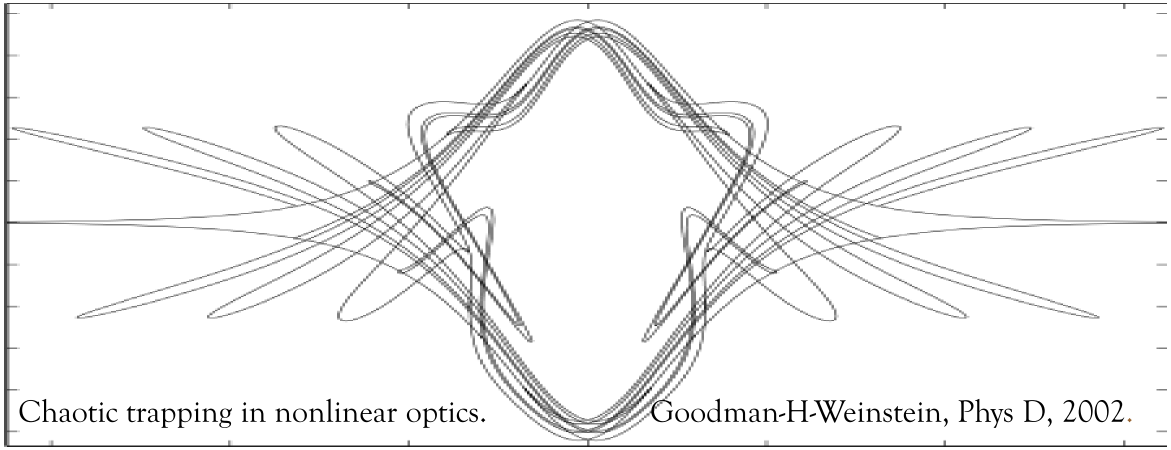
122+ years of nonlinear dynamics: More is different and less is more*

Philip Holmes, Princeton University

… being a historical tour through the woods and meadows:
from Poincaré to Smale via van der Pol, Birkhoff, Cartwright, Levinson, Kolmogorov, Arnold and Moser, etc., and beyond, almost to the present day.



*With apologies to Philip Anderson and Ludwig Mies van der Rohe,

and thanks to many students, postdocs, colleagues, critics, carpers, and, not least: NSF, DoE, NIMH, NIH, Burroughs-Wellcome.

U Toronto Physics Colloquium, March 22nd, 2007.

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- A prize and a mistake: the discovery of chaos. (1885-1899, Paris and Stockholm).
- The horseshoe (1959-60, IAS and Rio).
- Some of what led to the horseshoe (1927-49, Holland, England and the U.S.).

53
- Some of what followed: KAM, catastrophes, center manifolds, unfoldings, local and global bifurcations (1954-2007, USSR and various other locations).

$$75 + 53 \neq 122$$

I: More is different: Global behavior

King Oscar's Prize, 1885-1890

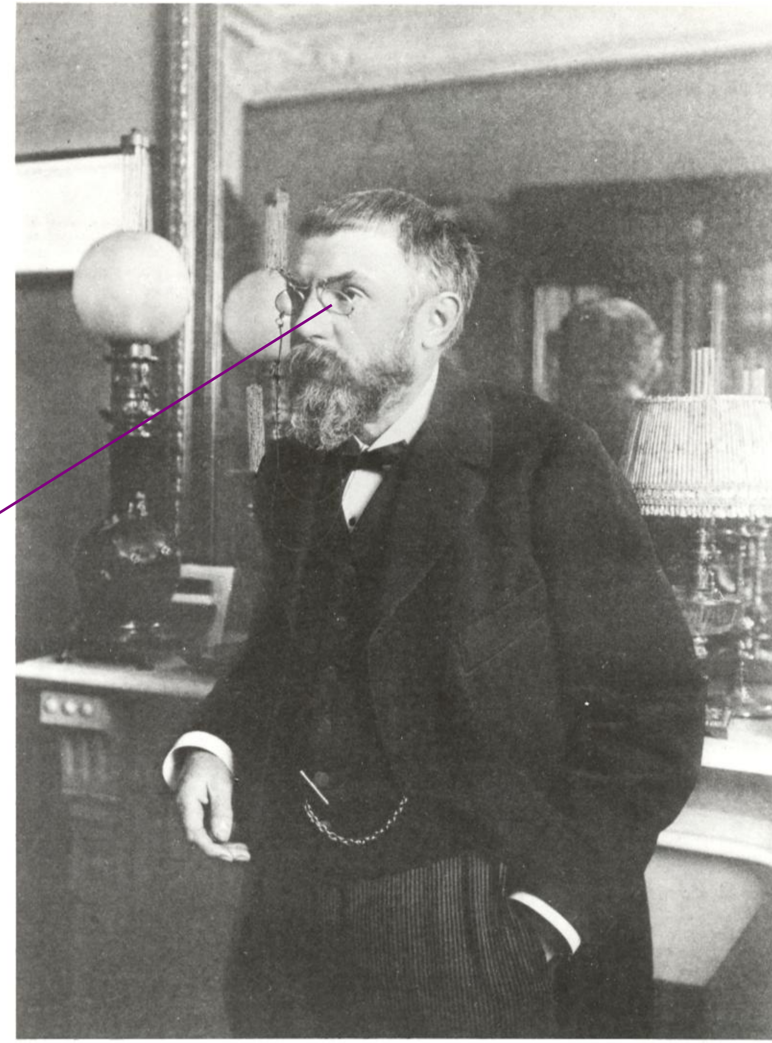
KING OSCAR'S PRIZE
(Problem 1 of 4)

GIVEN A SYSTEM of arbitrarily many mass points that attract each other according to Newton's laws, try to find, under the assumption that no two points ever collide, a representation of the coordinates of each point as a series in a variable that is some known function of time and for all of whose values the series converges uniformly.

This problem, whose solution would considerably extend our understanding of the solar system, would seem capable of solution using analytic methods presently at our disposal; we can at least suppose as much, since Lejeune Dirichlet communicated shortly before his death to a geometer of his acquaintance [Leopold Kronecker], that he had discovered a method for integrating the differential equations of Mechanics, and that by applying this method, he had succeeded in demonstrating the stability of our planetary system in an absolutely rigorous manner. Unfortunately, we know nothing about this method, except that the theory of small oscillations would appear to have served as his point of departure for this discovery. We can nevertheless suppose, almost with certainty, that this method was based not on long and complicated calculations, but on the development of a fundamental and simple idea that one could reasonably hope to recover through preserving and penetrating research.

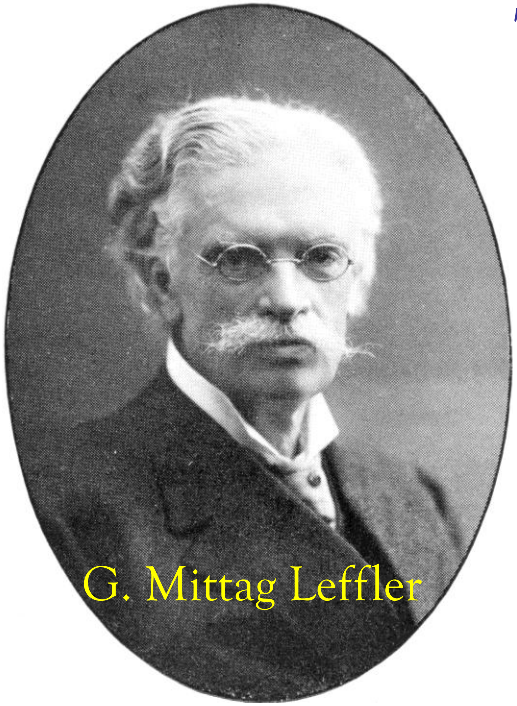
In the event that this problem nevertheless remains unsolved at the close of the contest, the prize may also be awarded for a work in which some other problem of Mechanics is treated in the manner indicated and solved completely.

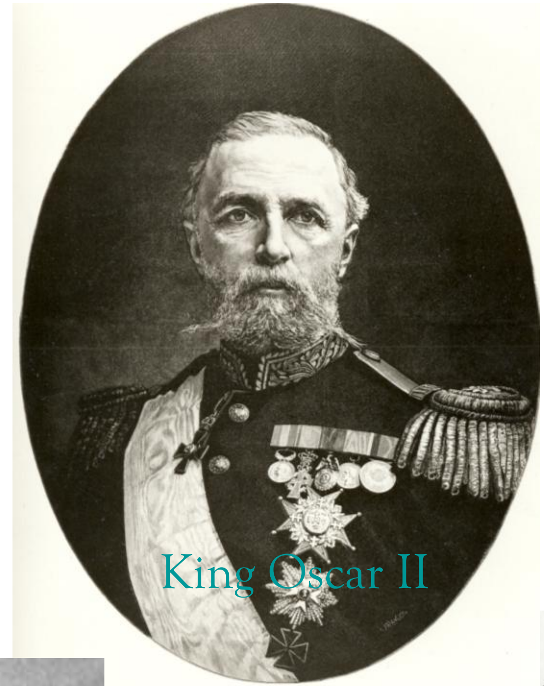
Acta Mathematica, vol. 7, 1885-86.

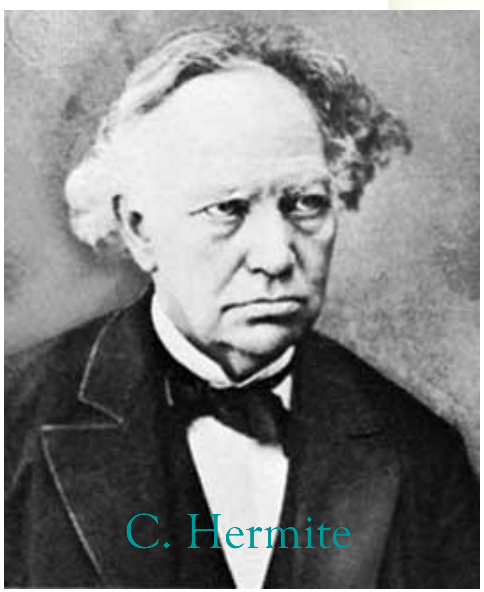


Henri Poincaré (1854-1912)

The King and his jury

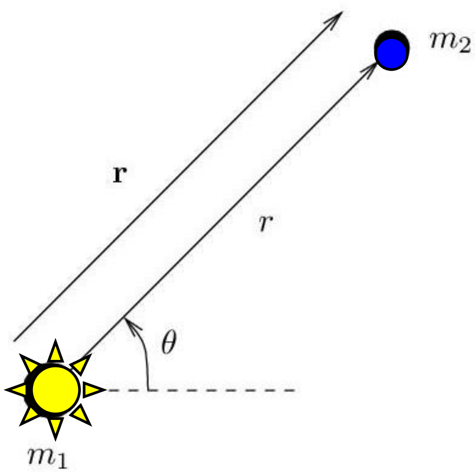


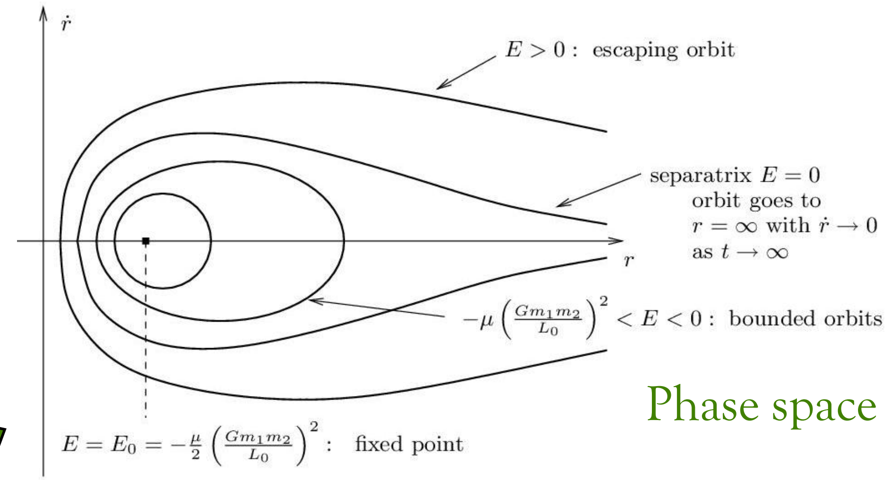




Two bodies good

Newton (and Euler) integrated the differential equations for two bodies (Sun + Earth) and found the elliptical orbits of Kepler, and they showed that the inverse square law also predicted Kepler's first and third laws. They found celestial order:



