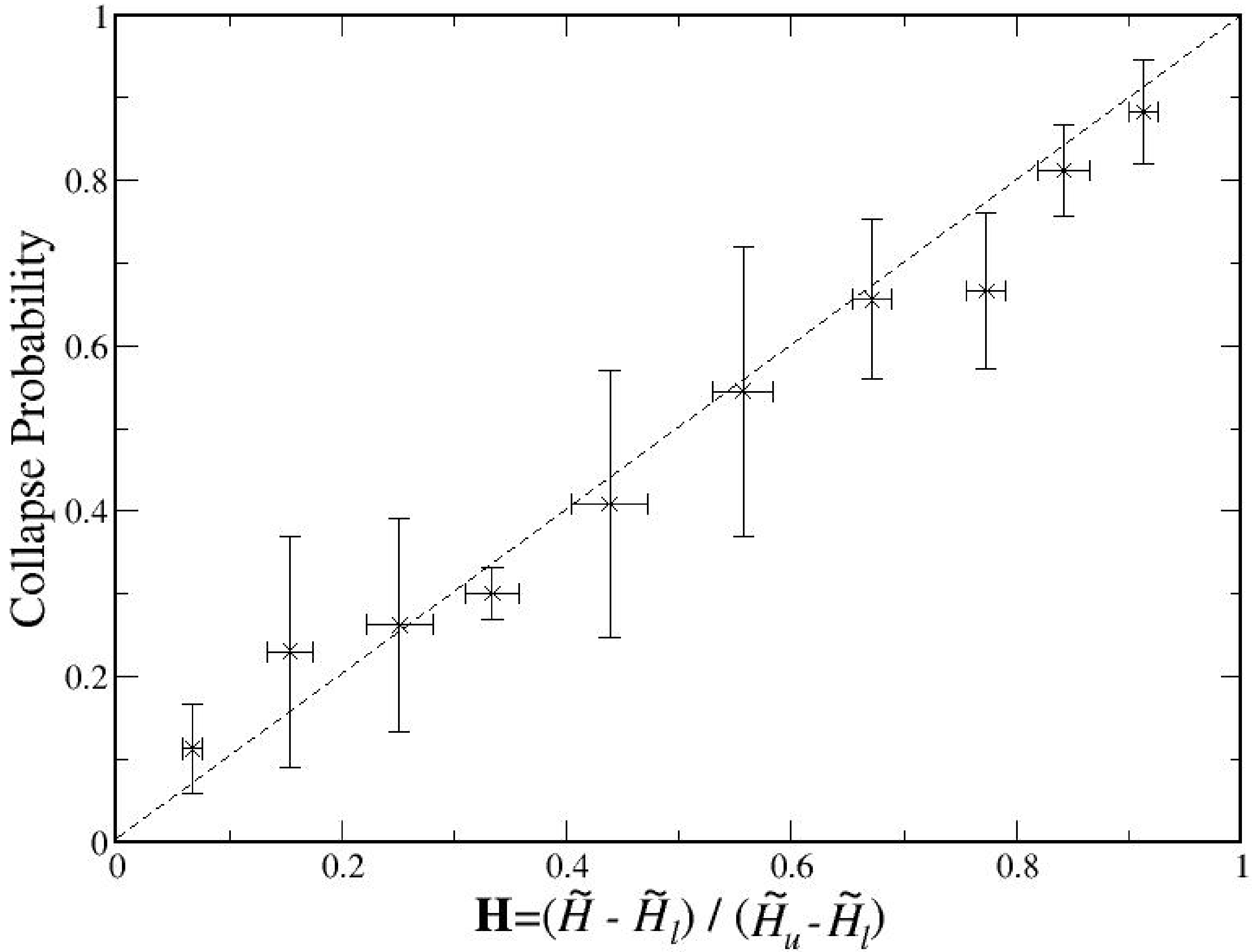




Transition from Solid to Flow

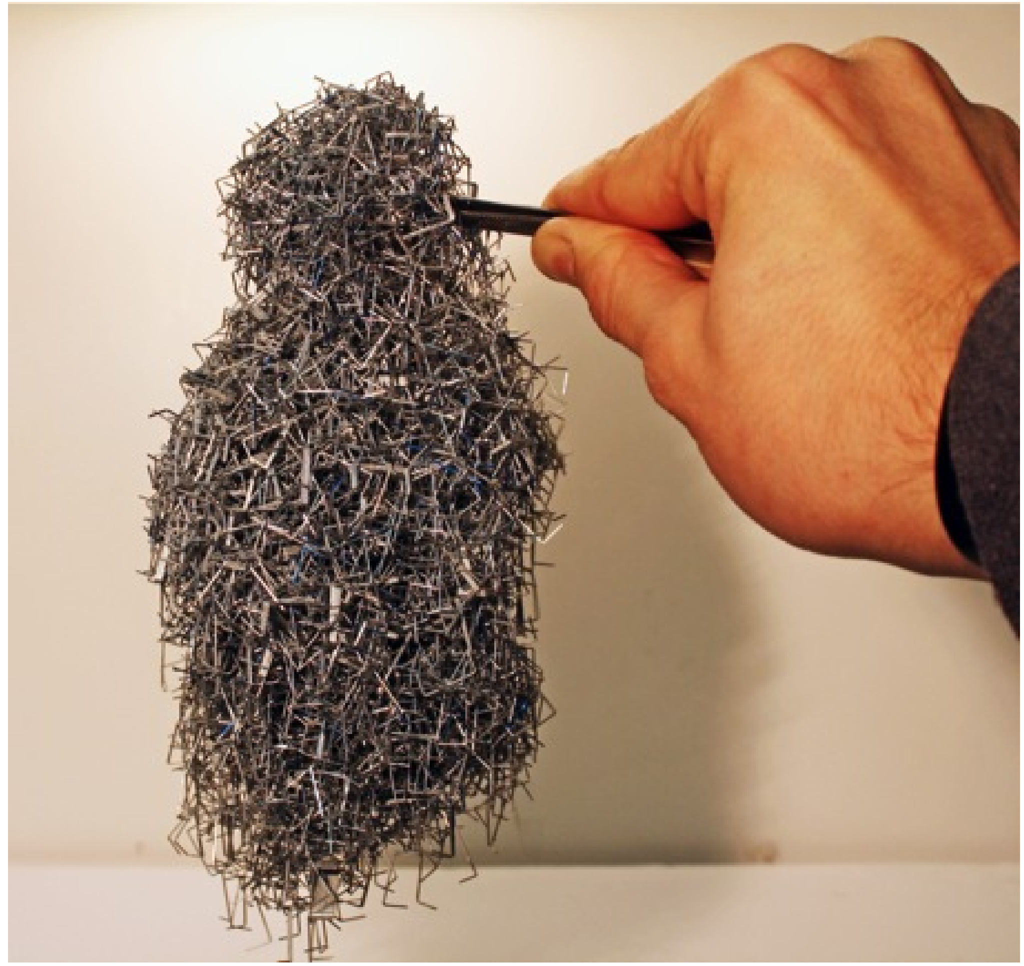


Linear Increase in Collapse Probability



Staples

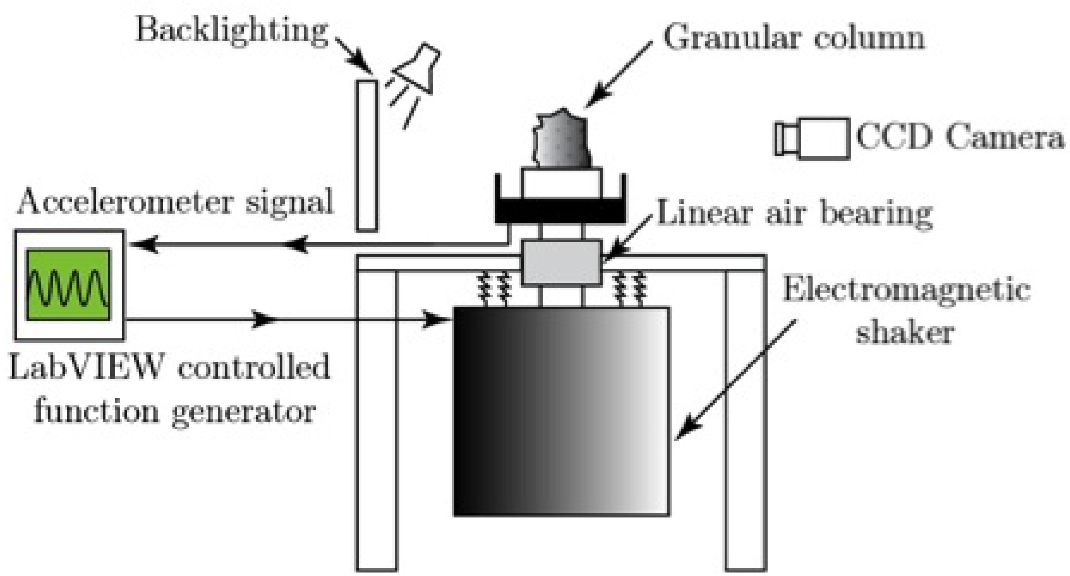
Nick Gravish, SVF, David Hu, Dan Goldman (2012)



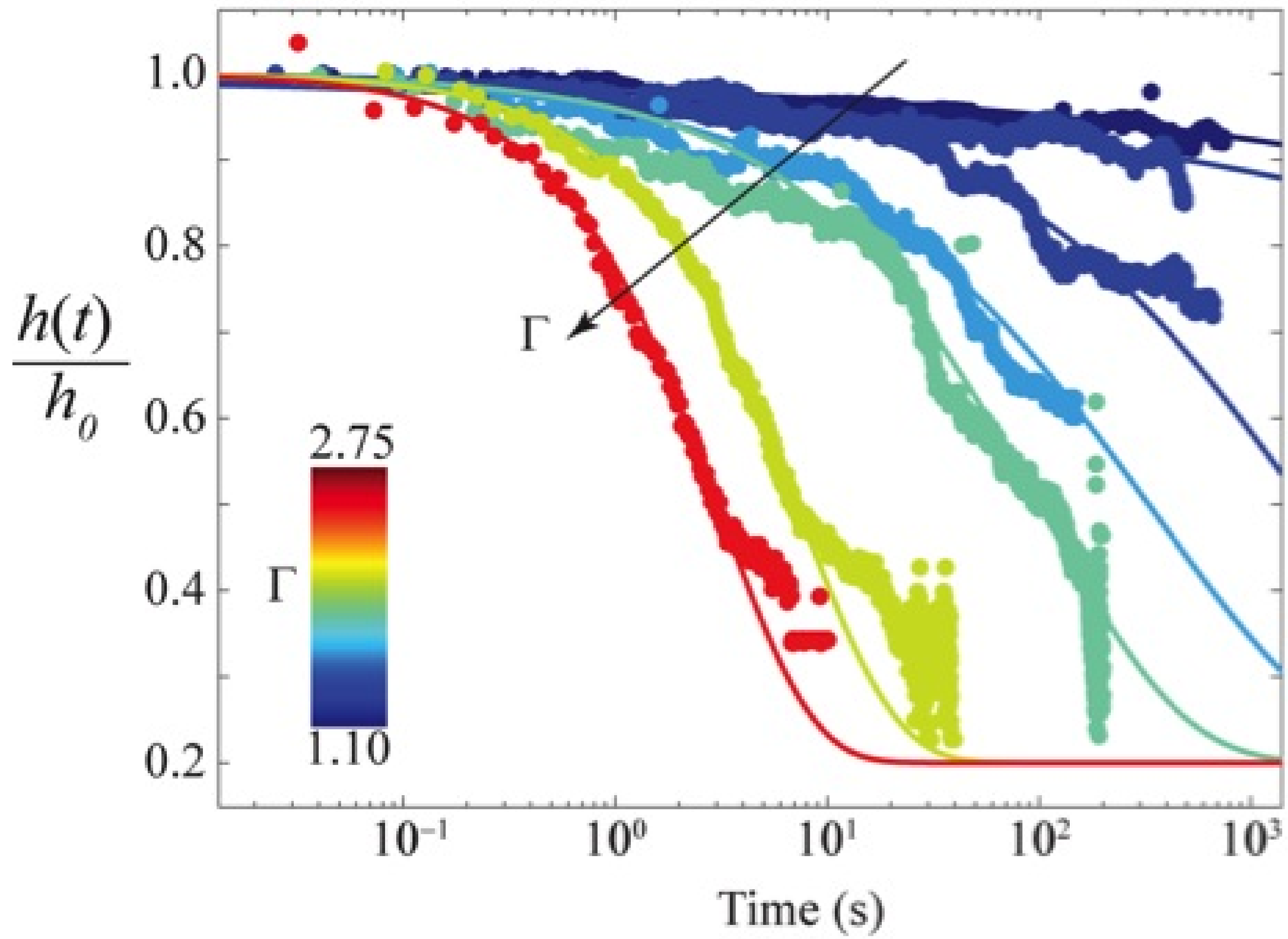
Biological geometric cohesion: ant rafts

- Rafts contain $\sim 10^5$ ants, essential for colony survival
- Cohere through interlocked limbs (stable even when dead)

Vibration induced melting

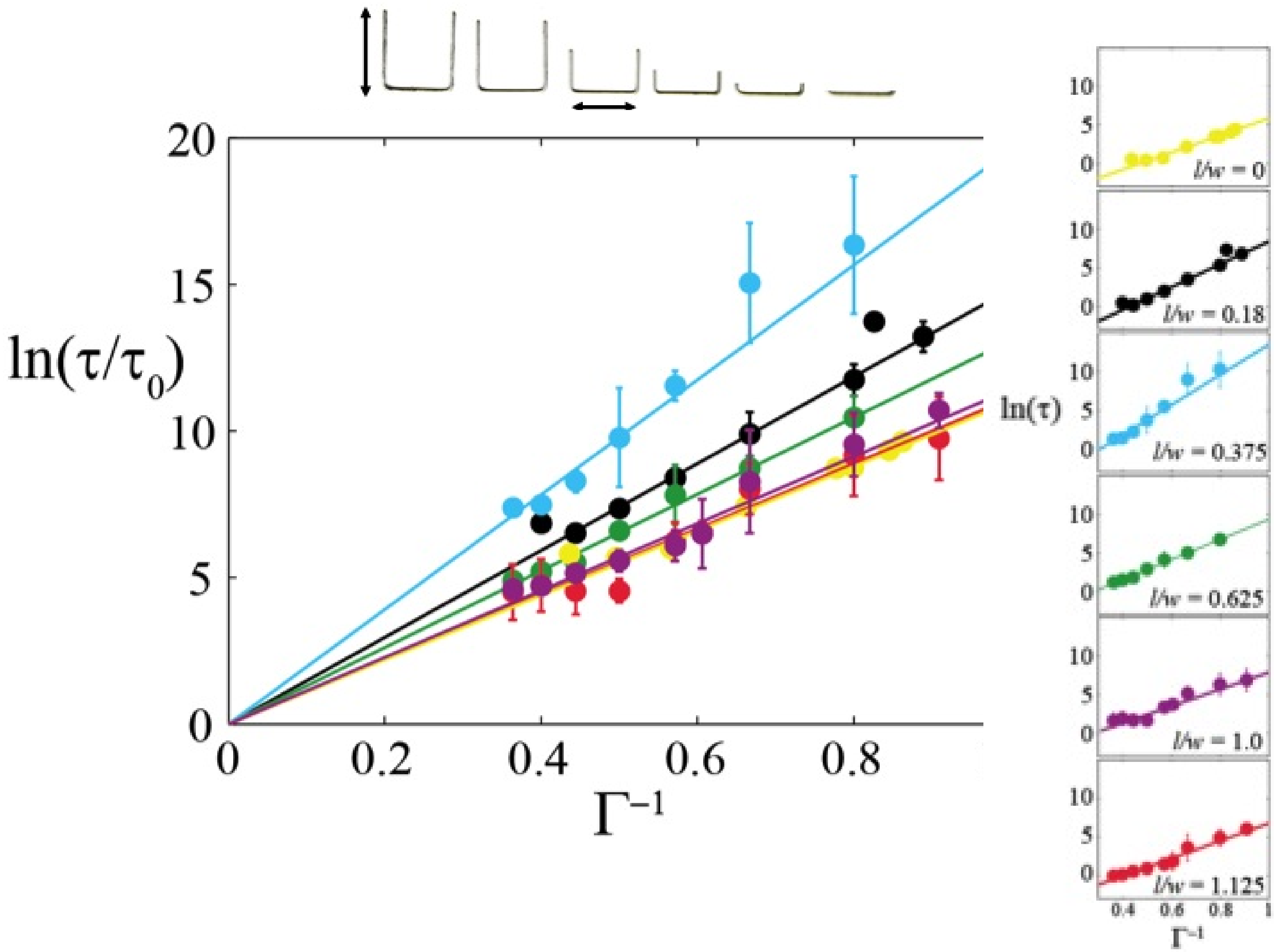


Height relaxes as stretched exponential $h(t) \propto \exp[-(t/\tau)^\beta]$



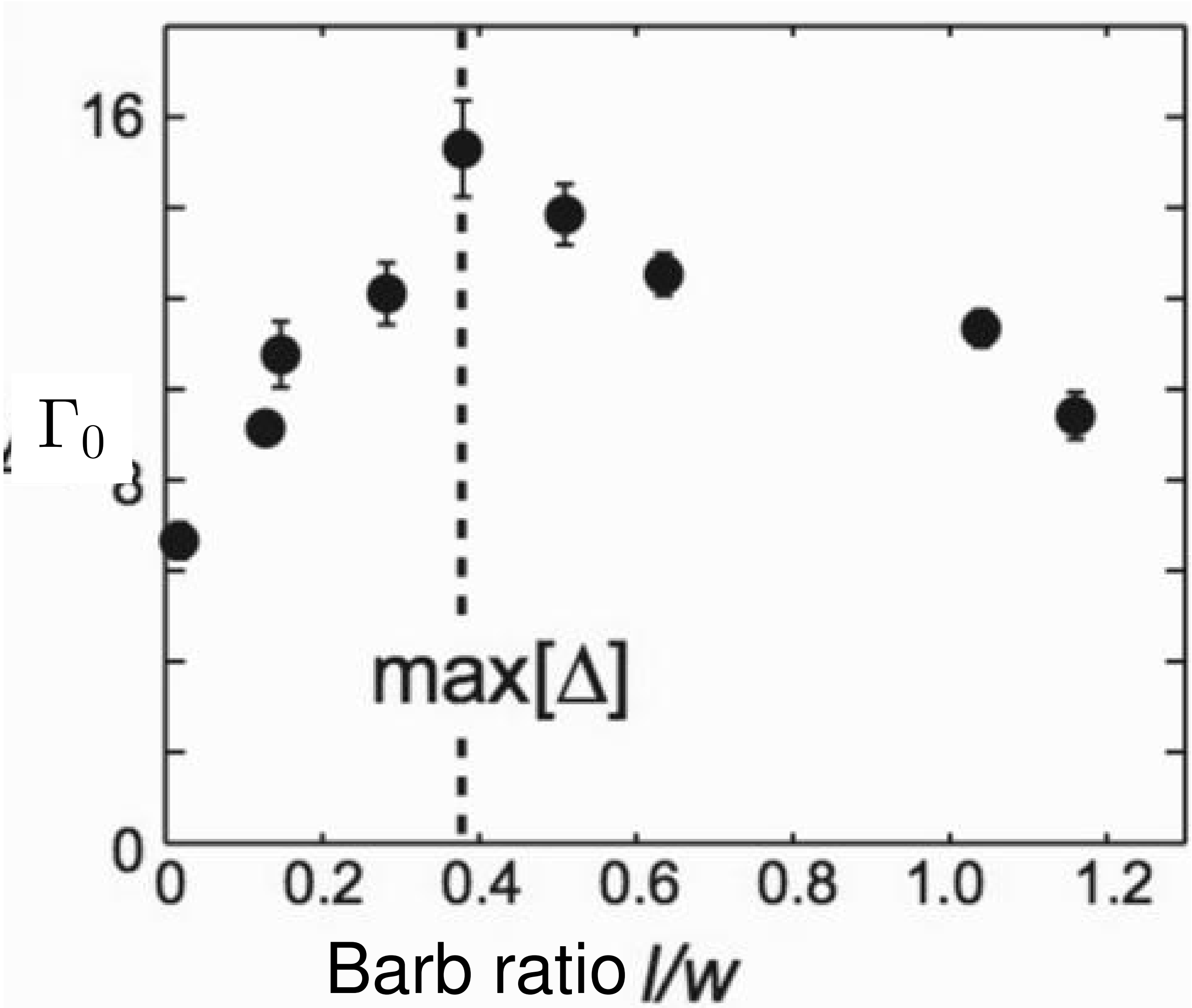
- Large τ: rigid pile not shaking hard

Timescale τ vs. vibration acceleration Γ: $\tau \propto \exp[\Gamma_0/\Gamma]$

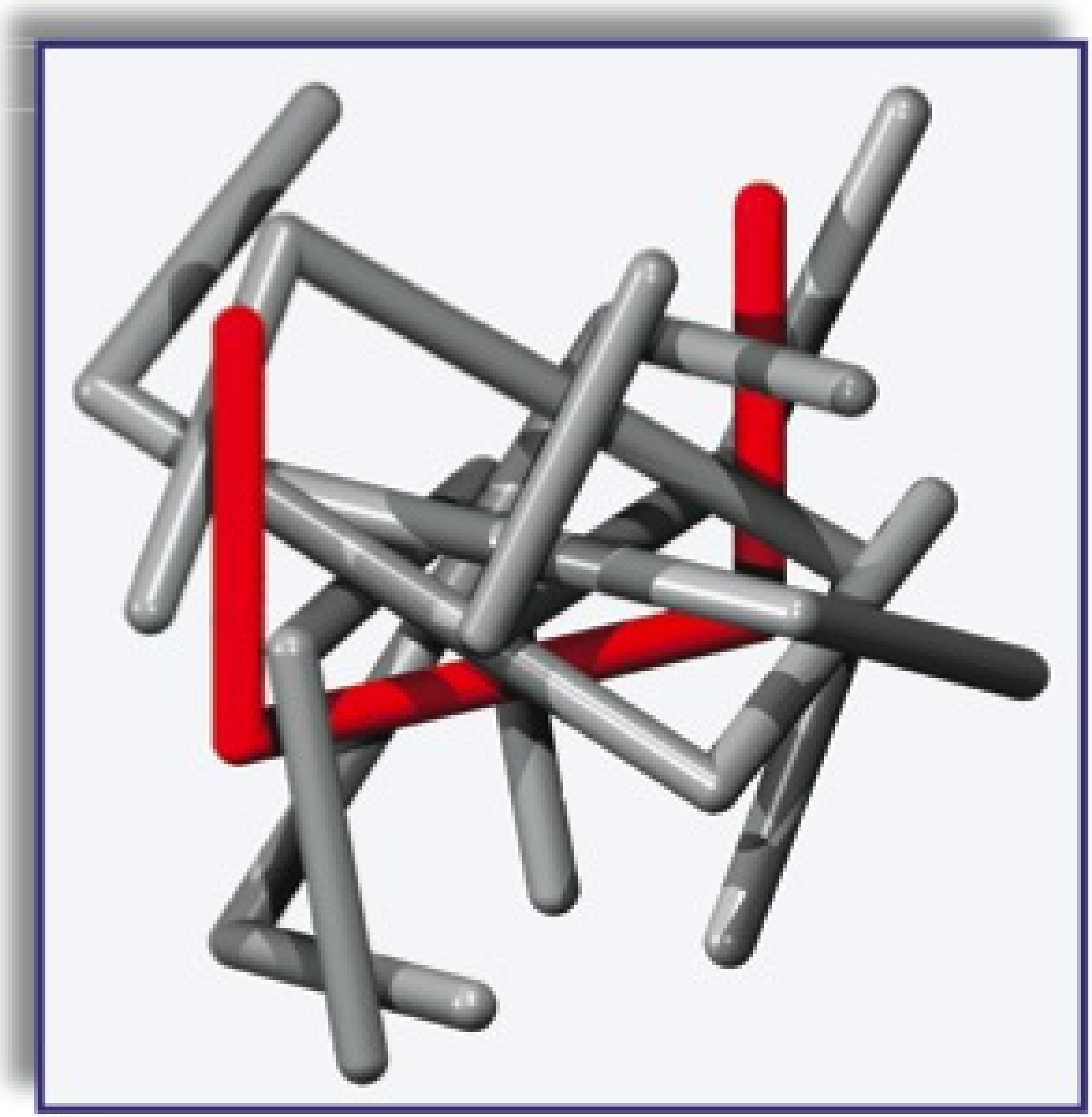


- Large Γ_0 : pile much more resistant to less vigorous shaking

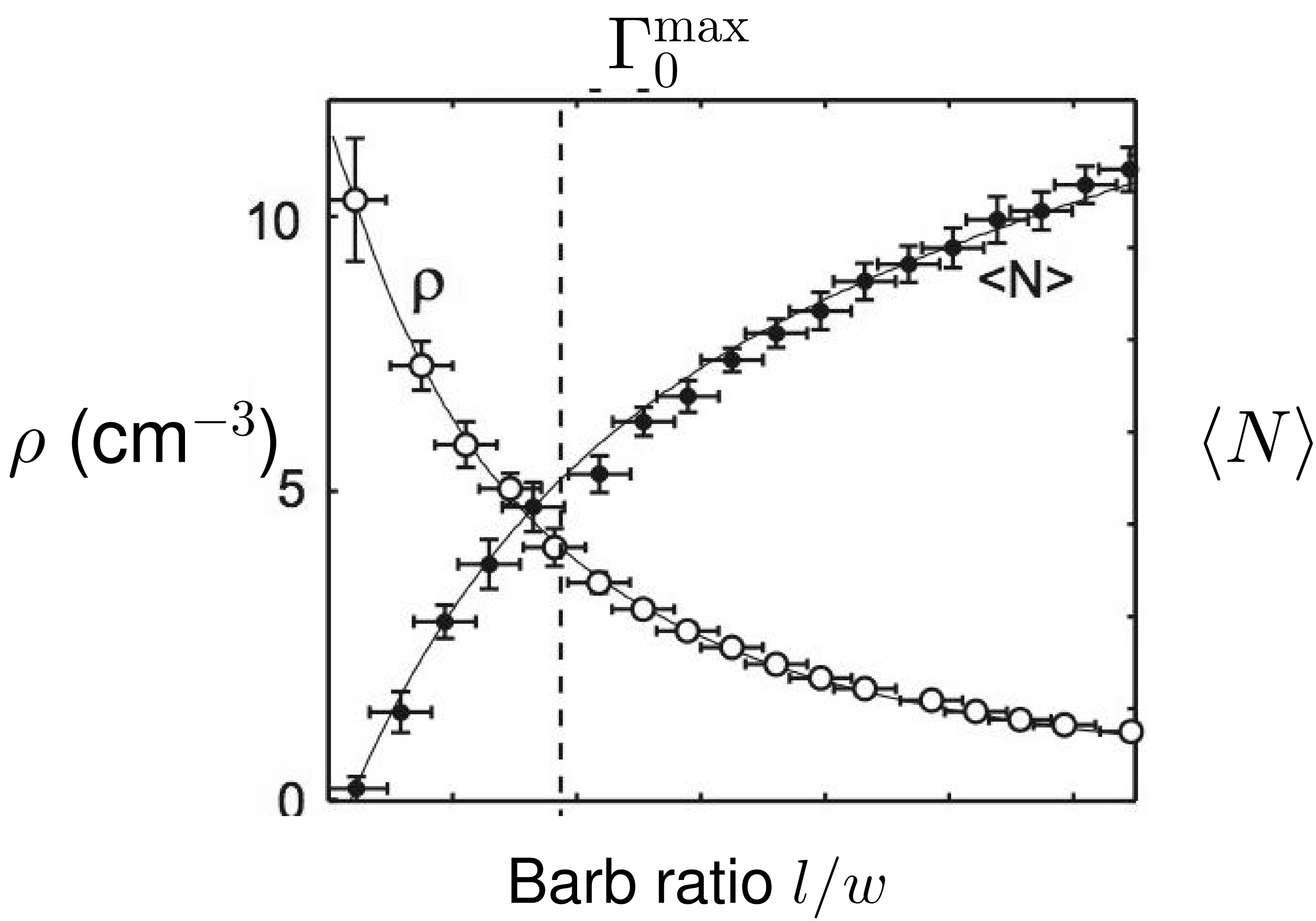
Optimally Rigid Barb Ratio



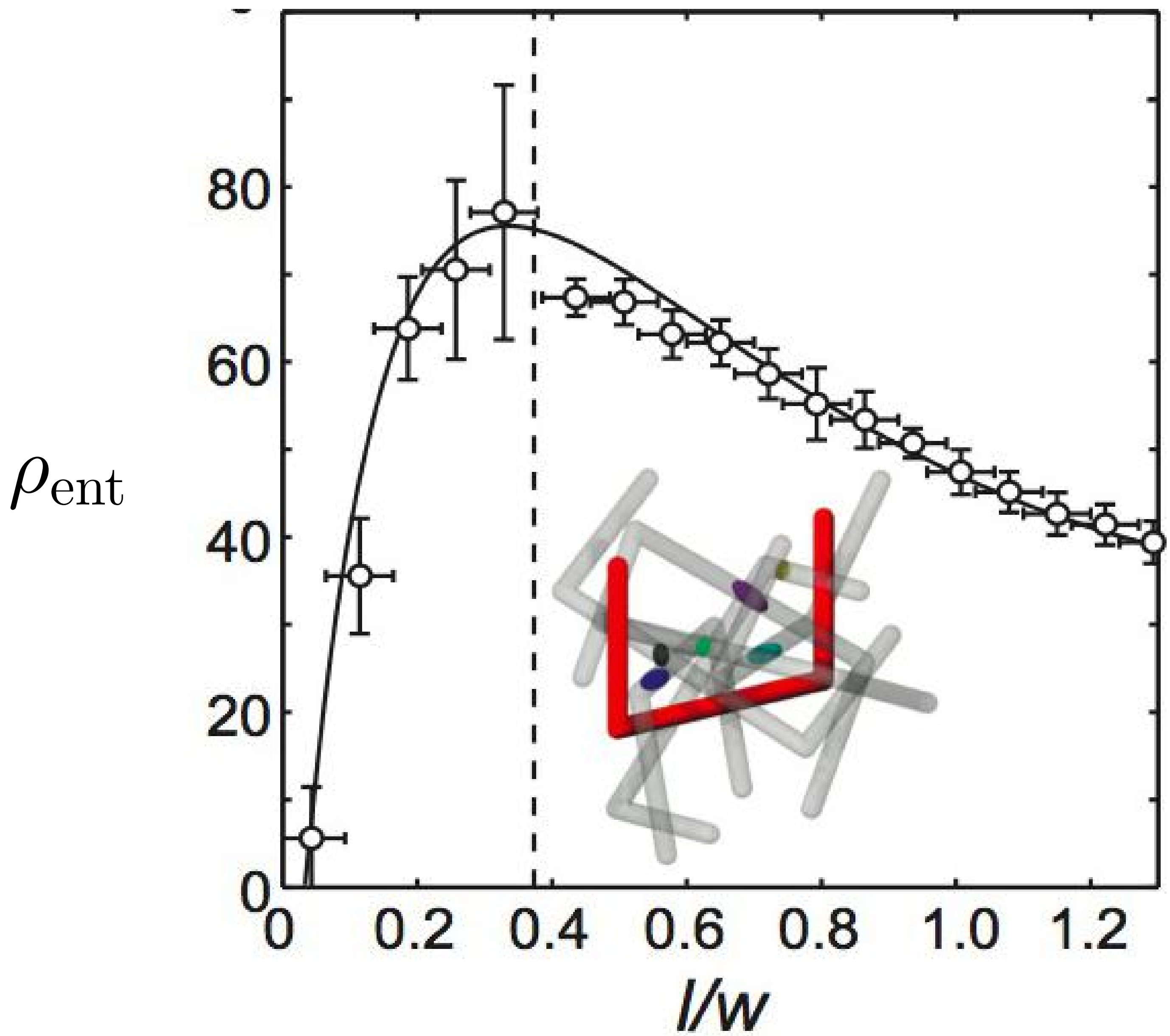
Entanglement Number



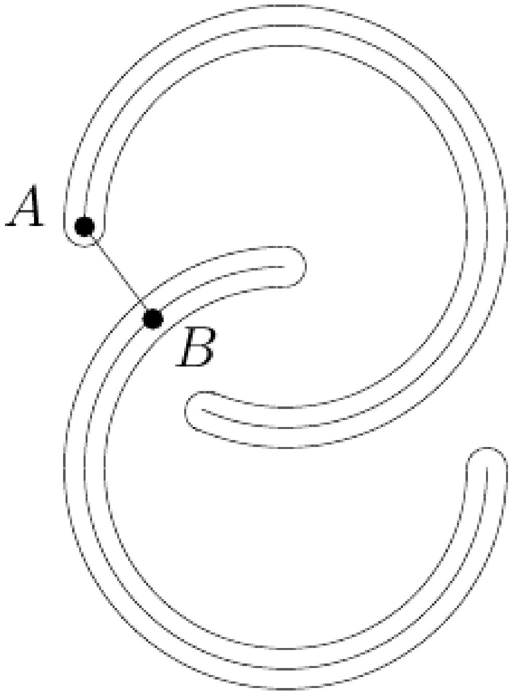
Nonlinear decay/growth of packing fraction, entanglement



Entanglement density peaks with Γ_0