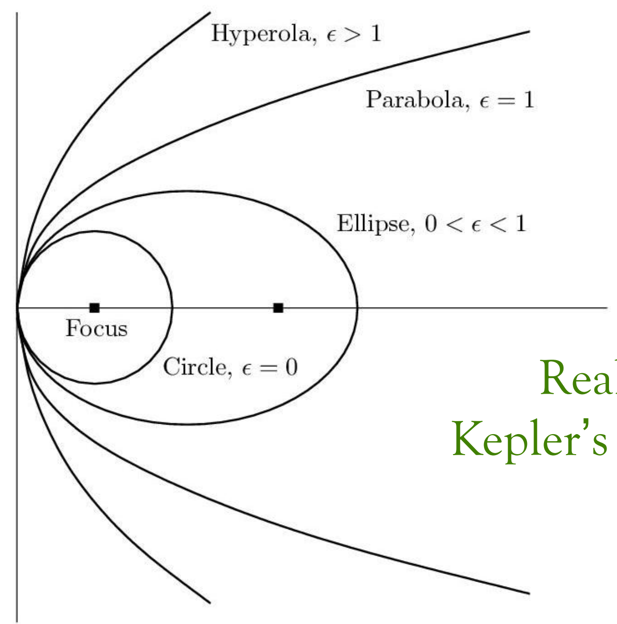


Phase space

Conservation of linear and angular momenta and energy

$$\mu r^2\dot{\theta} \overset{def}{=} L_0 = \text{const.}$$

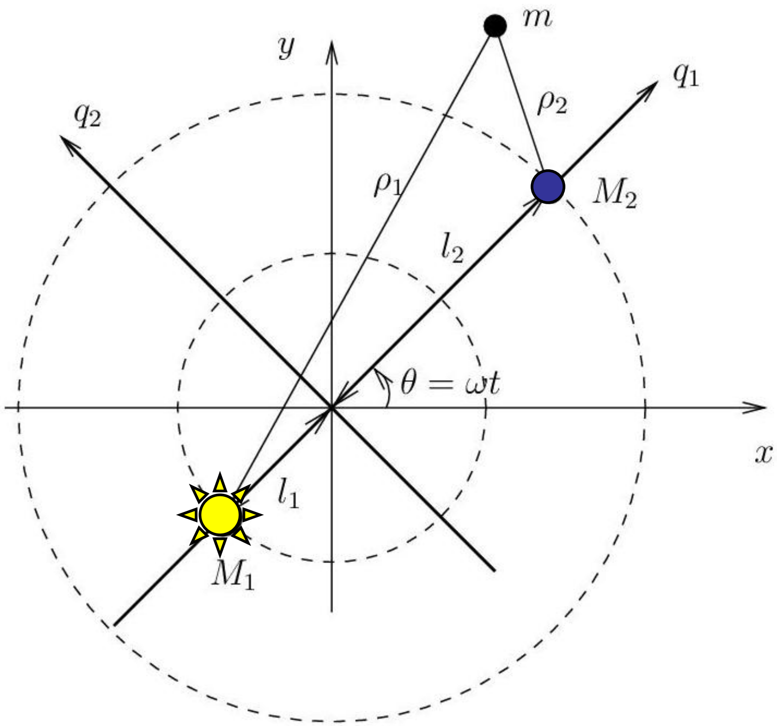
$$\underbrace{\frac{\mu}{2}\dot{r}^2}_{\text{`radial' kinetic energy}} + \underbrace{\frac{L_0^2}{2\mu r^2} + V(r)}_{\text{effective potential}} = E = \text{const.}$$

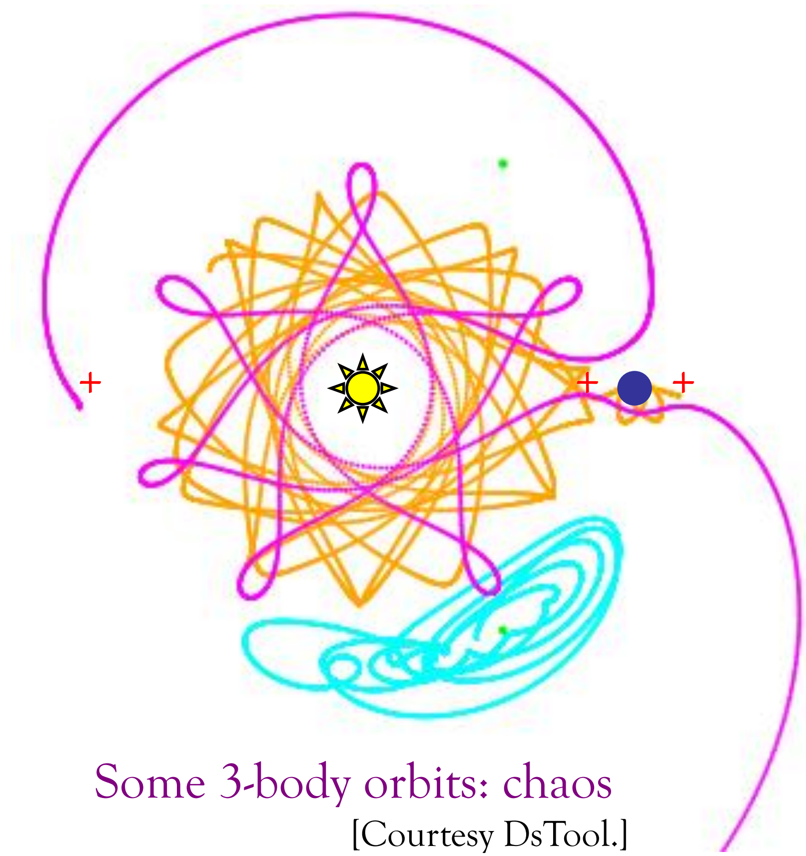


Real space:
Kepler's conic sections

Three bodies bad

Newton struggled unsuccessfully with the "problem of the moon" (Sun + Earth +Moon).
This was idealised as the restricted, planar, circular, 3-body problem:





Some 3-body orbits: chaos
[Courtesy DsTool.]

Newton was unable to solve it, and nor could Euler, Lagrange, Laplace, Poisson, …
… and nor could Poincaré, in the end,

but he did quite
a lot anyway:

Poincaré's prize paper

SUR LE

PROBLÈME DES TROIS CORPS

ET LES

ÉQUATIONS DE LA DYNAMIQUE

PAR

H. POINCARÉ
À PARIS.

MÉMOIRE COURONNÉ
DU PRIX DE S. M. LE ROI OSCAR II
LE 21 JANVIER 1889.

Acta Mathematica 13, 1-270, 1890.

(The first "textbook"
in ***Dynamical systems.***)

Poincaré's prize paper: contents

TABLE DES MATIÈRES.

270 pages!

Poincaré's prize paper: results

Introduction.

Le travail qui va suivre et qui a pour objet l'étude du problème des trois corps est un remaniement du mémoire que j'avais présenté au Concours pour le prix institué par Sa Majesté le Roi de Suède. Ce remaniement était devenu nécessaire pour plusieurs raisons. Pressé par le temps, j'avais dû énoncer quelques résultats sans démonstration; le lecteur n'aurait pu, à l'aide des indications que je donnais, reconstituer les démonstrations qu'avec beaucoup de peine. J'avais songé d'abord à publier le texte primitif en l'accompagnant de notes explicatives; mais j'avais été amené à multiplier ces notes de telle sorte que la lecture du mémoire serait devenue fastidieuse et pénible.

J'ai donc préféré fondre ces notes dans le corps de l'ouvrage, ce qui a l'avantage d'éviter quelques redites et de faire mieux ressortir l'ordre logique des idées.

Je dois beaucoup de reconnaissance à M. Phragmén qui non seulement a revu les épreuves avec beaucoup de soin, mais qui, ayant lu le mémoire avec attention et en ayant pénétré le sens avec une grande finesse, m'a signalé les points où des explications complémentaires lui semblaient nécessaires pour faciliter l'entière intelligence de ma pensée. Je lui dois la forme élégante que je donne au calcul de S_i^m et de T_i^m à la fin du § 12. C'est même lui qui, en appelant mon attention sur un point délicat, m'a permis de découvrir et de rectifier une importante erreur.

Dans quelques-unes des additions que j'ai faites au mémoire primitif, je me borne à rappeler certains résultats déjà connus; comme ces résultats sont dispersés dans un grand nombre de recueils et que j'en fais un fréquent usage, j'ai cru rendre service au lecteur en lui épargnant de fastidieuses recherches; d'ailleurs je suis souvent conduit à appliquer ces théorèmes sous une forme différente de celle que leur auteur leur avait d'abord donnée et il était indispensable de les exposer sous cette nouvelle forme. Ces théorèmes acquis, dont quelques-uns sont même classiques

sont développés, à côté de quelques propositions nouvelles, dans le chapitre 1er (1ère partie).

Je suis bien loin d'avoir résolu complètement le problème que j'ai abordé. Je me suis borné à démontrer l'existence de certaines solutions particulières remarquables que j'appelle solutions périodiques, solutions asymptotiques, et solutions doublement asymptotiques. J'ai étudié plus spécialement un cas particulier du problème des trois corps, celui où l'une des masses est nulle et où le mouvement des deux autres est circulaire; j'ai reconnu que dans ce cas les trois corps repasseront une infinité de fois aussi près que l'on veut de leur position initiale, à moins que les conditions initiales du mouvement ne soient exceptionnelles.

Comme on le voit, ces résultats ne nous apprennent que peu de chose sur le cas général du problème; mais ce qui peut leur donner quelque prix, c'est qu'ils sont établis avec rigueur, tandis que le problème des trois corps ne paraissait jusqu'ici abordable que par des méthodes d'approximation successive où l'on faisait bon marché de cette rigueur absolue qui est exigée dans les autres parties des mathématiques.

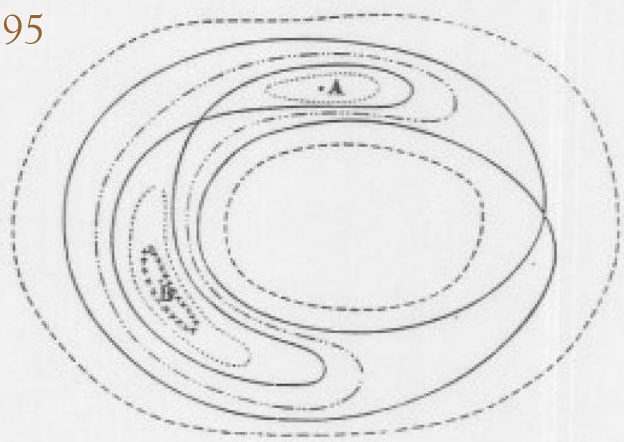
Mais j'attirerai surtout l'attention du lecteur sur les résultats négatifs qui sont développés à la fin du mémoire. J'établis par exemple que le problème des trois corps ne comporte, en dehors des intégrales connues, aucune intégrale analytique et uniforme. Bien d'autres circonstances nous font prévoir que la solution complète, si jamais on peut la découvrir, exigera des instruments analytiques absolument différents de ceux que nous possédons et infiniment plus compliqués. Plus on réfléchira sur les propositions que je démontre plus loin, mieux on comprendra que ce problème présente des difficultés inouies, que l'insuccès des efforts antérieurs avait bien fait pressentir, mais dont je crois avoir mieux encore fait ressortir la nature et la grandeur.

J'ai fait voir également que la plupart des séries employées en mécanique céleste et en particulier celles de M. Lindstedt qui sont les plus simples, ne sont pas convergentes. Je serais désolé d'avoir par là jeté quelque discrédit sur les travaux de M. Lindstedt ou sur les recherches plus profondes de M. Gyldén. Rien ne serait plus éloigné de ma pensée. Les méthodes qu'ils proposent conservent toute leur valeur pratique. On sait en effet le parti qu'on peut tirer dans un calcul numérique de l'emploi des séries divergentes et la série fameuse de Stirling en est un

Doubly asymptotic (homoclinic) orbits

Fig. 7.

p 195



La première question à traiter est la suivante: les courbes en trait plein, intersections des surfaces asymptotiques avec $y_1 = 0$, sont-elles aussi des courbes fermées? Il est clair qu'il en serait ainsi si les séries s_1 et s_2 étaient convergentes. Car les courbes en trait pointillé différeraient alors aussi peu qu'on voudrait des courbes en trait plein; la distance d'un point de la courbe pleine à la courbe pointillée tendrait vers 0 quand p croîtrait indéfiniment.

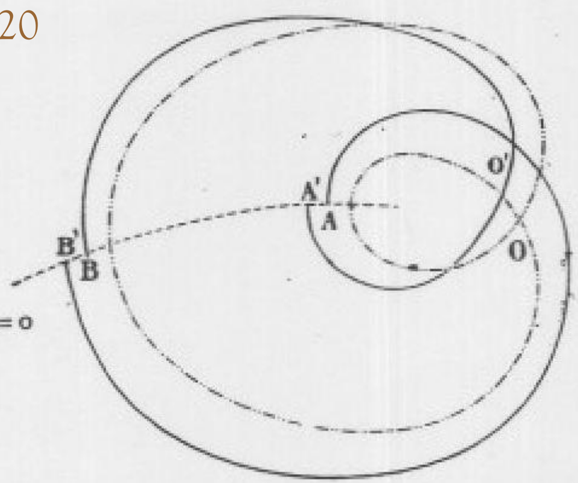
Je vais montrer sur un exemple simple qu'il n'en est pas ainsi. Soit:

$$\boxed{-F = p + q^2 - 2\mu \sin^2\frac{y}{2} - \mu\varepsilon \cos x\varphi(y),}$$

….. p 223 ….