Entanglement density of arcs

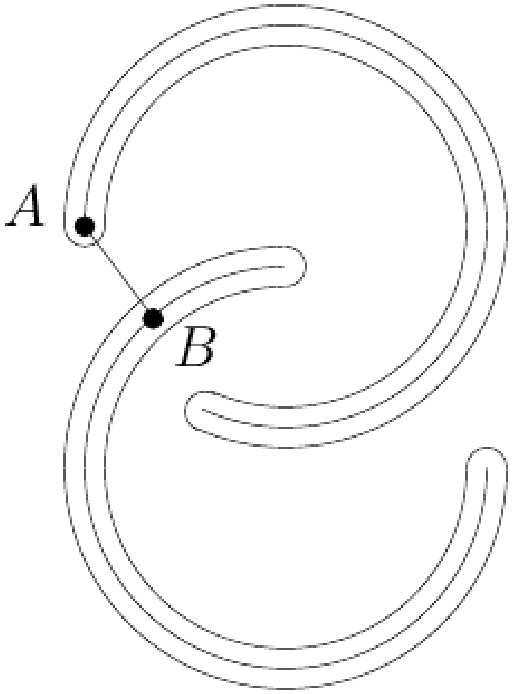


- no jamming at $\theta_{subtended} = 0, 2\pi$
- can "easily" find distance between two arcs

$$(x_c^A, y_c^A, \theta_A), \quad (x_c^B, y_c^B, \theta_B)$$

$$\begin{aligned} D_{AB}^2 \equiv & ((x_c^A + r\cos\theta_A) - (x_c^B + r\cos\theta_B))^2 \\ & +((y_c^A + r\sin\theta_A) - (y_c^B + r\sin\theta_B))^2 \end{aligned}$$

Entanglement density of arcs



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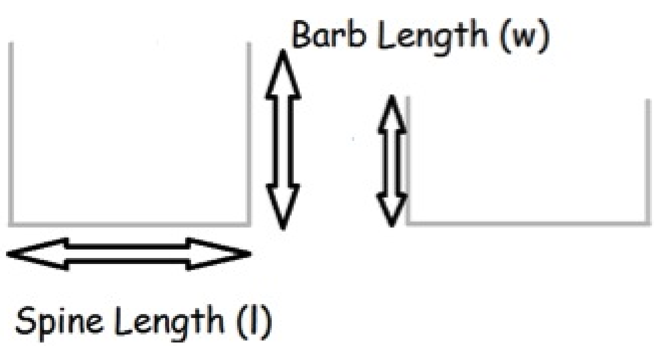
Lagrangian formulation actually *easier*. Minimize

$$D^2 = (x_A - x_B)^2 + (y_A - y_B)^2$$

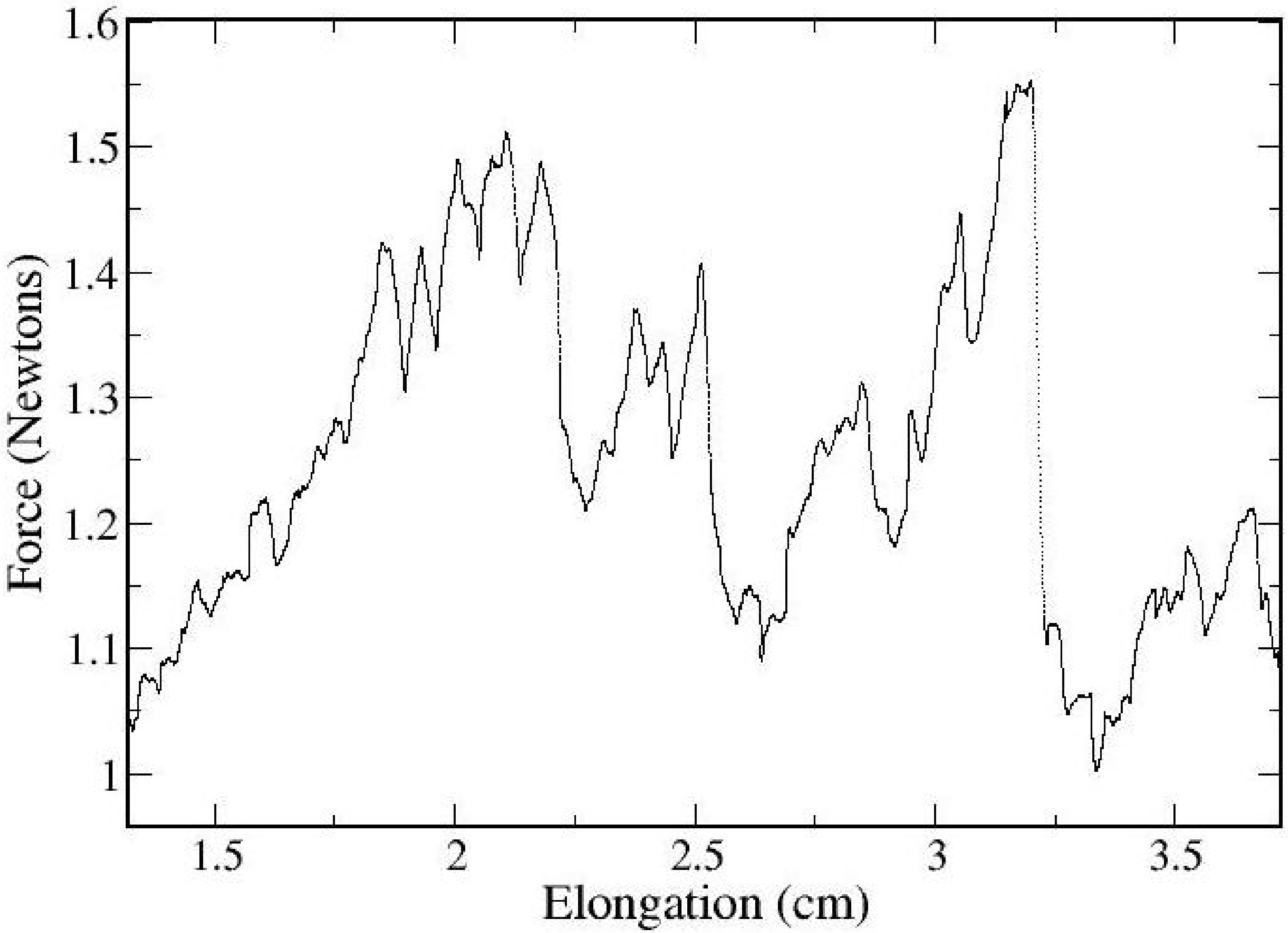
subject to constraints: $(x_{A/B}, y_{A/B})$ lie on circle A/B

Rheology of Geometrically Cohesive Granular Materials

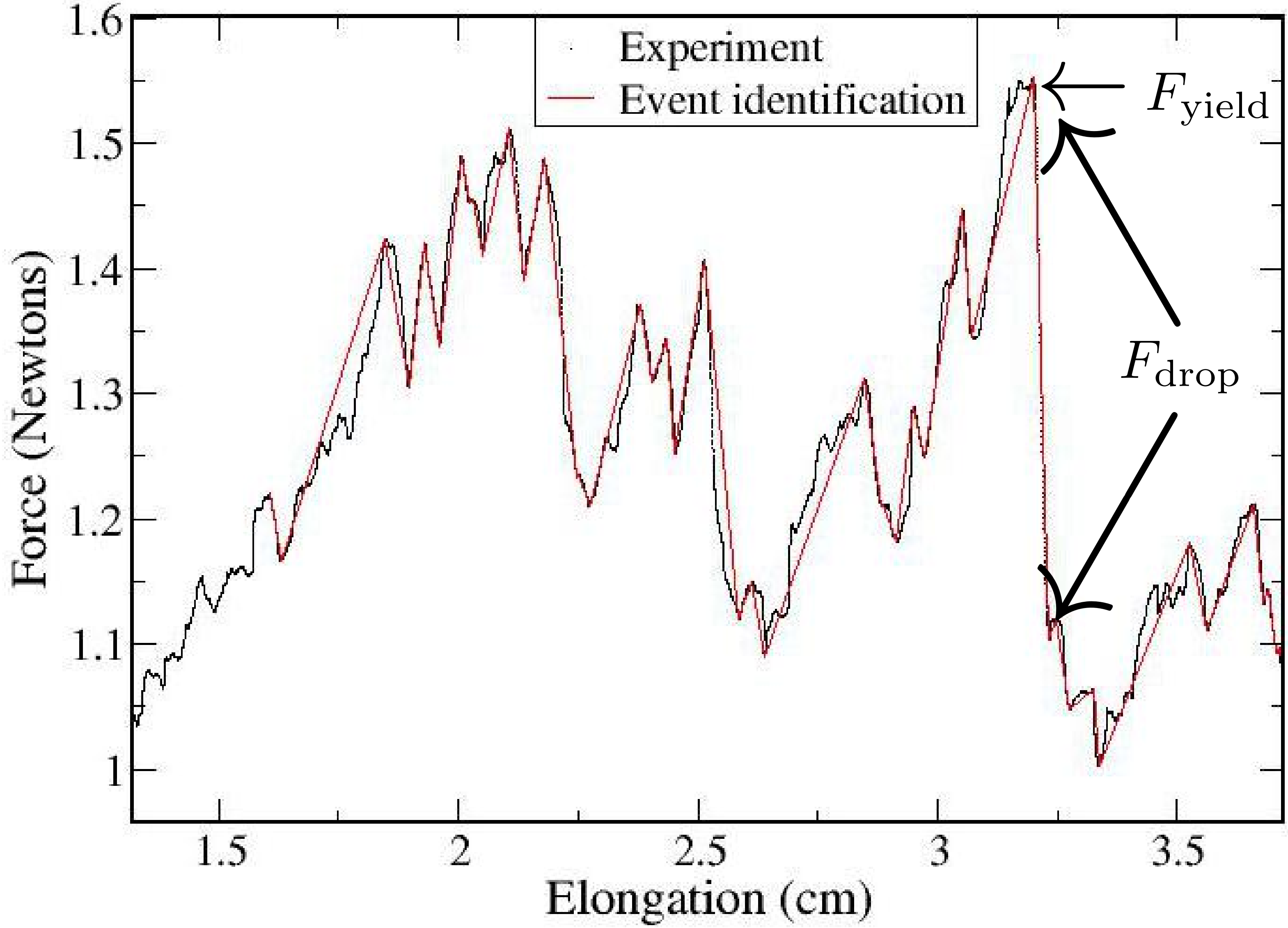
- $l = 1.3$cm
- $w = 0.64$cm
- $l/w = 0.5$
- $d = 0.1$cm
- $\phi \approx 0.24$
- $D = 3.1$cm
- $L = 2 - 11$cm



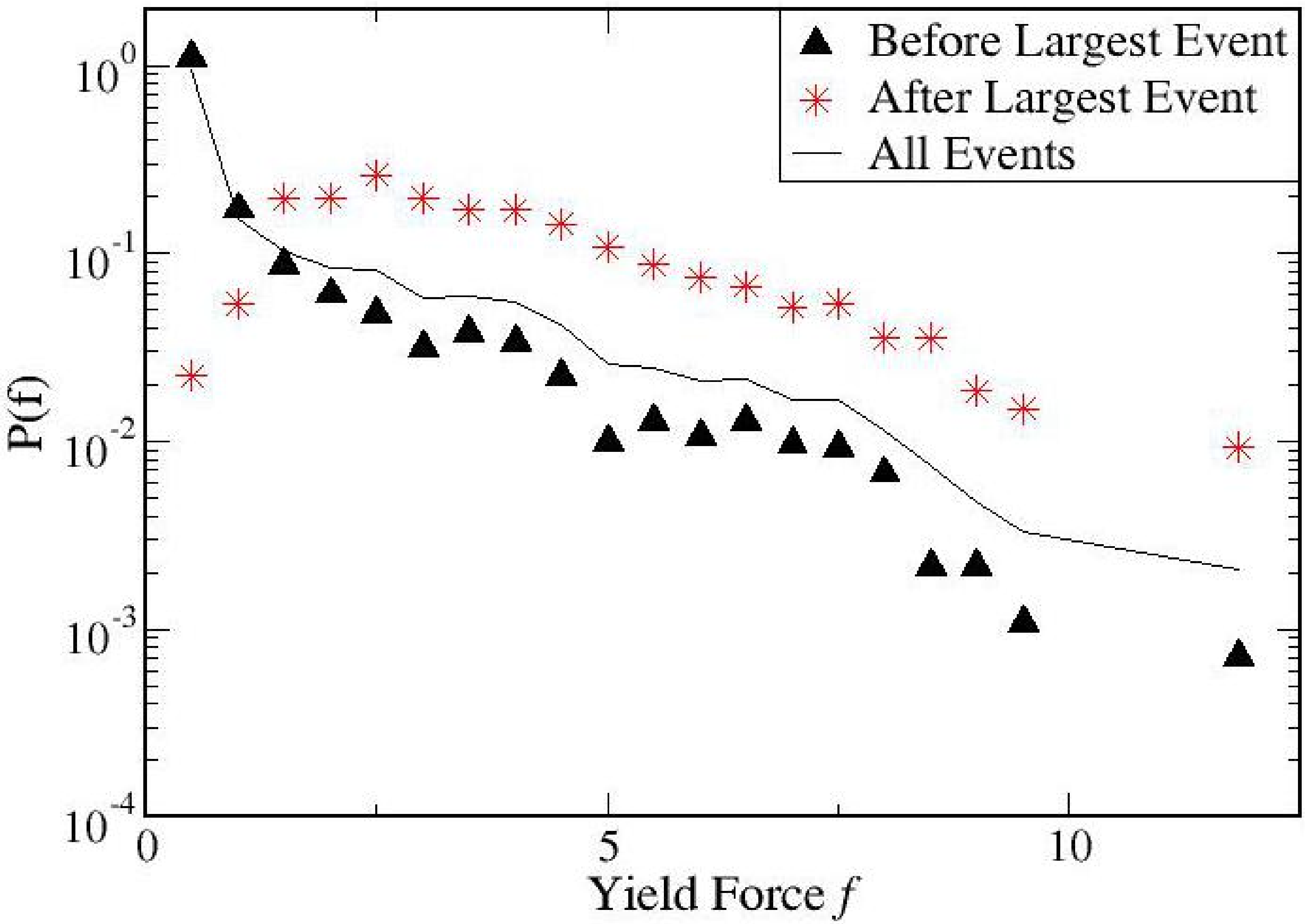
Canonical stick-slip behavior



Event details

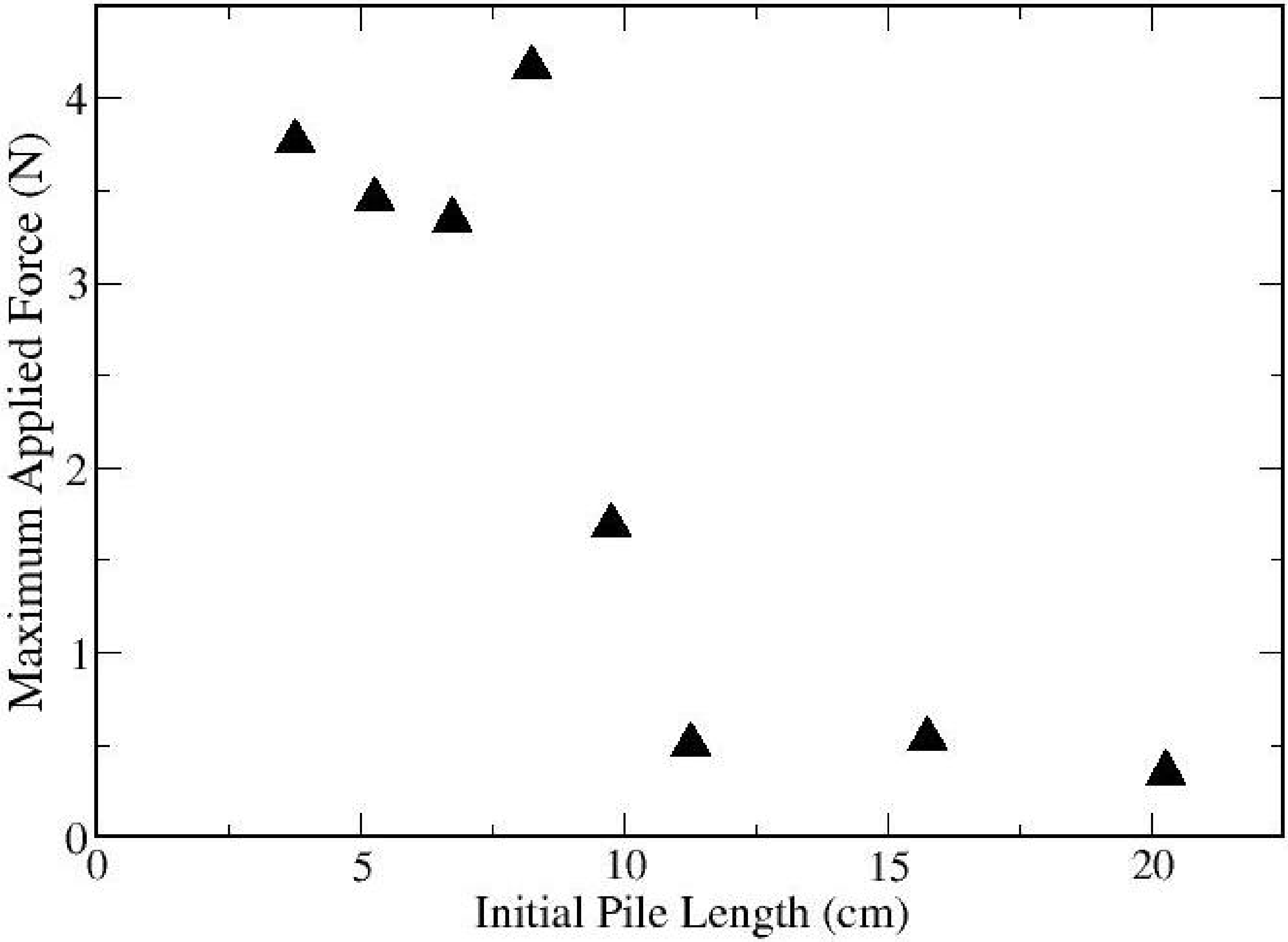


Memory of “major” event



- Treat all 7543 slip events as independent failures

Longer piles significantly weaker



Weibullian "weakest link" statistics (Ken Kamrin)

- Basic assumptions
  - $*$ multiple small units δL, if any one fails, sample fails
  - $*$ probability of a unit failing is $P_f \propto F^m \delta L$

- Success probability of N elements: $S = \Pi\left[1 - \alpha F^m \delta L\right]$

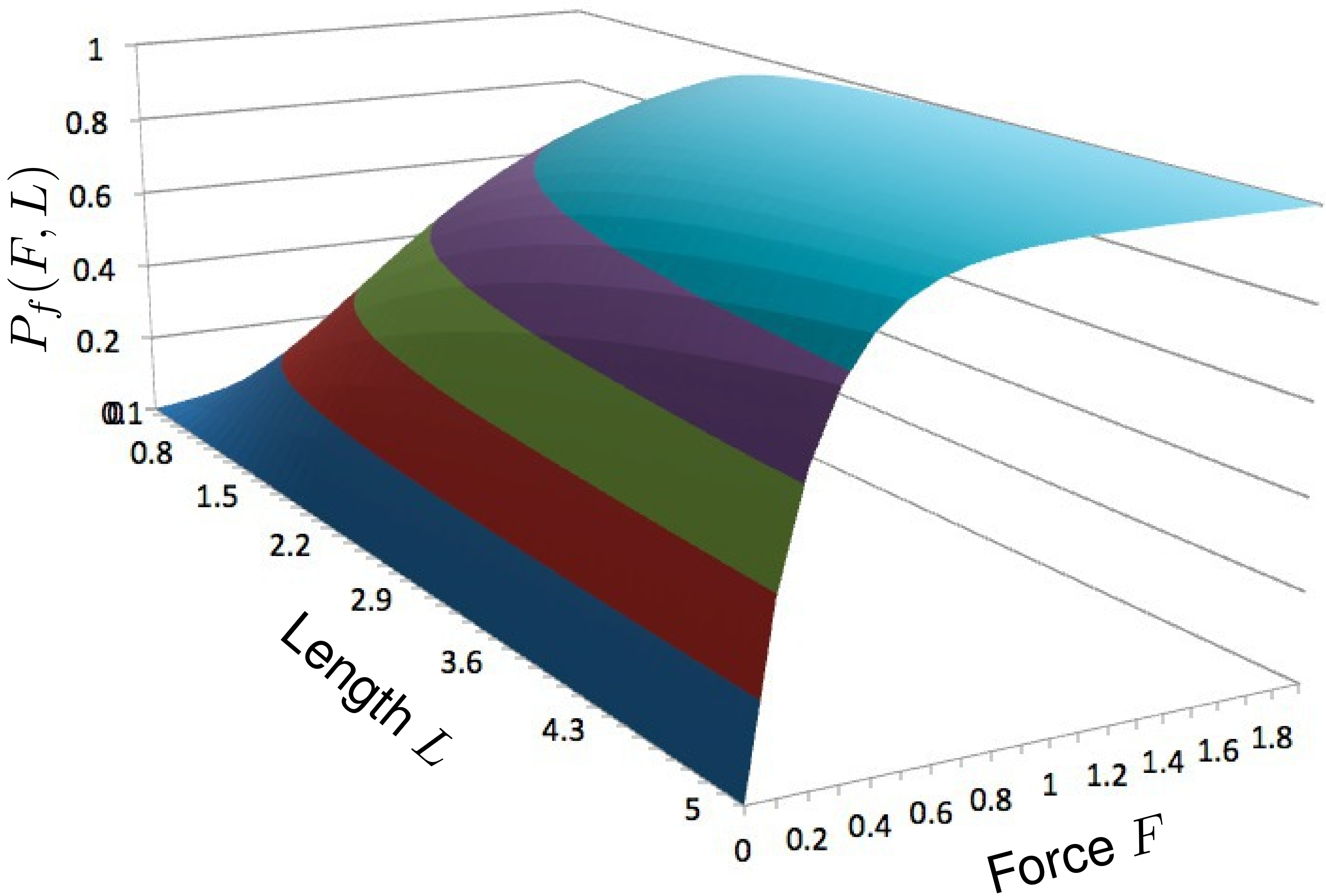
$$\begin{aligned} \ln S &= \sum \ln\left[1 - \alpha F^m \delta L\right] \approx \sum -\alpha F^m dL \\ \Longrightarrow S &\approx \exp\left[-\alpha L F^m\right] \end{aligned}$$