Fig. 9.

p 220



il viendra:

$$J = 4\alpha\sqrt{2\mu}\int_0^\infty \frac{t^{-\alpha}\,dt}{1+t^2}.$$

Faisons $t^2 = u$, on aura:

$$J = 2\alpha\sqrt{2\mu}\int_0^\infty \frac{u^{-\frac{\alpha+1}{2}}\,du}{1+u} = \frac{2\pi\alpha\sqrt{2\mu}}{\cos\frac{\alpha\pi}{2}} = \frac{-8\pi i}{e^{\frac{\pi}{\sqrt{2\mu}}} + e^{-\frac{\pi}{\sqrt{2\mu}}}}.$$

Donc J n'est pas nul; donc les courbes $BO'B'$ et $AO'A'$ ne sont pas fermées; donc les séries s_1 et s_2 ne sont pas convergentes, non plus que

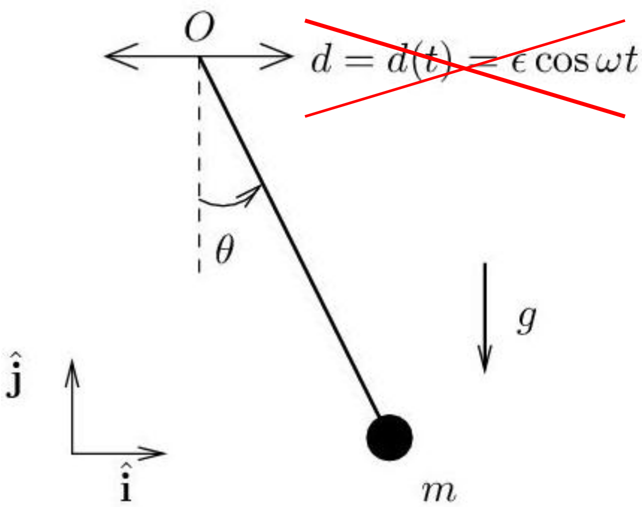
The Melnikov integral, up to a constant!

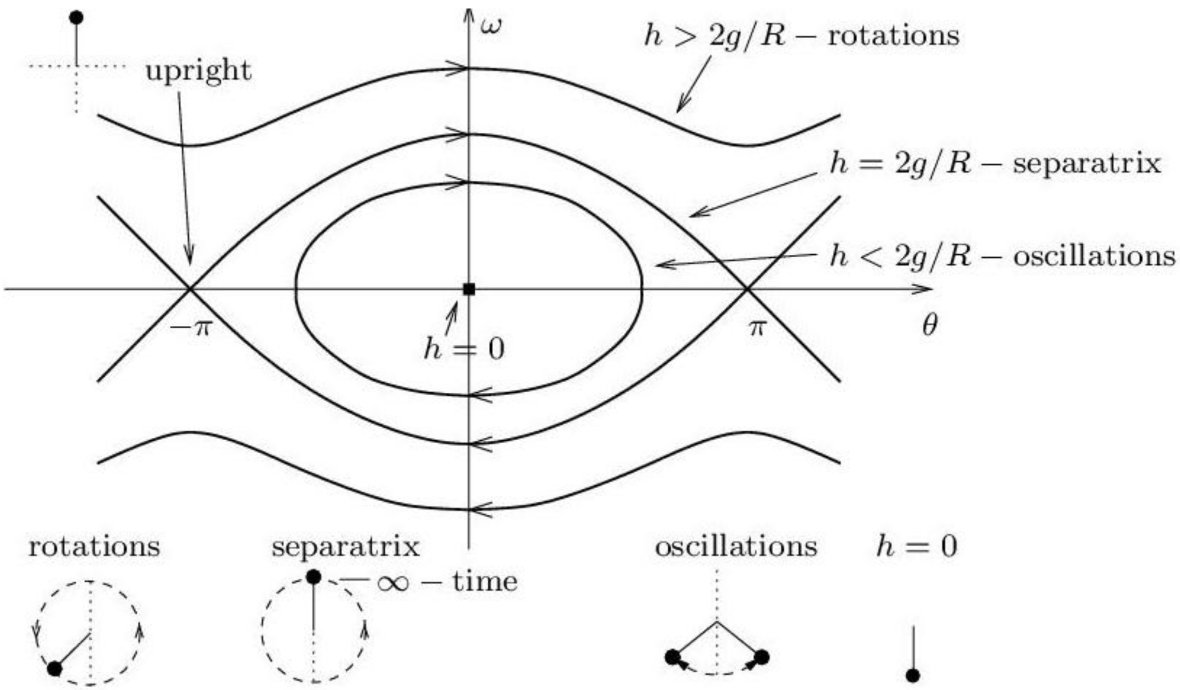
[V.K. Melnikov, Trans. Moscow. Mat. Soc. 12. 1-57, 1963.]

A simple pendulum is simpler

… and tells most of our story:

$$\dot{\theta} = \omega$$
$$\dot{\omega} = -\frac{g}{l}\sin\theta$$





With a motionless support, as in the 2-body problem, conservation of energy

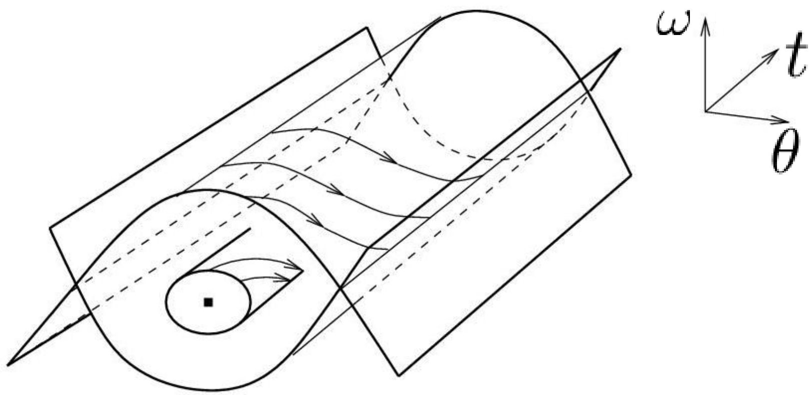
$$E = \frac{ml^2\omega^2}{2} + mgl(1 - \cos\theta) = \text{const.}$$

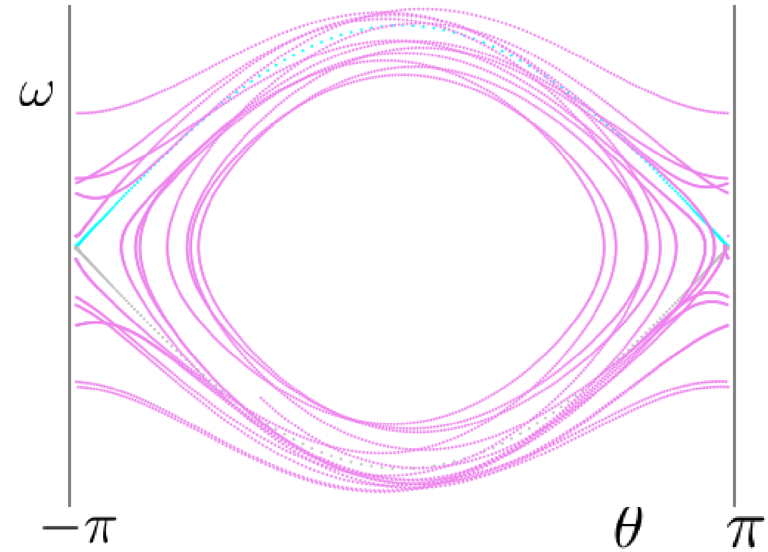
allows us to plot ordered sets of periodic orbits and a separatrix (a doubly-asymptotic orbit).

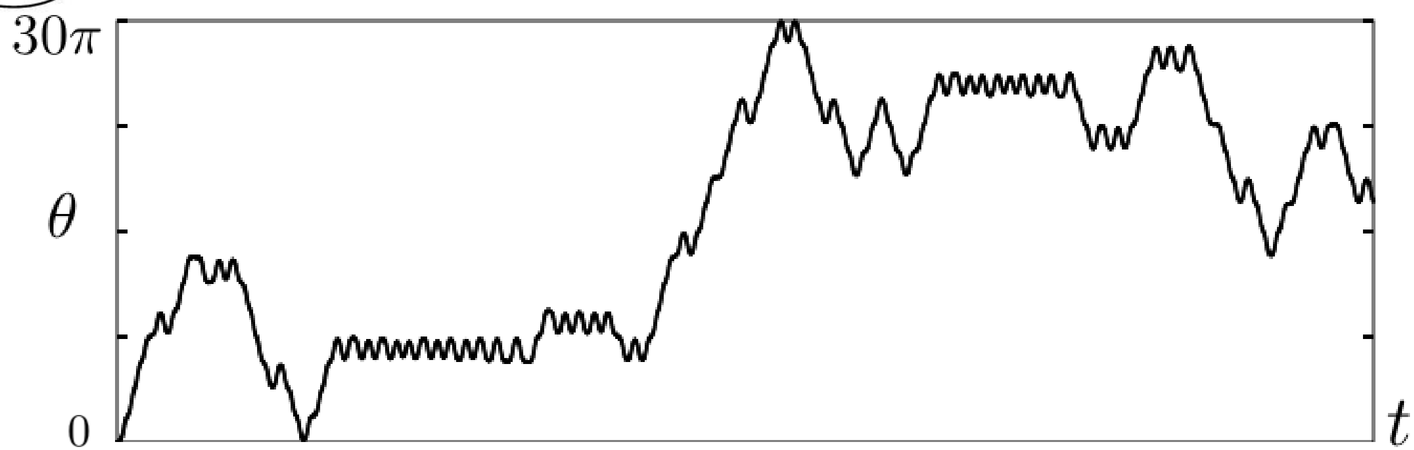
... but add a small external oscillation

or couple to a second oscillator (two degrees of freedom):

$$\dot{\theta} = \omega$$
$$\dot{\omega} = -\frac{g}{l}\sin\theta + \boxed{\frac{\epsilon}{l}\cos\theta\cos t}$$

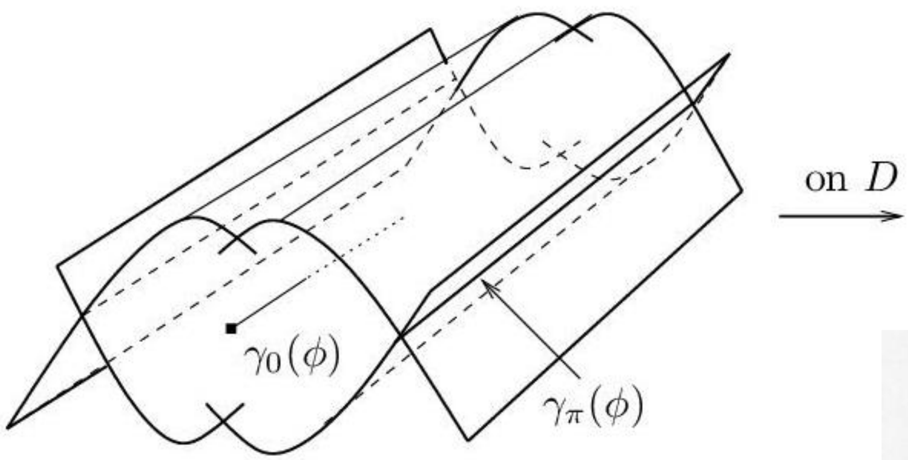


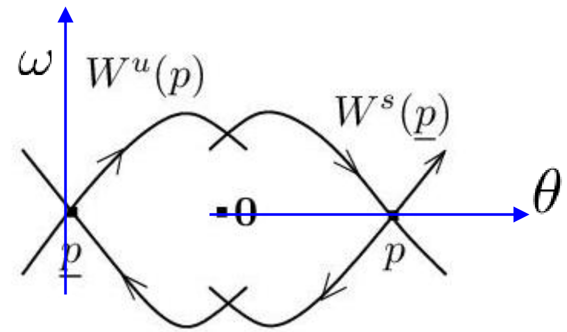


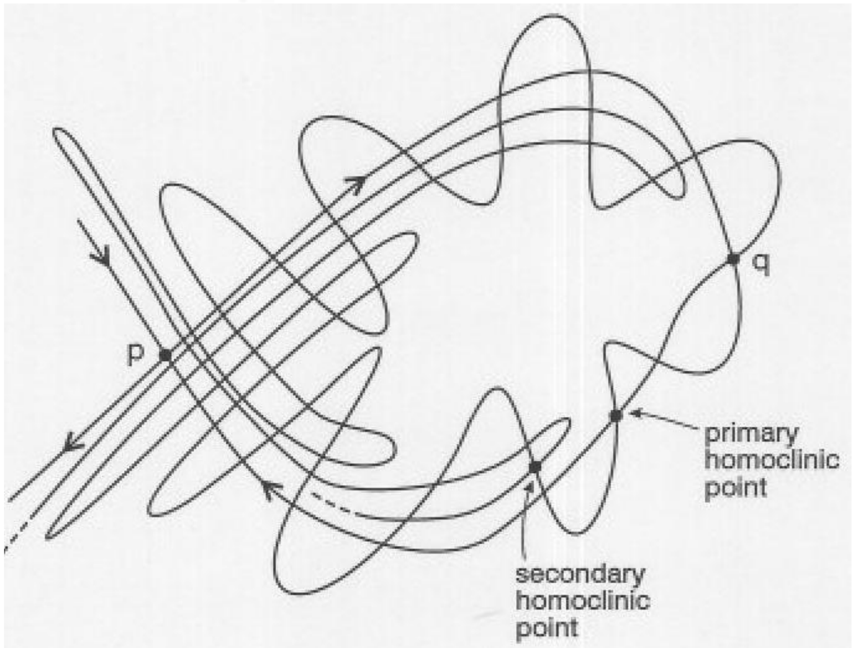


... and the separatrix splits so that orbits can wander between librations and rotations, giving sensitive dependence and chaos!