- Probability of failure *if* force F and length L:

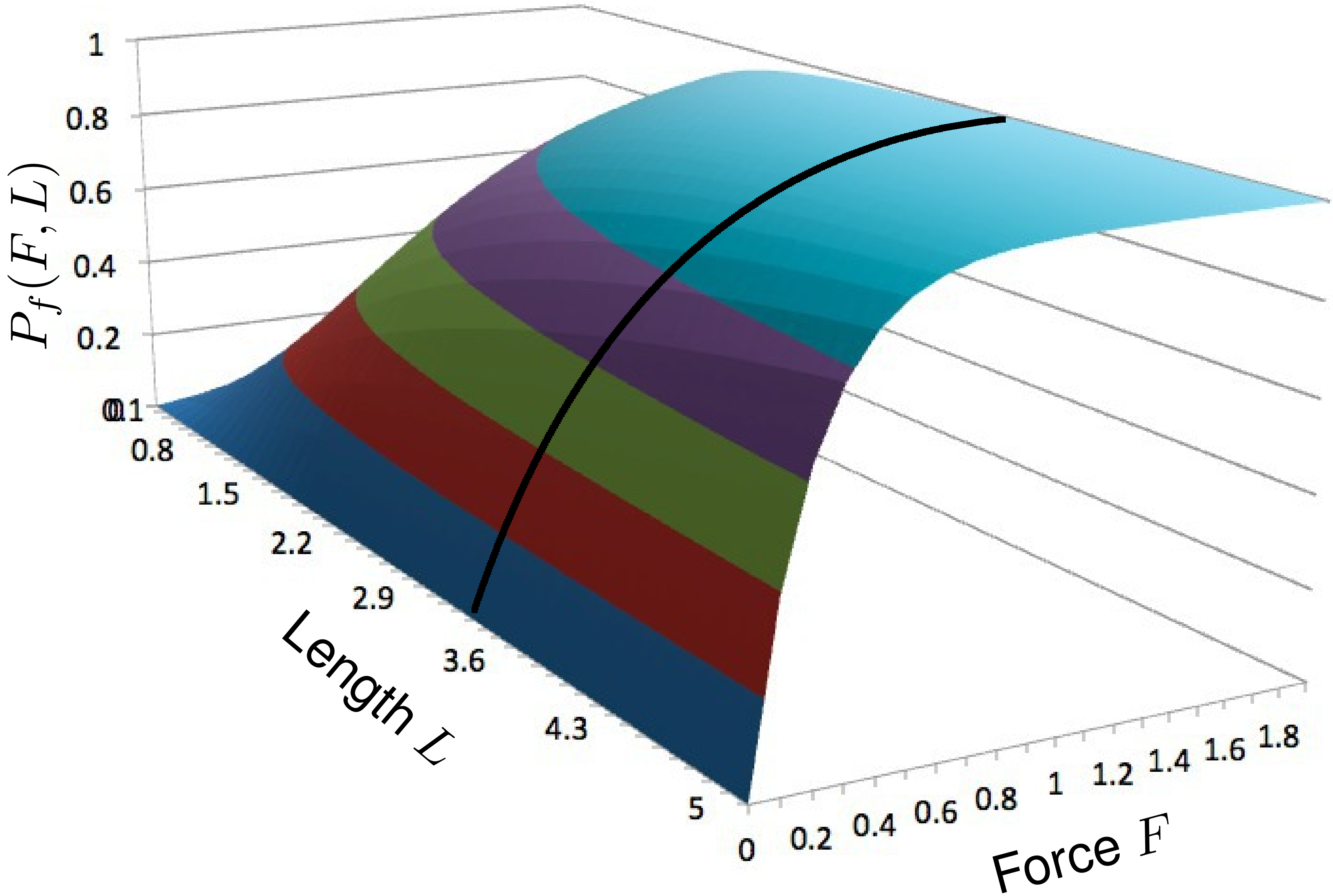
$$P_f(F, L) = 1 - e^{-\alpha L F^m}$$

Cumulative Failure distribution function

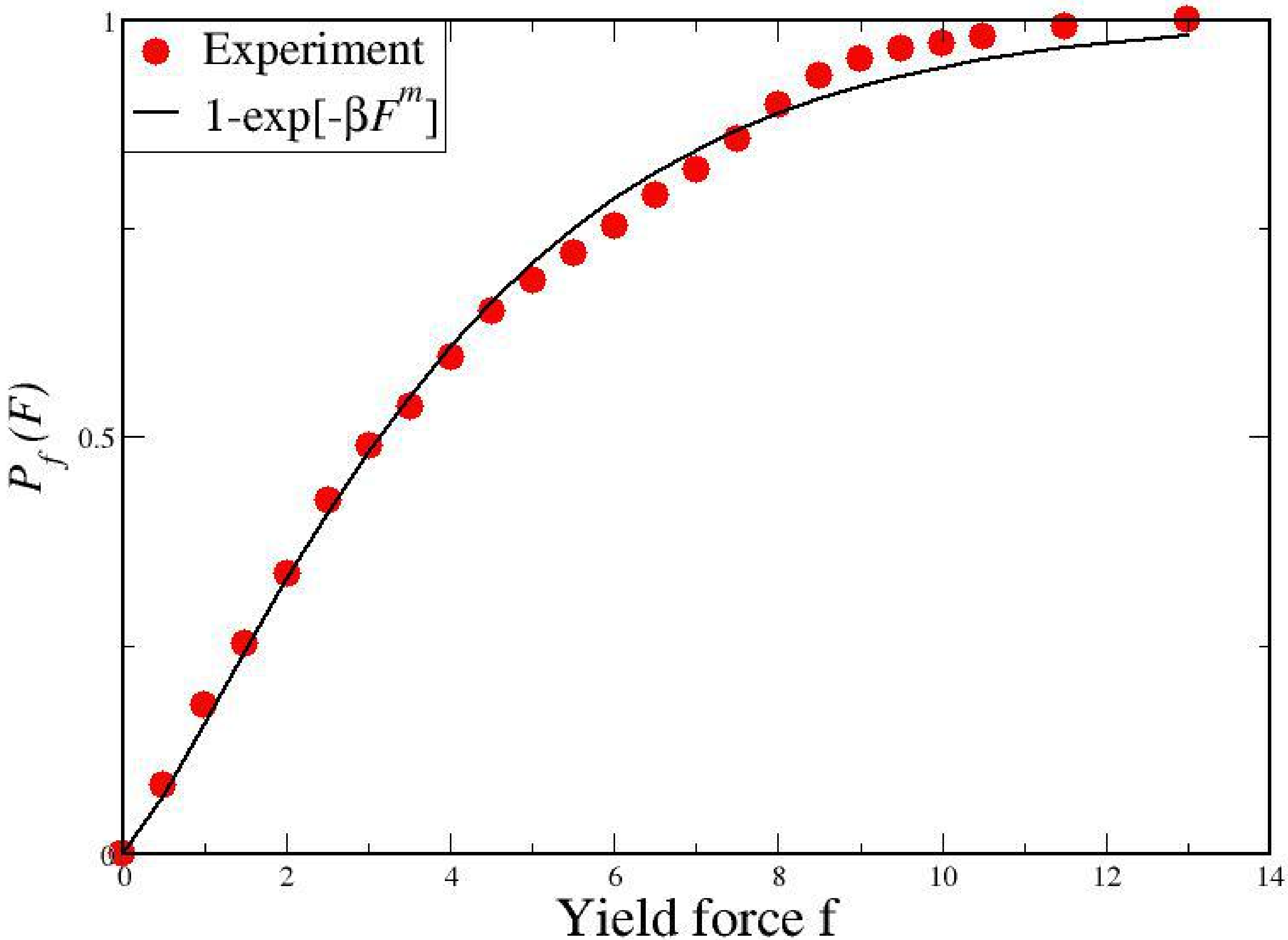
$$P_f(F, L) = 1 - e^{-\alpha L F^m}$$



Prediction #1 (fixed length): $P_f(F) = 1 - e^{-\beta F^m}$

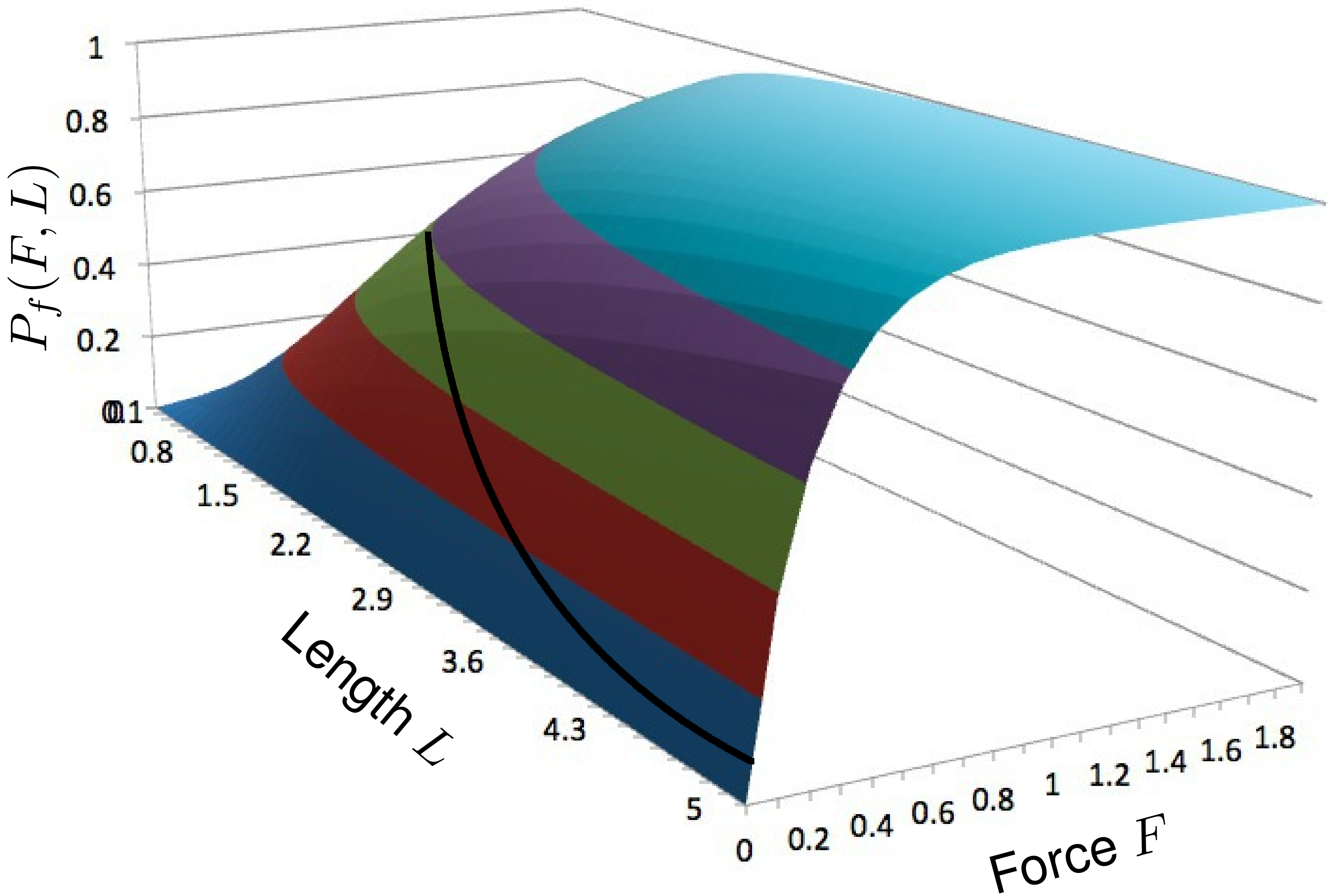


Fixed length failure probability distribution: $m = 1$



Prediction #2: Mean yield force $\sim L^{-1}$

- $< F_Y >= \int_0^\infty \alpha L F e^{-\alpha L F} dF \sim L^{-1}$



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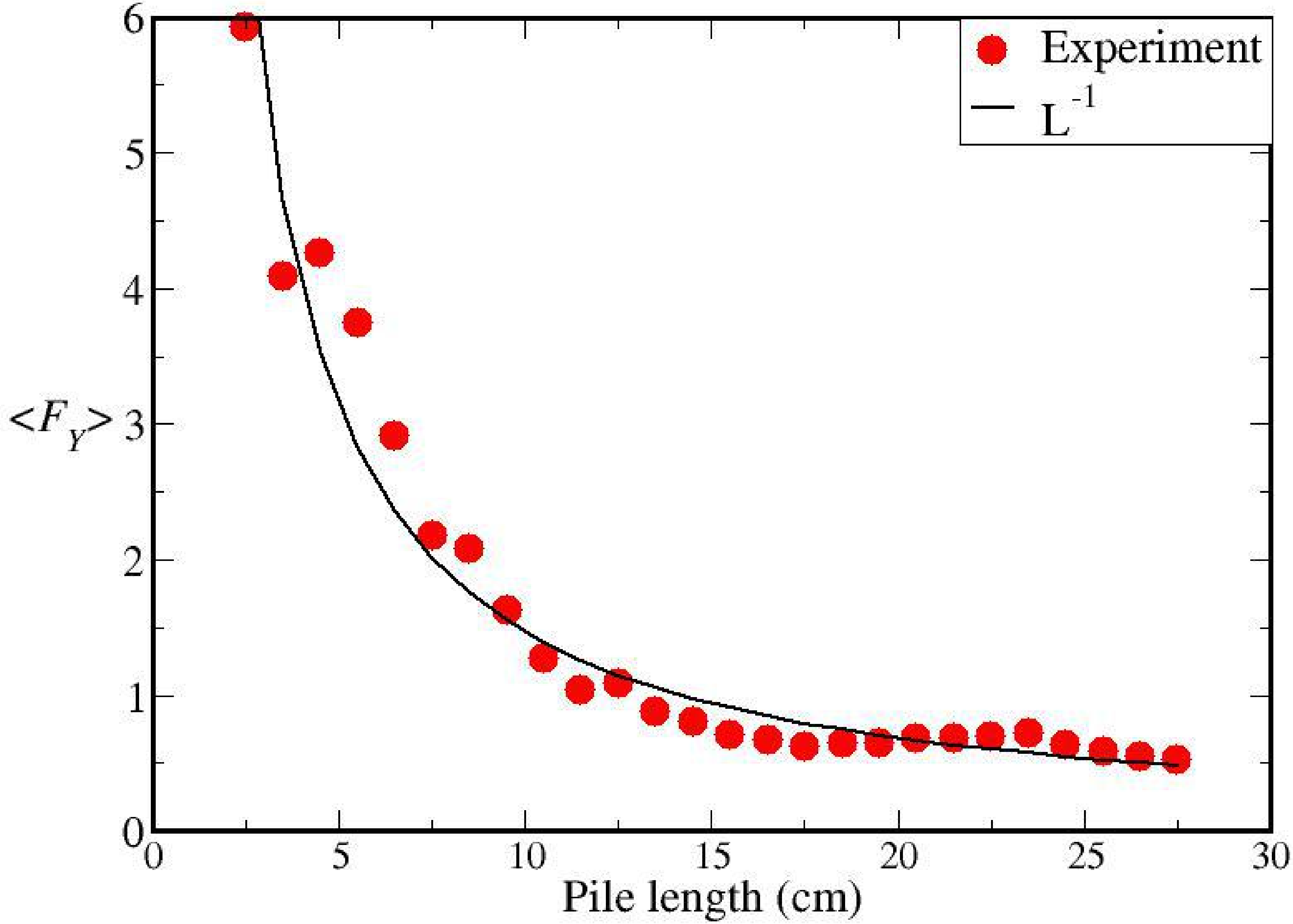