In three dimensional (θ, ω, t)–phase space orbits can tie themselves in knots! This is best seen in the (θ, ω) cross-section via a **Poincaré map**:







The stable and unstable manifolds intersect in doubly asymptotic or homoclinic points. An infinite, but discrete set survives the perturbations. Shortly, we'll see what their presence implies for nearby orbits.

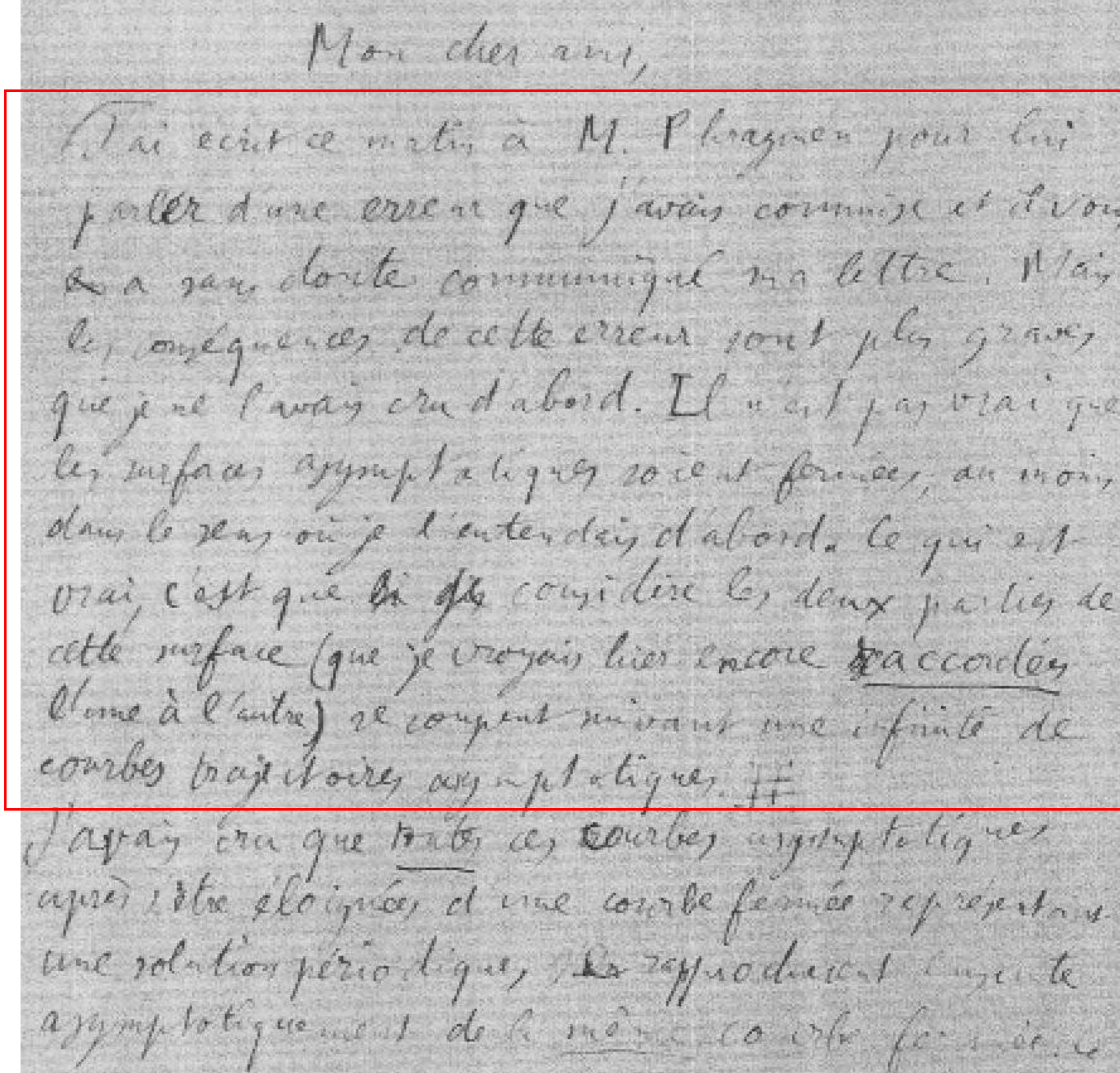
But now we return to ….

Poincaré's mistake: Phragmen's questions,

and Poincaré's response to Mittag-Leffler:

L.E. Phragmen: Acta's copy editor, proof reader.

H. Gylden: Mittag-Leffler's nemesis.



Mon cher ami,

J'ai écrit ce matin à M. Phragmen pour lui parler d'une erreur que j'avais commise et il vous a sans doute communiqué ma lettre. Mais les conséquences de cette erreur sont plus graves que je ne l'avais cru d'abord. Il n'est pas vrai que les surfaces asymptotiques soient fermées, au moins dans le sens où je l'entendais d'abord. Ce qui est vrai, c'est que si je considère les deux parties de cette surface (que je croyais hier encore raccordées l'une à l'autre) se coupent suivant une infinité de courbes trajectoires asymptotiques.

J'avais cru que toutes ces courbes asymptotiques après s'être éloignées d'une courbe fermée représentant une solution périodique, se rapprocheraient ensuite asymptotiquement de la même courbe fermée.

Paris, Dec 1st 1889:

qui est vrai c'est qu'il y en a une infinité qui jouissent de cette propriété.

Je ne vous dissimulerai pas le chagrin que me cause cette découverte. Je ne sais d'abord si vous pensez encore que les résultats qui subsistent, à savoir l'existence des solutions périodiques, celle des solutions asymptotiques, la théorie des exposants caractéristiques, la non-existence des intégrales uniformes et la divergence des séries de M. Lindstedt méritent encore la haute récompense que vous avez bien voulu leur accorder.

D'autre part, de grands remaniements vont devenir nécessaires et je ne sais si on n'a pas commencé à tirer le mémoire; j'ai télégraphié à M. Phragmén.

En tout cas, je ne puis mieux faire que de confier mes perplexités à un ami aussi dévoué que vous l'avez toujours été.

Je vous en écrirai plus long quand j'y verrai un peu plus clair dans mes affaires.

Veuillez agréer, mon cher ami, avec mes bien sincères excuses, l'assurance de mon entier dévouement,

Poincaré

The final version was Submitted in Jan 1890.
Only two months for corrections and new material!

Poincaré's mistake: the original paper

SUR LE

PROBLÈME DES TROIS CORPS

ET LES

ÉQUATIONS DE LA DYNAMIQUE

PAR

H. POINCARÉ

À PARIS.

MÉMOIRE COURONNÉ

DU PRIX DE S. M. LE ROI OSCAR II

LE 21 JANVIER 1889.

AVEC DES NOTES

PAR L'AUTEUR.

Introduction.

Sur le problème des trois corps et les équations de la dynamique.

Le présent mémoire a été entrepris pour répondre à la première des quatre questions du concours; mais les résultats que j'ai obtenus sont tellement incomplets que j'aurais hésité à les publier si je ne savais que l'importance et la difficulté du problème donne de l'intérêt à tout ce qui s'y rapporte et qu'on ne peut attendre une solution définitive que d'une longue série d'efforts successifs.

Les immortels fondateurs de la mécanique céleste ont cherché à résoudre le problème des n corps par approximations successives. A cet effet, ils ont développé la solution suivant les puissances croissantes des masses et exprimé chaque terme du développement par une série de sinus et de cosinus. Leur succès montre suffisamment que cette méthode était la plus convenable pour les premières approximations.

Parmi les résultats qu'ils ont obtenus, un des plus remarqués est celui qui se rapporte à la stabilité du système solaire. LAPLACE et POISSON sont parvenus à démontrer qu'en tenant compte seulement des premières et des secondes puissances des masses, les grands axes des orbites ne subissent que des variations périodiques. On a cru longtemps que le fait était général et on en a même cherché une démonstration directe; c'était une erreur. Dès qu'on tient compte des troisièmes puissances des masses, on voit apparaître des termes séculaires dans le développement des grands axes.

Ainsi la méthode dont nous venons de parler devient insuffisante quand on veut pousser l'approximation un peu loin. Les séries auxquelles elle conduit contiennent non seulement des termes purement trigonométriques de la forme:

$$A \sin \alpha t \quad \text{ou} \quad B \cos \alpha t,$$

non seulement des termes mixtes de la forme:

$$At^m \sin \alpha t \quad \text{ou} \quad Bt^m \cos \alpha t,$$

Poincaré's mistake: how the paper changed

Deuxième partie.

Equations de la dynamique et problème des n corps.

Notes.

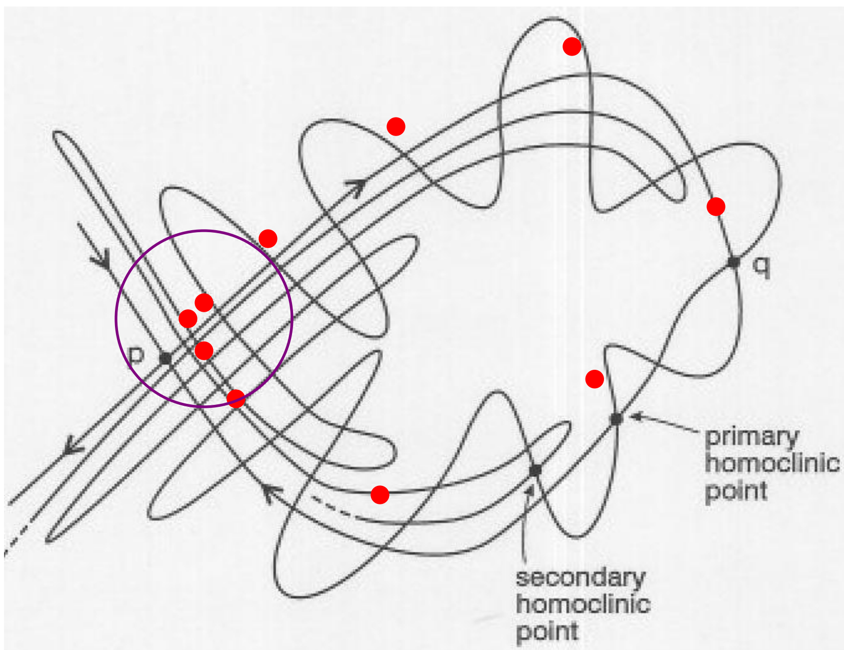
The "notes" prompted by Phragmen's questions were all incorporated into the text and an entirely new part appeared.

In two months Poincaré laid the foundations of "chaos theory."

[With thanks to June Barrow-Green, *Poincaré and the Three Body Problem*, AMS/LMS, 1997. Also see F. Diacu and PH *Celestial Encounters*, PUP, 1996.]

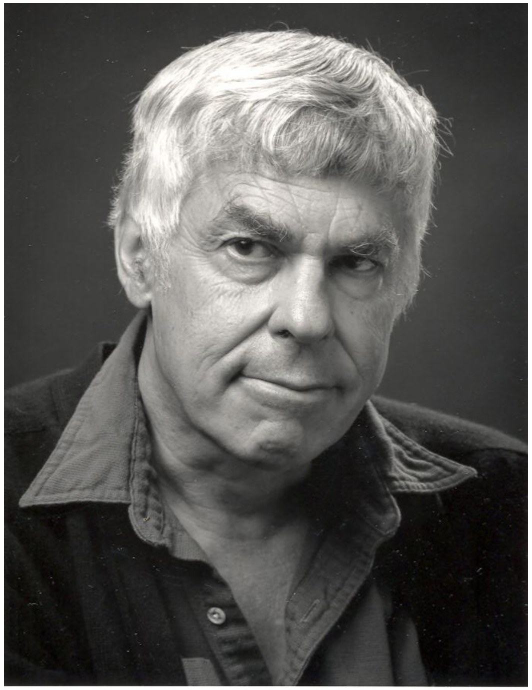
Global Behavior: Towards the horseshoe

George Birkhoff proved that near a homoclinic point there is an infinite set of periodic points, including points with arbitrarily long periods (they "mark time" near the saddle point). [*Dynamical Systems*, AMS, 1927].
In 1913 Birkhoff had proved Poincaré's "last geometric theorem."



And that's far from all that's near a homoclinic point! It implies …

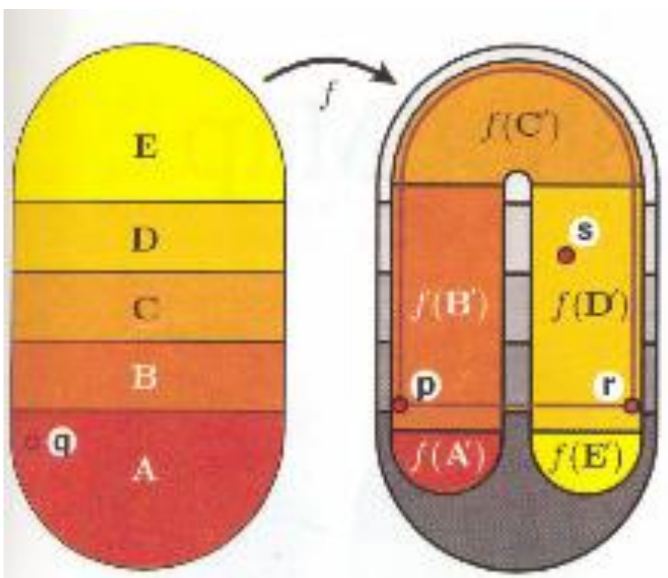
Smale's Horseshoe

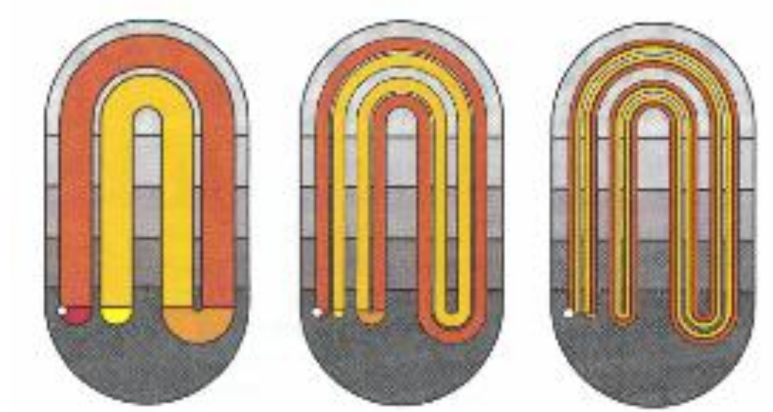


At IAS in 1959, having turned the sphere inside out and solved the Poincaré conjecture in n > 4 dimensions, Stephen Smale started thinking about dynamical systems. Norman Levinson had told him about early work on forced relaxation oscillations that suggested that his conjecture about structurally stable systems having only finitely many periodic orbits might be incorrect. ▪ At IMPA in Rio, Smale made pictures of possible Poincaré maps and realised that he could define a structurally stable map with infinitely many periodic orbits and much more: a chaotic invariant set. Two years later Lee Neuwirth (Bebe's dad) helped Smale define the form of the map that we now know:

Iterate!

First and

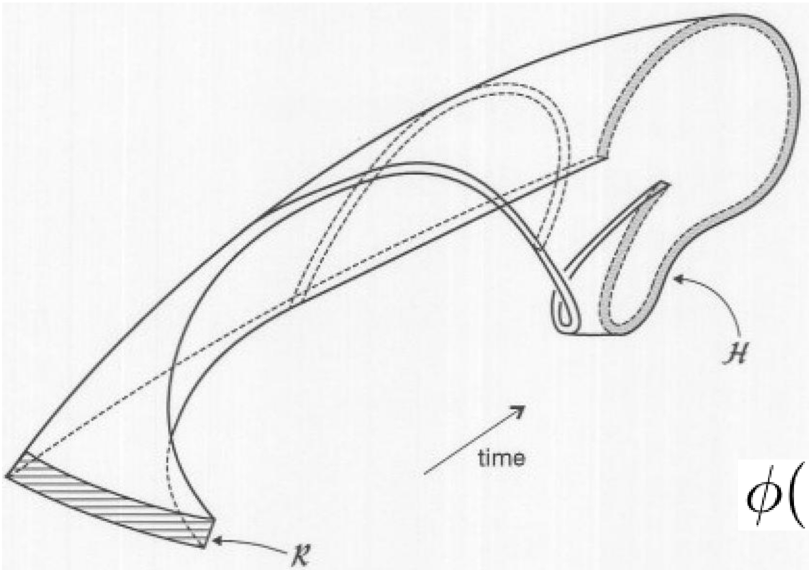


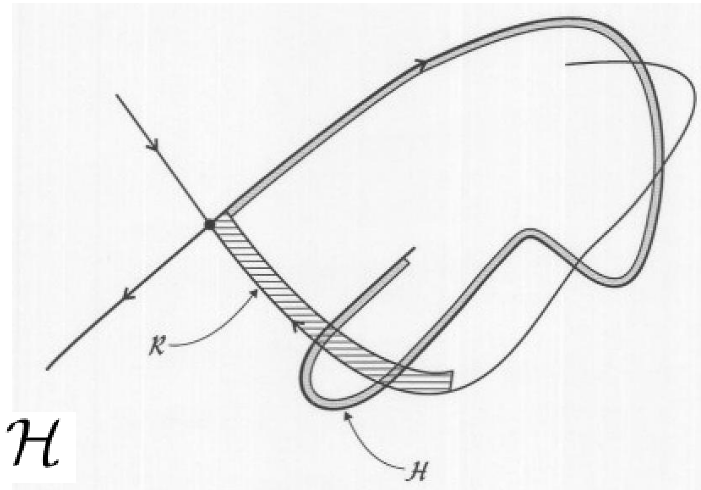


… second, third, and fourth go round ….

(M. Shub, AMS Notices)

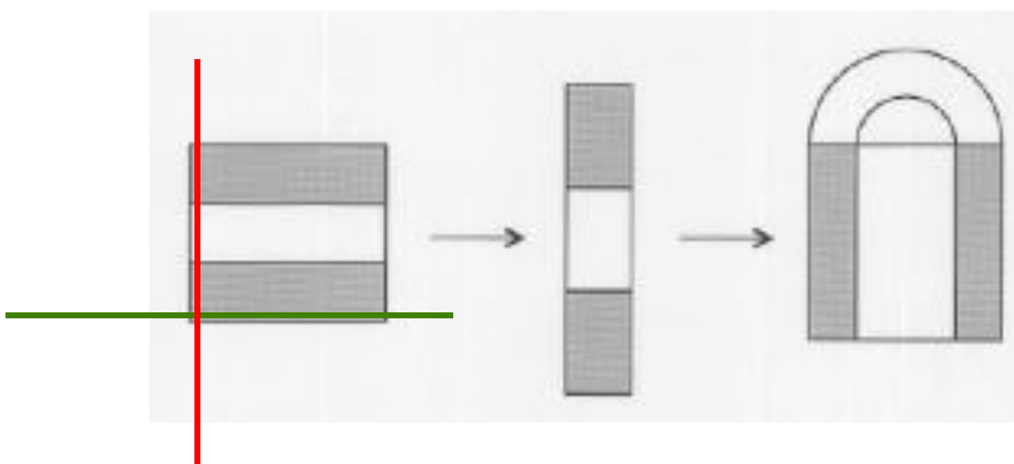
How the flow makes the map:





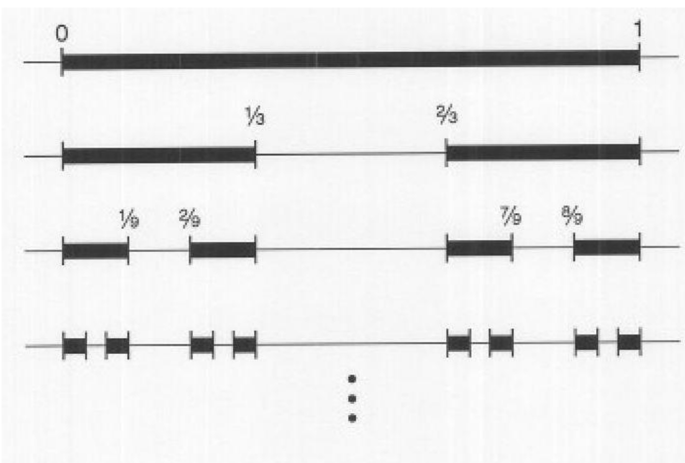
$$\phi(T) : \mathcal{R} \to \mathcal{H}$$

... idealise and make it piecewise linear:

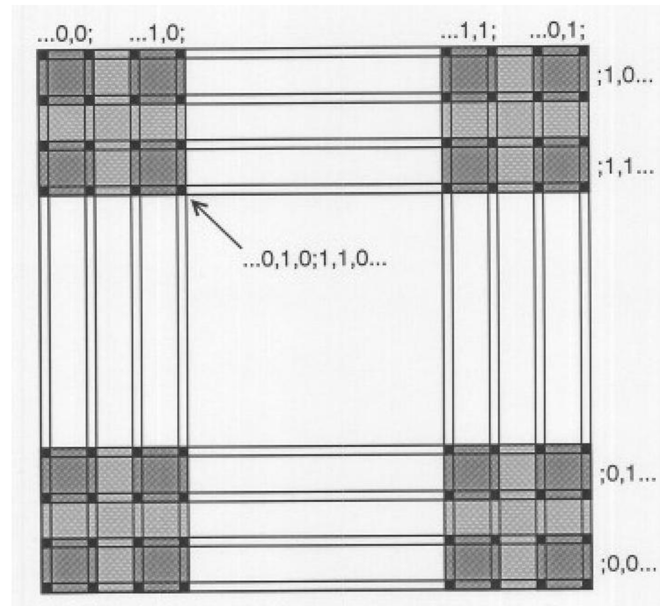


Cantor sets

The set X of points that never leaves the central square is a Cantor set: uncountable, perfect, containing no open sets, every point an accumulation point. Georg Cantor had invented these beasts to give analysts nightmares. Smale coded the infinite set with the two “letters” 0 and 1:



Middle third Cantor set



The horseshoe

This translates the nasty geometry of X into symbolic dynamics: words in a two letter alphabet: out of chaos came order.

It wasn’t the first such idealised model ...

The value of abstraction: Cat map or bat map?

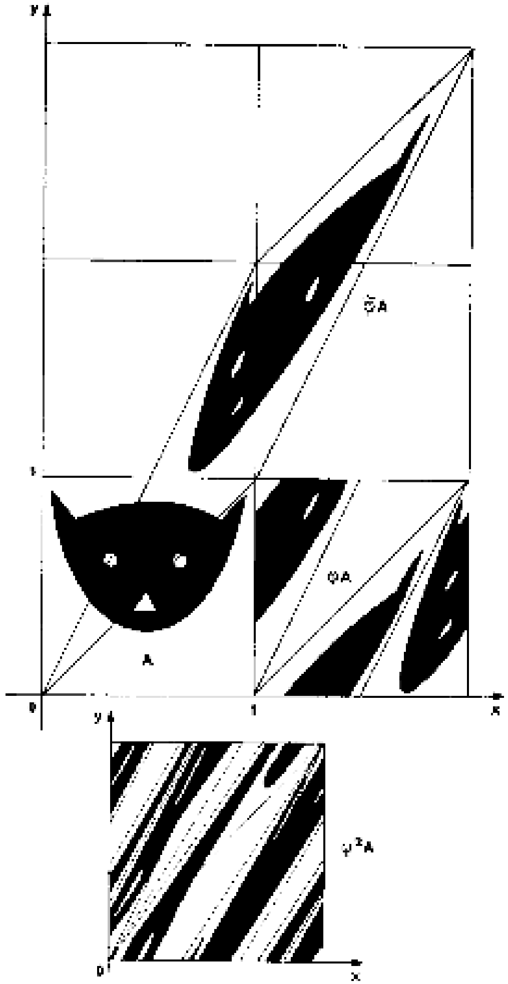
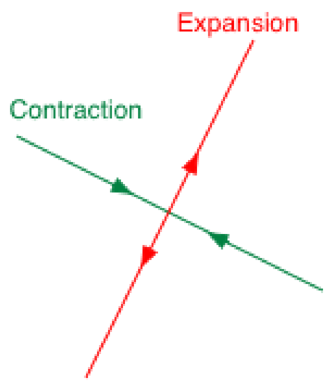


Figure 1.17



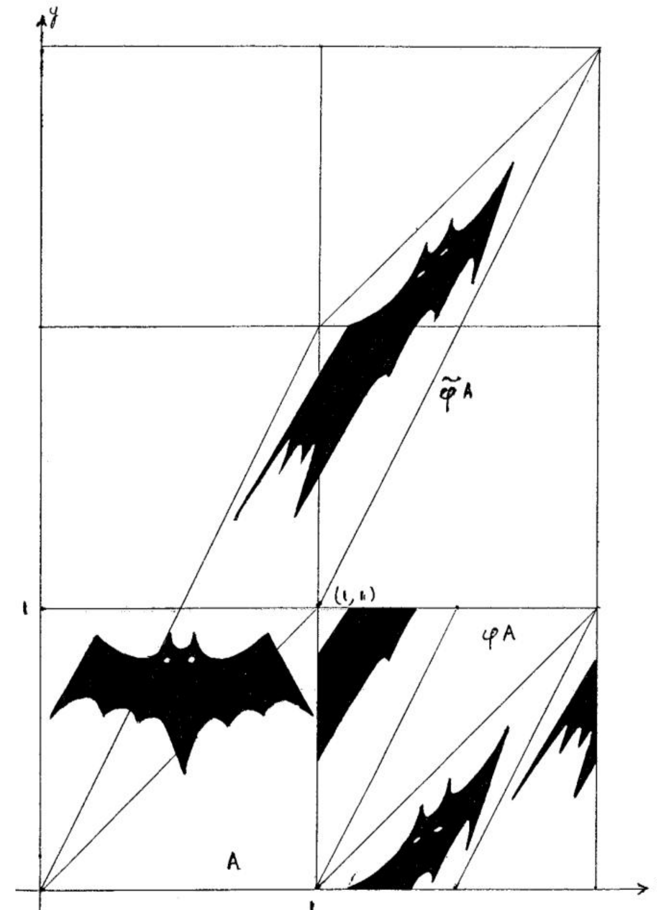


Figure 2.28

… or hyperbolic toral automorphism.