Prediction #3: Collapse all data onto single curve

$$P_f(F,L) = 1 - e^{-\alpha L F^m} \Longrightarrow \frac{-1}{L} \log\left[1 - \frac{P_f(F,L)}{P_f(L)}\right] \propto F$$

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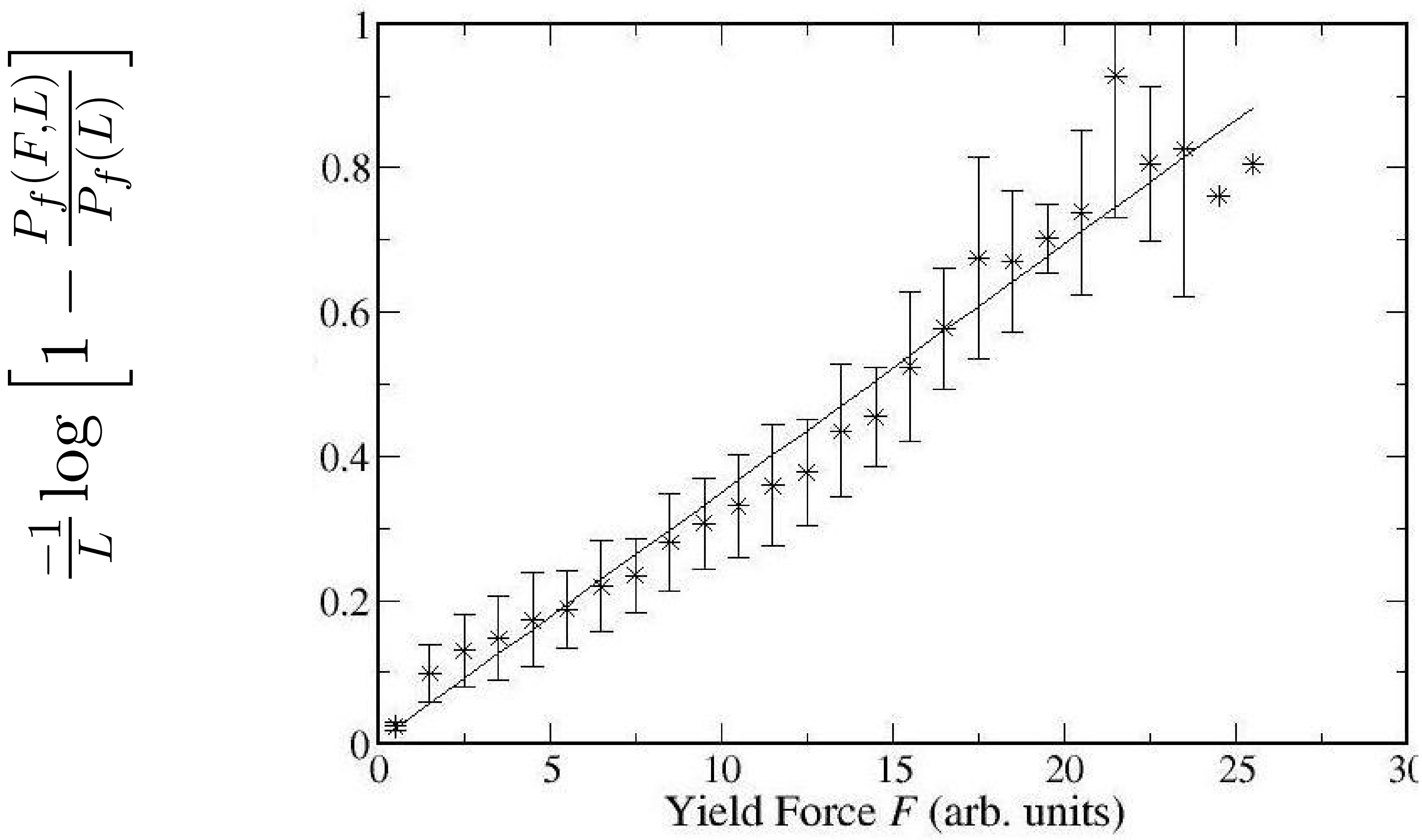
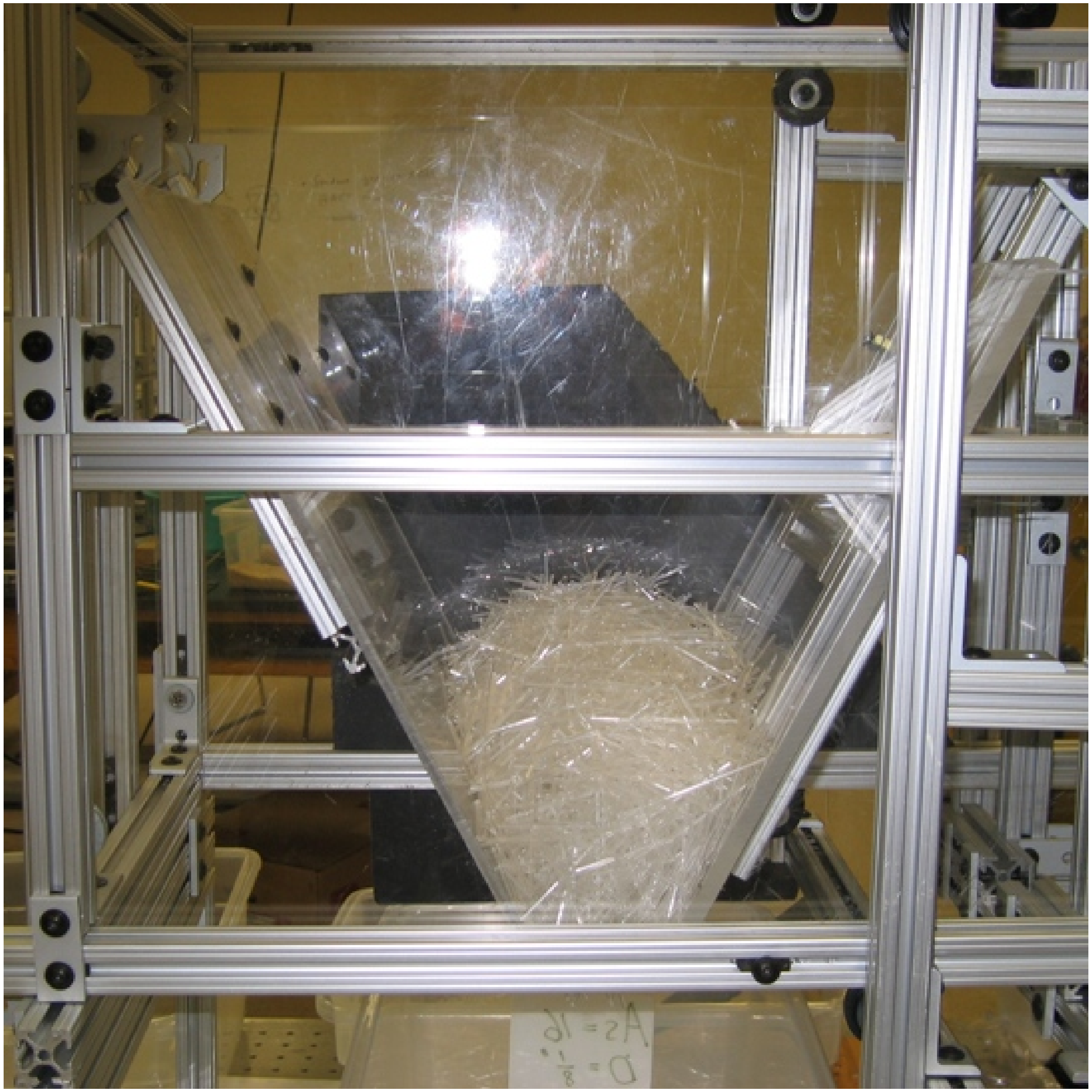
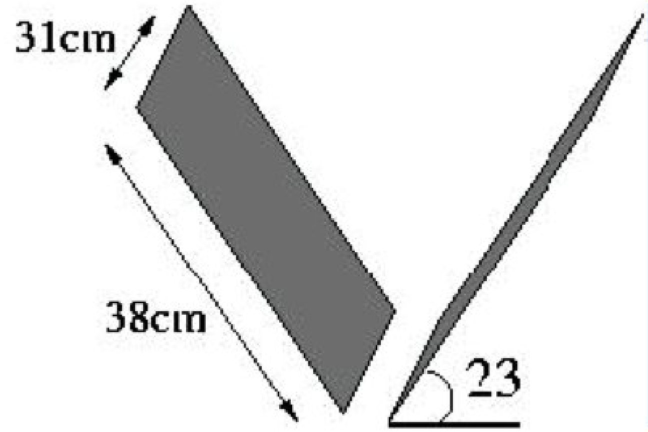


Conclusions

- Geometrically Cohesive Granular Materials (GCGM) exhibit solid-like and stick-slip *extensional* rheology
- statistical/thermodynamic models show some success in explaining behavior
- extensional rheology well-modeled by weakest-link theory that assumes yield probability proportional to applied force