Notes by A. Avez on *Ergodic Theory of Dynamical Systems*, University of Minnesota, School of Mathematics, (1966).

Thanks to David Chillingworth.

From V.I. Arnold, A. Avez, *Ergodic Problems of Classical Mechanics* (1968)

Thanks to Clancy Rowley.

Levinson pointed the way to the horseshoe ...

Vol. 50, No. 1, January, 1949

Annals of Math 50, 1949.

A SECOND ORDER DIFFERENTIAL EQUATION WITH SINGULAR SOLUTIONS

By Norman Levinson

(Received June 22, 1948)

1. Cartwright and Littlewood [5] have announced some remarkable results for the differential equation

(1.0) $$\ddot{x} + k(x^2 - 1)\dot{x} + x = b\lambda k \cos \lambda t$$

where k is a large constant and b and λ are constants. While only a sketch of the method of Cartwright and Littlewood has appeared, the authors state that their proof is difficult.

Here we shall consider an equation which exhibits the same singular behavior but for which the proof is considerably simplified by an artifice (described in §2). A summary of our results has already appeared [9]. We consider

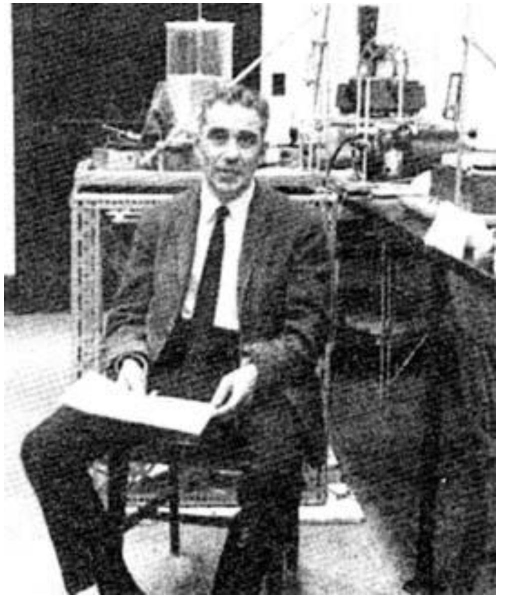
(1.1) $$\ddot{y} + p(y)\dot{y} + y = c \sin t$$

where $p(y)$ is a certain polynomial and c is a constant restricted to belong to a certain set of intervals.

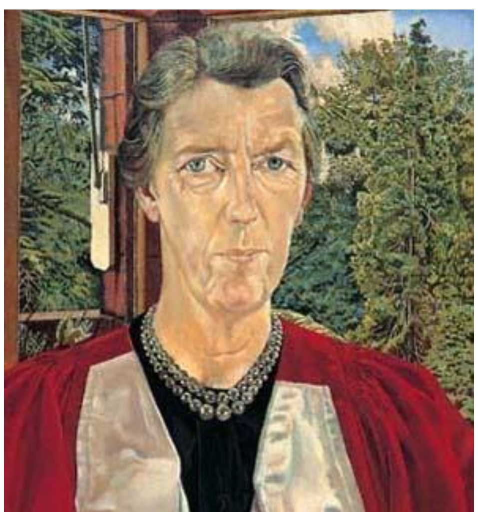
Among the solutions of (1.1) there is a family F of remarkably singular structure. Solutions, $y(t)$, of F have a maximum value of approximately 3. If the maximum occurs at $t = t_1$ then for $t > t_1$ and so long as $y > 1$, y is approximately of the form

(1.2) $$(3 - b)e^{-\rho(t-t_1)} - b \cos t$$

where b is a constant, $0 < b < 1$, and the constant $\rho > 0$ is small. Thus for $t > t_1$, y aside from the cosine term decreases slowly. When y reaches the value 1 then it falls, within an interval of t at most 2π in length, to its minimum value of approximately -3. It then repeats its behavior, with opposite sign, slowly rising to $y = -1$ and from there rapidly reaching a maximum close to 3 again. This general pattern is repeated over and over again. (The reader will probably find it helpful to make a sketch of the solution described in this and the following paragraphs.)



… via Cartwright and Littlewood's work



ON NON-LINEAR DIFFERENTIAL EQUATIONS OF THE SECOND ORDER:

I. THE EQUATION $\ddot{y}-k(1-y^2)\dot{y}+y=b\lambda k\cos(\lambda t+a)$, k LARGE

M. L. Cartwright *and* J. E. Littlewood†.

1. In the present short preliminary survey we confine ourselves, to fix ideas, to equations of the form

$$\ddot{y}+f(y)\dot{y}+g(y)=p(t),$$

where f, g are real and analytic for real y, p is real and analytic and has period ϖ in (real) t, and $\underline{\lim} f>0$ as $y\to\pm\infty$. A specially important case is that of $g(y)\equiv y$.

There is some general theory of such equations. A trajectory (or "motion") with initial conditions $y(t_0)=\xi$, $\dot{y}(t_0)=\eta$ (ξ, η real) at some fixed $t=t_0$ is said to have the point $P=(\xi,\eta)$ as "representative point". If ξ', η' are the values of y, $\dot{y}$ at $t=t_0+\varpi$, the transformation T from P to $P'=(\xi',\eta')=TP=T(\xi,\eta)$ is (1, 1) and continuous (in fact analytic).

With the condition $\underline{\lim} f>0$ and suitable conditions on g (fulfilled for $g=y$), every trajectory is bounded as $t\to\infty$, and T transforms a large

† Received 12 December, 1945; read 13 December, 1945.

J. London Math Soc 20, 180-189, 1945.
(from WW II work on radar)

that can be normalized as

$$\text{(E)} \qquad \ddot{y} - k(1-y^2)\dot{y} + y = b\lambda k \cos(\lambda t + a)\dagger.$$

k must not be small, or we are in case (ii); the next possibility of simplification is to suppose it large, which gives us the equation of the title.

3. If $b > \frac{2}{3}$ and $k > k_0(b, \lambda)$‡, (E) shows the simplest possible behaviour: there is a stable p.m. of order 1, period $\varpi = 2\pi/\lambda$, to which every trajectory converges.

If, however, $b < \frac{2}{3}$, and k is large enough, (E) shows a rich variety of behaviour, some of it very bizarre§.

We have to exclude certain intervals of b; in order that these should be a small proportion of the whole interval $(0, \frac{2}{3})$, we need to introduce an arbitrarily small positive δ. There then exist $\epsilon_\delta = \epsilon(\lambda, \delta)$, small with δ, and $k_0 = k_0(\lambda, \delta)$, with the following properties. If $k \geqslant k_0$, there is a set of excluded intervals in $(0, \frac{2}{3})$, including among them $(0, \delta)$ and $(\frac{2}{3}-\delta, \frac{2}{3})$, of total length ϵ_δ at most. The remainder of $(0, \frac{2}{3})$ is also a set of intervals, $\mathfrak{B}$, say; this varies with k, but has length at least $\frac{2}{3}-\epsilon_\delta$. $\mathfrak{B}$ divides into two parts (roughly equal), $\mathfrak{B}_1$ and $\mathfrak{B}_2$.

When b belongs to an interval I_1 of $\mathfrak{B}_1$, (E) has a set of stable subharmonics of order $2n+1$‖, and most¶ trajectories converge each to some one of these. n is constant in I_1 and is of order $(\frac{2}{3}-b)k$.

When b belongs to an interval I_2 of $\mathfrak{B}_2$, (E) has a set of stable subharmonics of order $2n+1$, and another of order $2n-1$; most trajectories converge each to some member of one of the two sets. It possesses a further set Σ, infinite in number, of p.m. of a great variety of "structures" (described in more detail later). It possesses further a set X, of the power

† See B. van der Pol, *Proc. Institute of Radio Engineers*, 22 (1934), 1051–1086, § IX, 1080–1082. Some graphical solutions are given by D. L. Herr, *Proc. Institute of Radio Engineers*, 27 (1939), 396–402.

‡ This is the simplest instance of a point that should be emphasized. We never assert that behaviour is more and more nearly such and such as k increases, always that it is *exactly* such and such so soon as k exceeds a certain k_0 [here $k_0(b, \lambda)$]. In fact k is not "large", but only "large enough".

§ Our faith in our results was at one time sustained only by the experimental evidence that stable sub-harmonics of two distinct orders did occur. [See B. van der Pol and J. van der Mark, *Nature*, 120 (1927), 363.] It is this that leads to the startling consequences; the consequences themselves relate to non-stable motions (which the experiments naturally did not reveal).

‖ Any p.m. of order m shifted a period ϖ is another one of the same order, and so gives rise to a "set", m in number.

¶ The general sense of "most" is fairly obvious: to define it precisely would occupy too much space.

of the continuum, of non-periodic limiting trajectories, of the type described† as "discontinuous recurrent". If we denote the sets of representative points P in the (ξ, η) plane also by Σ and X, then every point of Σ is a limit point of points of Σ and also a limit point of points of X. A point of Σ is thus non-stable, and is clearly a highly singular, or multiple‡ f.p. The number n (which is of the order of k) is constant in I_2. Moreover the set K and its subsets Σ, X remain topologically equivalent throughout I_2. Thus a point of Σ remains "infinitely multiple" for all b of the interval I_2, contrary to the natural expectation that multiplicity would be confined to isolated values of b.

For b of an I_1 (of $\mathfrak{B}_1$) there is a set of non-stable subharmonics of order $2n+1$.

To complete the account of f.p. we observe finally that for all b of $(\delta, \frac{2}{3}-\delta)$ there is a single f.p. of order 1, and it is totally unstable. We shall call its representative point P_u, and denote by K_0 the set K less the point P_u.

As b increases (from δ to $\frac{2}{3}-\delta$), jumping the excluded intervals, the number n decreases (down to $O(\delta k)$). We have nothing to say about the *transitions* from one stable period $(2n+1)\varpi$ to two stable periods $(2n\pm 1)\varpi$ and *vice versa*; these take place in the excluded intervals§.

4. It follows from the famous "last geometrical theorem" of Poincaré‖ *inter alia*, that if a transformation T, which is (1, 1), continuous and area-preserving in the annulus between two curves, has f.p. of order n_1 on one curve and f.p. of a different order n_2 on the other, such that the points go round the curves once in n_1 and n_2 transformations respectively, then it has f.p. in the annulus of every order N such that m/N lies between $1/n_1$ and $1/n_2$ for some integer m; if $n_1 = 2n+1$ and $n_2 = 2n-1$, it has f.p. of orders $2n$, $4n\pm 1$, $6n\pm 1$, $8n\pm 1$, $8n\pm 3$, ... It seems generally taken for granted that an *annulus* is essential to such behaviour. But in our case of stable periods $(2n\pm 1)\varpi$ there is no annulus: K_0 is a connected set of zero area separating P_u from ∞, and the stable f.p. of K_0 are limit points of $T^m P$ (as $m \to \infty$) both for P near P_u and for P near ∞. There would seem, moreover, to be a much richer "fine structure" of

† G. D. Birkhoff, *Acta Math.*, 43 (1922), 1–119.

‡ *Cf.* L. Bieberbach, *Differentialgleichungen* (Berlin, 1930), 72–74, and J. Hadamard, *Bull. de la Soc. Math. de France*, 26 (1901).

§ We may, however, mention that as b increases through an interval I_2 the shorter period $(2n-1)\varpi$ extends its sphere of influence at the expense of the longer (and there is consistent behaviour in an I_1).

‖ See G. D. Birkhoff, *Dynamical systems* (New York, 1927), 165.

… which was sustained by Van der Pol (1889-1959)

E.M.F., $E_0 \sin \omega t$, is applied to it, currents and potential differences occur in the system the frequencies of which are whole submultiples of the frequency of the applied E.M.F., *e.g.* $\omega/2$, $\omega/3$, $\omega/4$ up to $\omega/40$.

To this end one can make use of the remarkable synchronising properties of relaxation-oscillations,

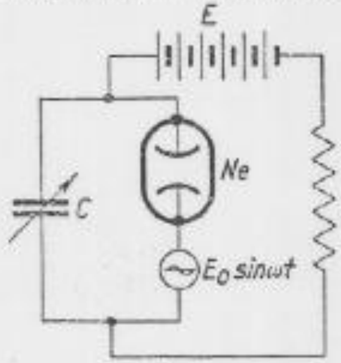


Fig. 1.

i.e. oscillations the time period of which is determined by the approximate expression $T = \tau/2\,CR$, a relaxation time (Balth. van der Pol, "On Relaxation Oscillations," *Phil. Mag.*, p. 978, 1926; also *Zeitschr. f. hochfreq. Technik*, 29, 114; 1927).

Let *Ne* in Fig. 1 represent a neon glow lamp, [illegible]

up to 1000/40 sec.$^{-1}$. In some recent experiments it was found possible to obtain a frequency demultiplication up to the ratio 1 : 1/200. Often an irregular noise is heard in the telephone receivers before the frequency jumps to the next lower value. However, this is a subsidiary phenomenon, the main effect being the regular frequency demultiplication. It may be noted that while the production of harmonics, as with frequency multiplication, furnishes us with tones determining the musical major scale, the phenomenon of frequency-division renders the musical minor scale audible. In fact, with a properly chosen 'fundamental' ω, the turning of the condenser in the region of the third to the sixth subharmonic strongly reminds one of the tunes of a bagpipe.

In conclusion, we give in Fig. 2 the measured time periods (which are thus found to be a series of discrete subharmonics) as a function of the setting of the condenser C. The dotted line in the figure gives the frequency with which the system oscillates in the absence of the applied alternating E.M.F. The shaded parts correspond to those settings of the condenser where an irregular noise is heard. In the actual experiment the resistance R was, for ease of adjustment, replaced by a diode. The experiment, however, succeeds just as well with an ohmic resistance R. Obviously the same experiment succeeds with all systems capable of producing relaxation-oscillations such as described in the papers quoted.

Balth. van der Pol.
J. van der Mark.

Natuurkundig Laboratorium der
N. V. Philips' Gloeilampenfabrieken.

The first devil's staircase?

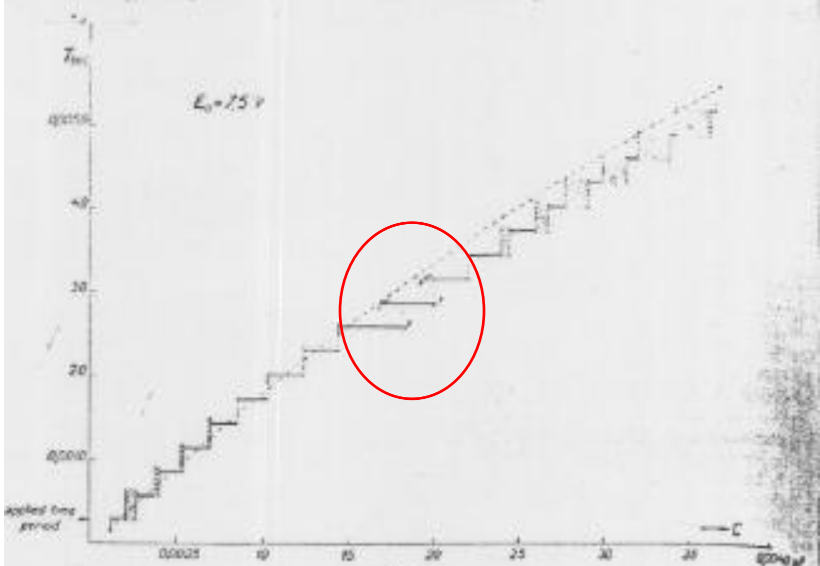
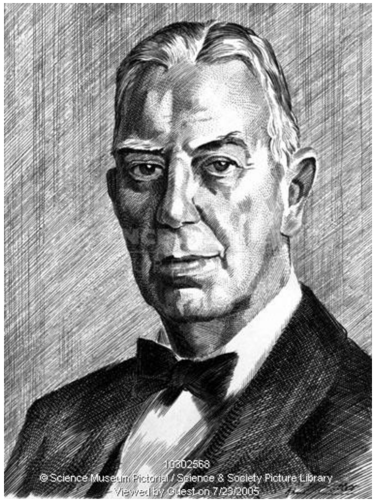


Fig. 2.

Van der Pol & Van den Mark
Nature 120, 363-364, 1927.

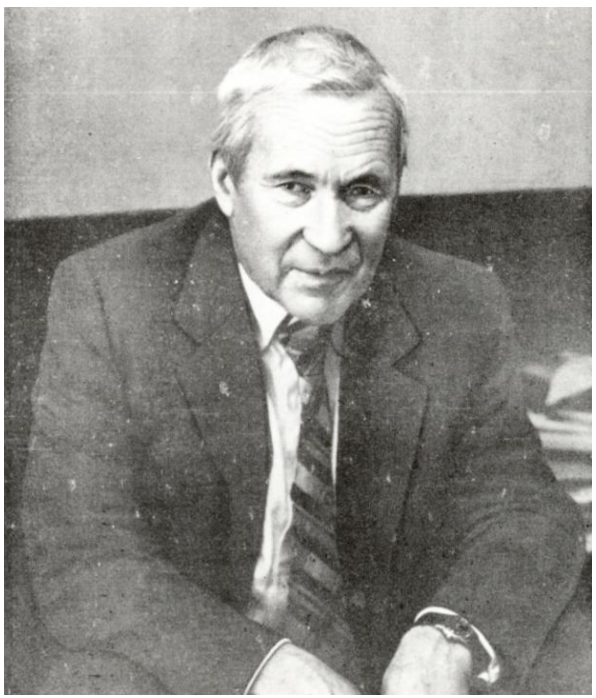
Balth. van der Pol.
J. van der Mark.
Natuurkundig Laboratorium der
N. V. Philips' Gloeilampenfabrieken.
Eindhoven, Aug. 5.



Meanwhile, in Moscow, Kolmogorov's

seminar was busy with celestial mechanics. In 1954 at the Mathematical Congress in Amsterdam he announced the K theorem. Moser, a recent graduate who had worked with C.L. Seigel, was asked to write a commentary for *Mathematical Reviews*. He began asking questions about details that he couldn't understand. Eventually he traveled to Moscow. Arnold, then a student of Kolmogorov, translated his lecture. … and so

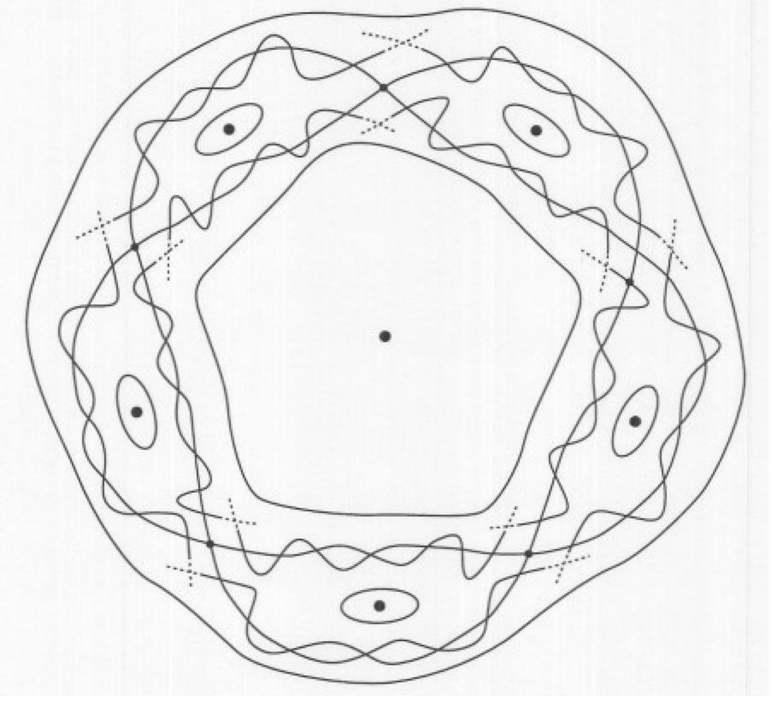
K + A + M = KAM.



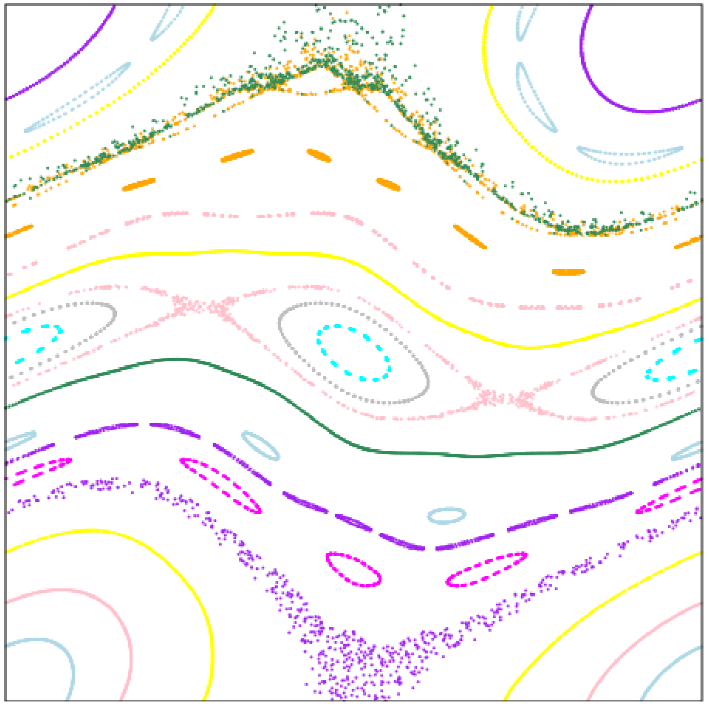
KAM theory is an ongoing story, but roughly speaking it ties together …

Order and Chaos

Integrable Hamiltonian systems like the simple pendulum, have families of invariant circles or tori. Under perturbations, a “thick” Cantor set of these survive, separated by gaps inhabited by homoclinic tangles and smaller tori and so on ad infinitum ….



$\sim$



Is this what Poincaré had glimpsed in December 1889?
In any case, now it’s everywhere, in heaven and on earth:

Celestial homoclinic chaos (touring the solar system)

CHAOS VOLUME 10, NUMBER 2 JUNE 2000

Heteroclinic connections between periodic orbits and resonance transitions in celestial mechanics