Flow of Particles Through Wedge Hoppers

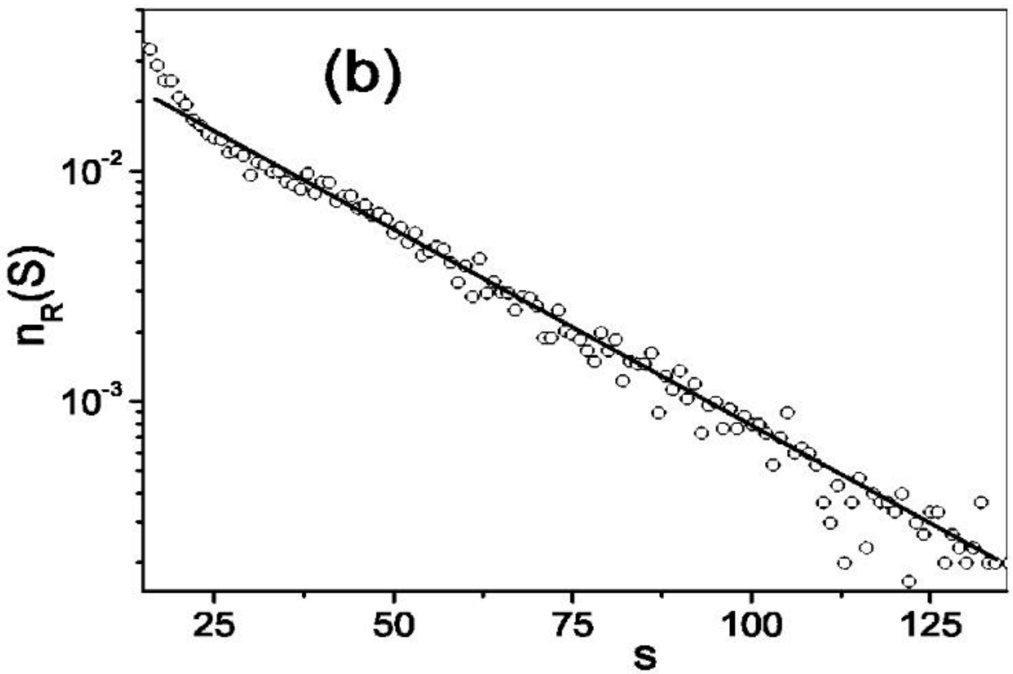


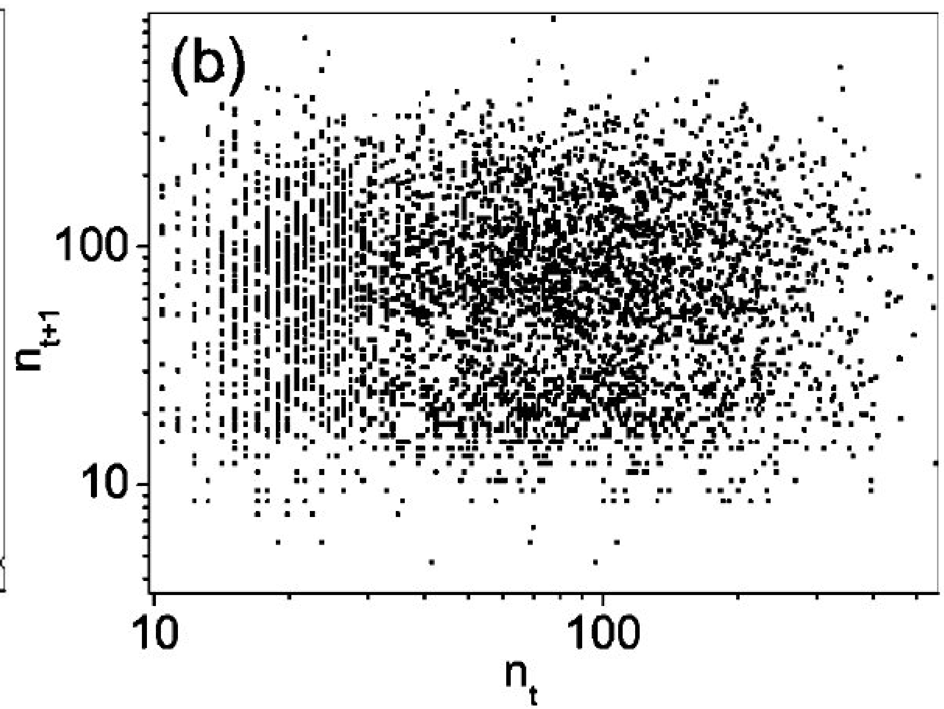


aperture: 0-50cm
rods: $d = 0.08 - 0.6$cm
$L = 0.6 - 8$cm
aspect ratio: $1 - 50$

Summer Saraf

Ordinary (round) Hoppers: Exit Mass Distribution Decays Exponentially

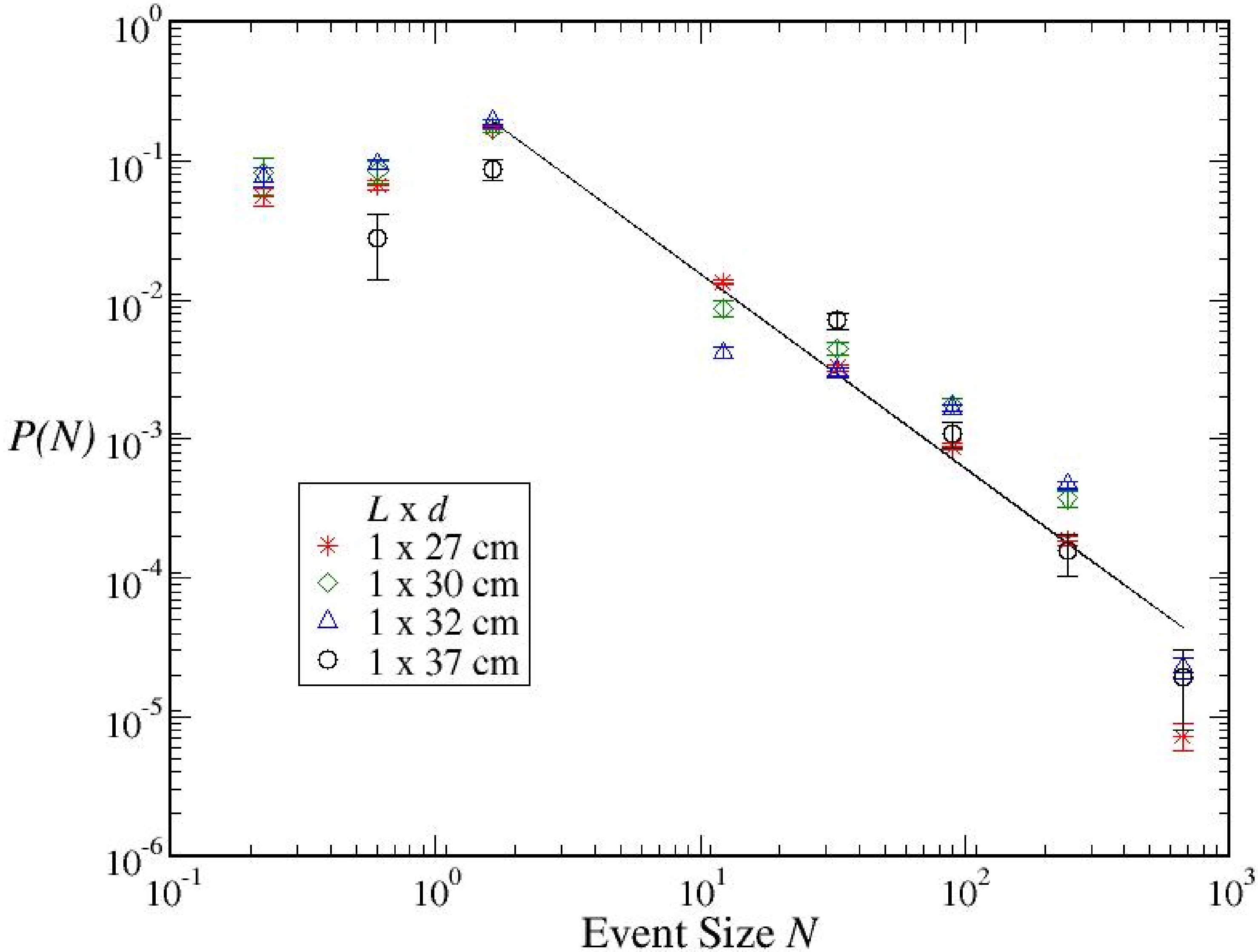




- Zuriguel (2005), distributions scale as $\langle s \rangle$ which may diverge as hopper aperture $R \to R_c \approx 5D$ (D=particle diameter)

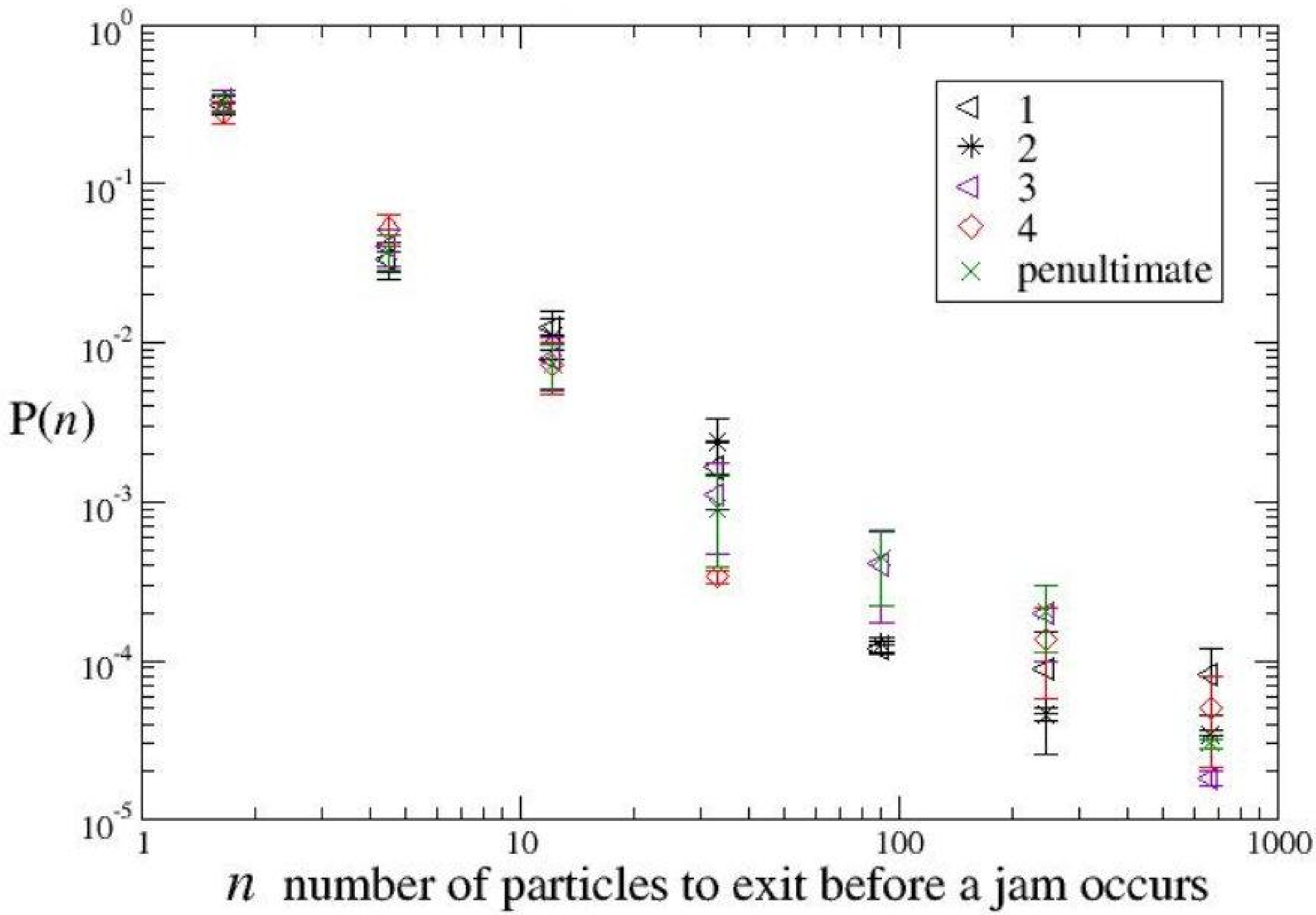
- no memory effects

3+ Decades of Power Law



Large events *more* common than expected

Independent of hit number (Aperture = 10d)



Random Walk Models

- Exponential decay implies no correlations:
  - $\ast$ 1 particle exits w/probability p
  - $\ast$ n particles: $P(n) = p \cdot p \ldots p \cdot (1-p) = p^n \cdot (1-p)$

$$\begin{aligned} \log P(n) &= \log p^n (1-p) \\ &= \log(1-p) + n \log p \end{aligned}$$

$$\begin{aligned} \exp\left[\log P(n)\right] &= e^{[\log(1-p) + n \log p]} \\ P(n) &= e^{\log(1-p)}\, e^{n \log p} \\ &= (1-p) e^{n \log p} \end{aligned}$$

$p < 1$ and so $\log p < 0$ and $P(N)$ exponentially decays

$\boxed{p}$

- generate random numbers until get one $> p$
- number of random numbers is "event size"
- do this many times, make histogram of event size

Simulation: March 2009

Wedge-shaped hopper w/uniform probability

p	p	p	p	p	p	p	p

Simulation: March 2009

Wedge-shaped hopper w/uniform probability

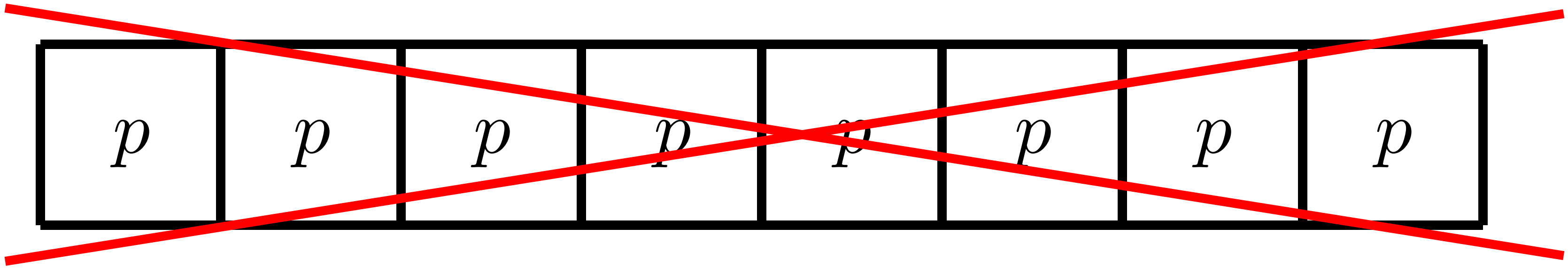
p	p	p	p	p	p	p	p

nonuniform probability: cell j has probability p_j

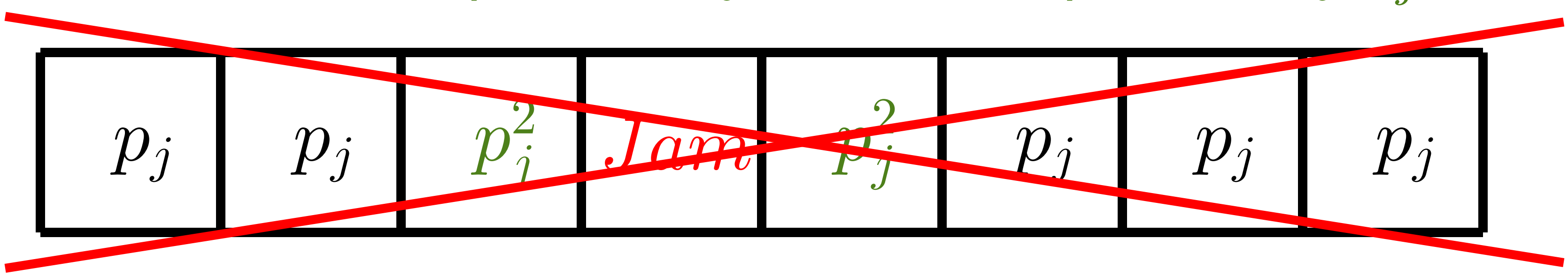
p_j	p_j	p_j^2	Jam	p_j^2	p_j	p_j	p_j

Simulation: March 2009

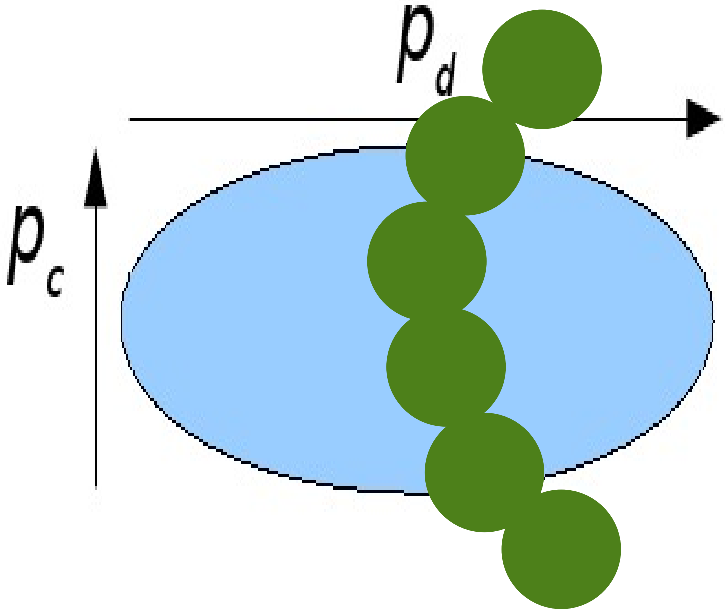
Wedge-shaped hopper w/uniform probability



nonuniform probability: cell j has probability p_j

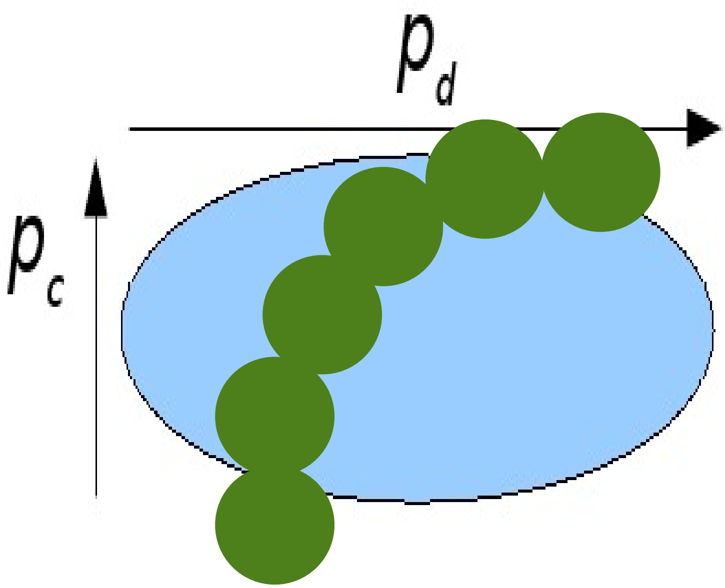


“String” orientation



$\perp$: low exit probability

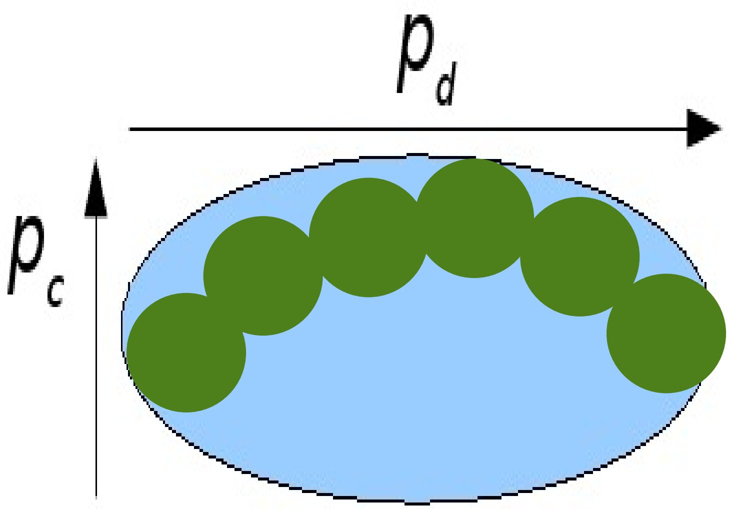
“String” orientation



$\perp$: low exit probability

$>$: larger exit probability

“String” orientation



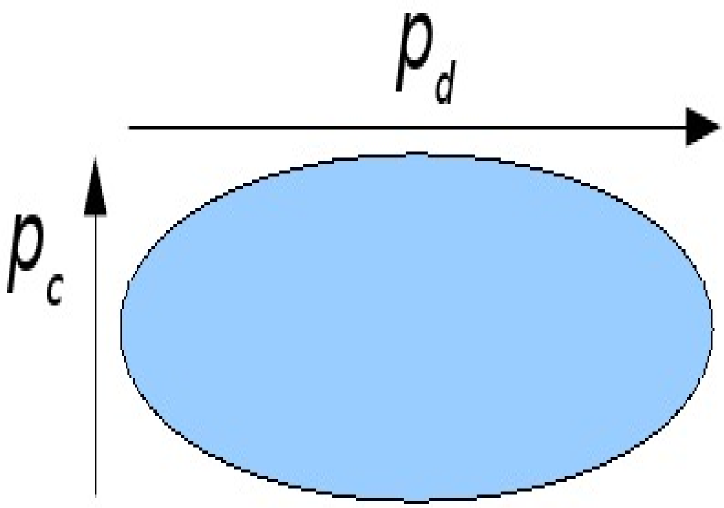
$\perp$: low exit probability

$>$: larger exit probability

$\|$: largest exit probability

Distribution of exit probabilities $p(\theta)$. Need to average over *string orientations* to find $\langle P(n) \rangle$.

Final Picture



$$P(N) = \int_{p_c}^{p_d} p^N (1-p)\, \mathrm{d}p$$

$$= \left[\frac{p^{N+1}}{N+1} - \frac{p^{N+2}}{N+2}\right]_{p_c}^{p_d}$$

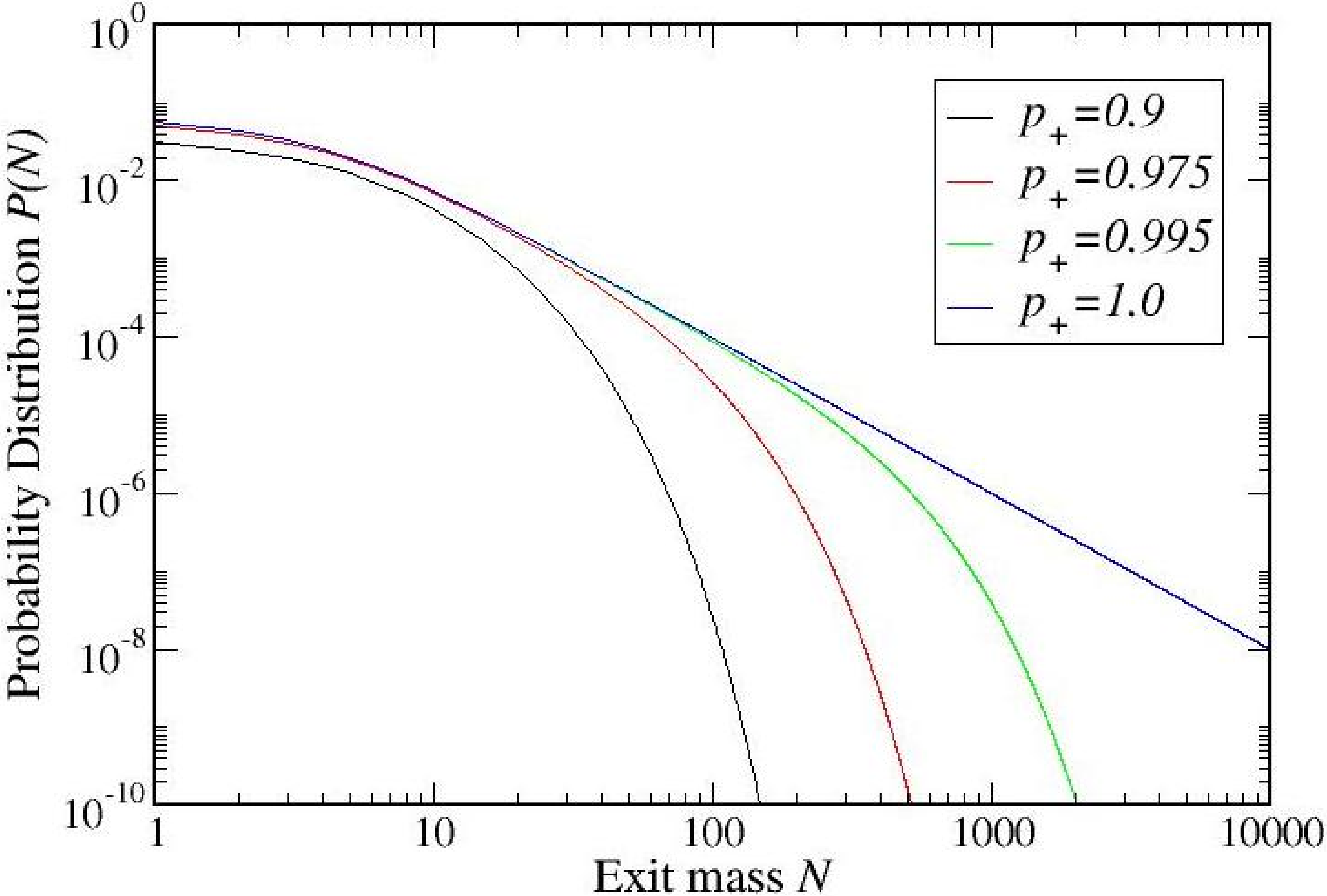
$$\approx \frac{p_d^{N+1}}{N+1} \approx \frac{\exp[N \ln p_d]}{1+N}$$

As long as $p_d < 1$, exponential decay.

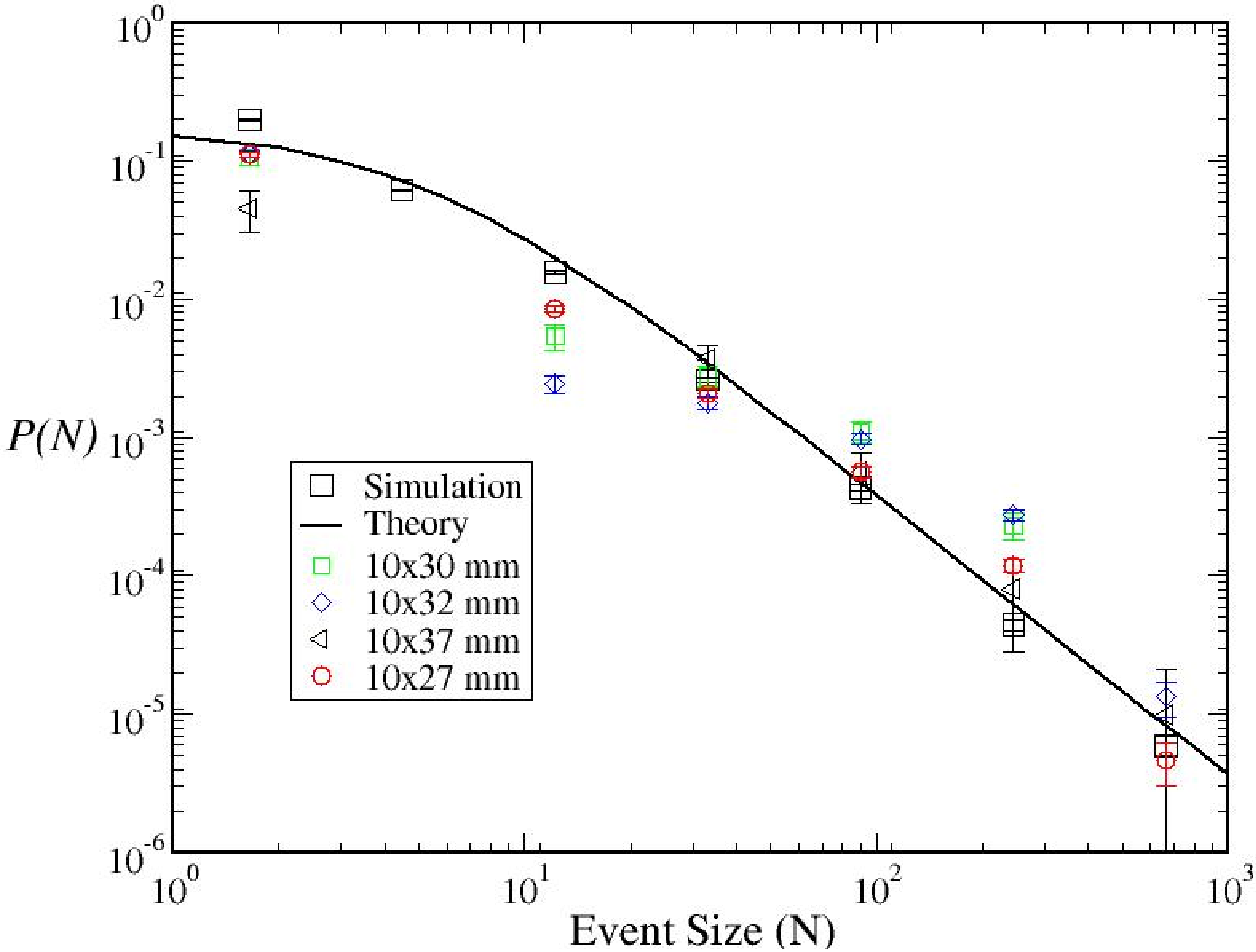
As soon as $p_d = 1$:

$$\frac{1}{N+1} - \frac{1}{N+2} \approx \frac{1}{N^2}$$

Transition to Power Law



Success!



P(N)
Event Size (N)
Simulation
Theory
10x30 mm
10x32 mm
10x37 mm
10x27 mm

Conclusions II

- Exit-mass probability distribution in wedge hoppers shows broad power-law tail
- Model that assumes characteristic length-scale (strings) with orientation dependent exit probability
  - ∗ Exponential or power-law tail depending on aperture geometry
- Model and experiment agree over many decades