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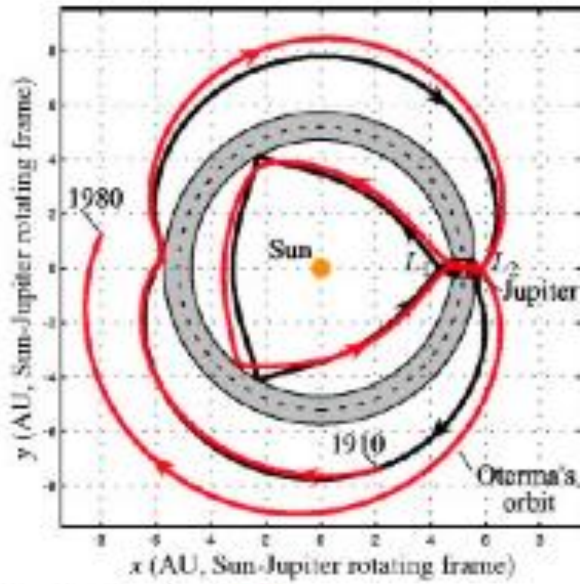
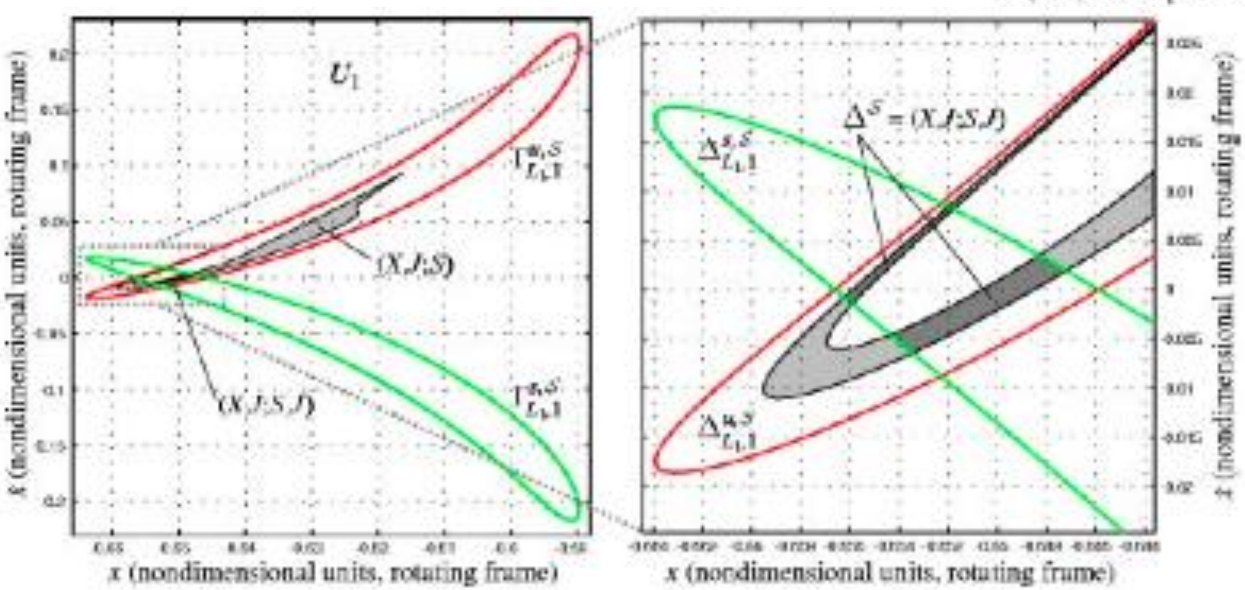
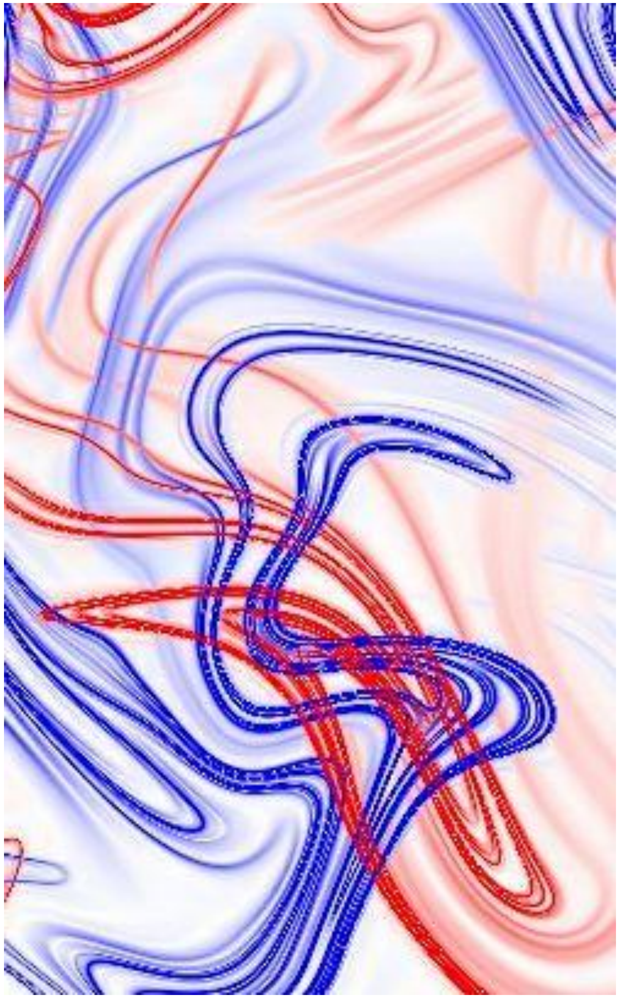


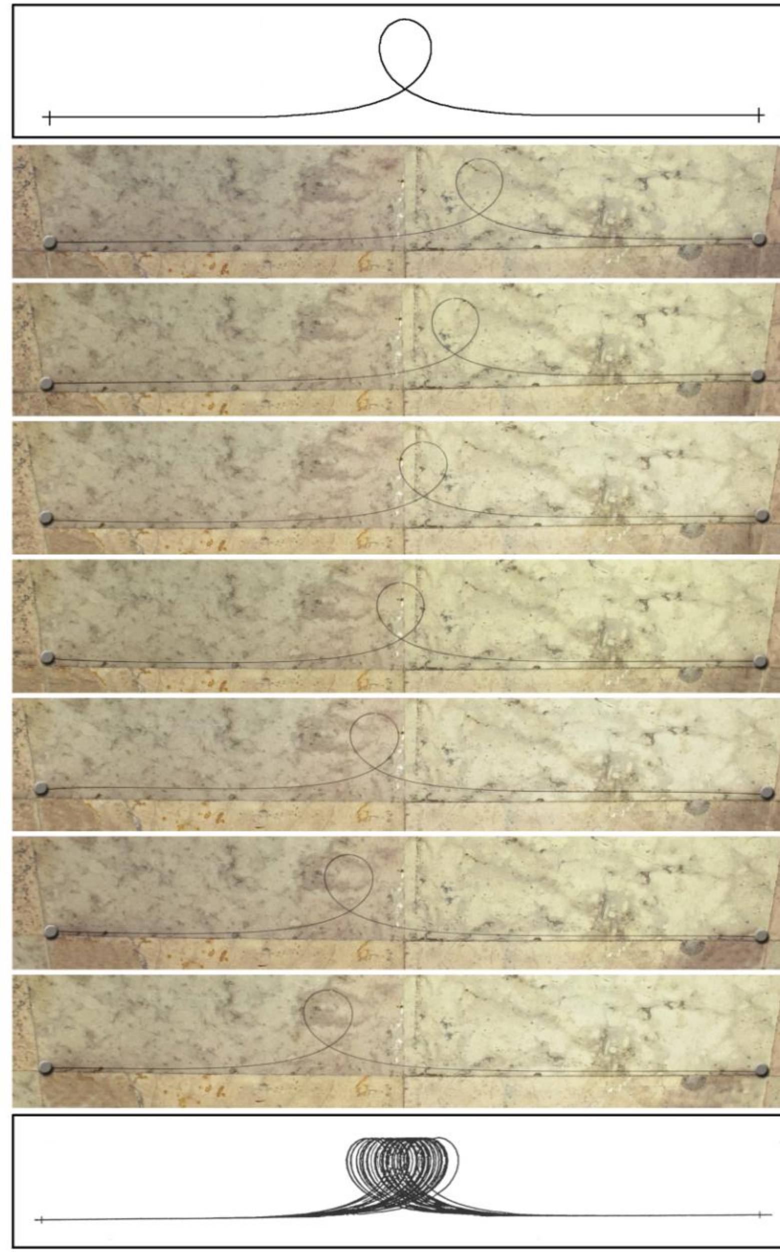


(a) (b)

Terrestrial homoclinic chaos 1 (it's not only in the stars)



Fluid mixing: Voth, Haller & Gollub, PRL, 88, #254501, 2002.



Buckling rods: Domokos-H, Proc. Roy. Soc. A, 459, 1535, 2003.

Terrestrial homoclinic chaos 2 (from planets to plants)

Tendril Perversion in Intrinsically Curved Rods

T. McMillen[1] and A. Goriely[1,2]

J. Nonlinear Sci. Vol. 12: pp. 241–281 (2002)

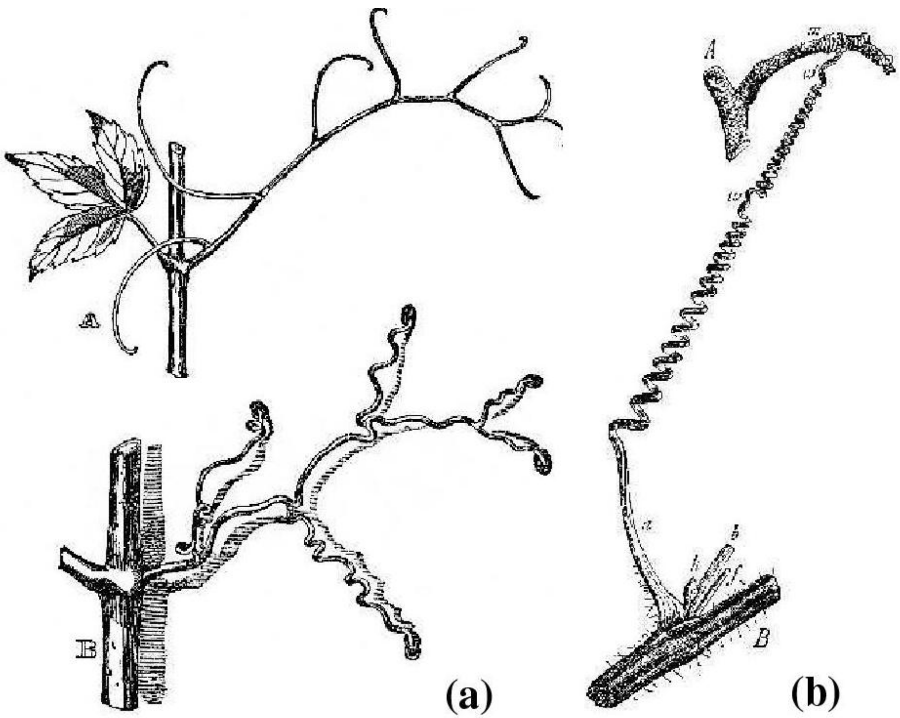


Fig. 3. (a) Growth of climbing plants (tendril bearers) as drawn by Darwin [14]. In the first stage (A), the tendrils are *circumnutating* until they find an attachment. In the second stage (B), the tendrils are attached and perversion sets in. (b) Another example of tendril perversion in Bryonia dioica. Illustration from Sachs's *Text-book of Botany* (1875).

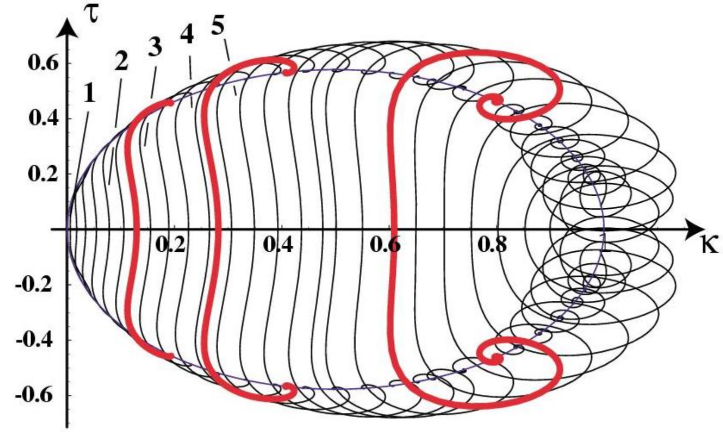


Fig. 21. Family of heteroclinic orbits. Projections in the $\kappa - \tau$ plane. The thick curves represent orbits computed by shooting, the thin curves are those computed by continuation. Values of the parameters are $K = 1$, $\Lambda = 1$, $\Gamma = 0.75$.

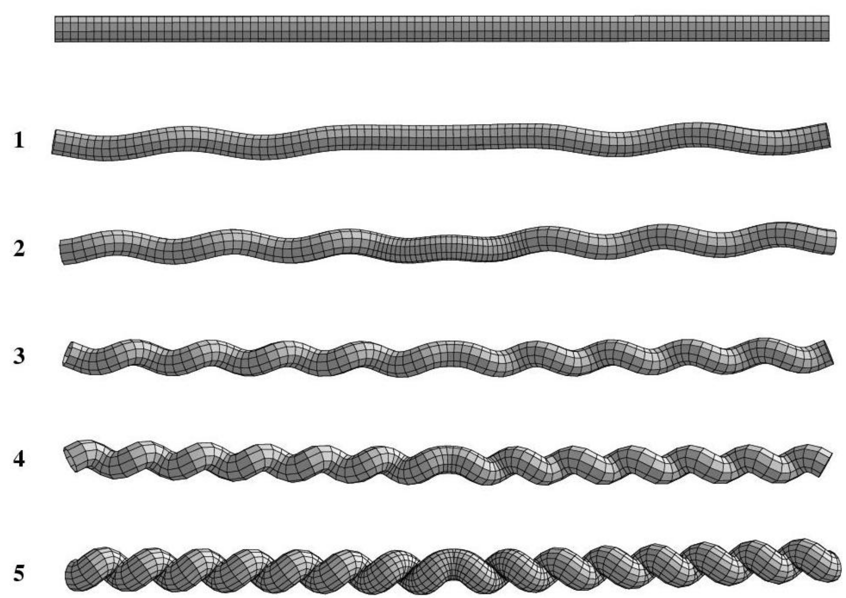


Fig. 22. Family of heteroclinic orbits. Perversions for varying values of tension. Values of the parameters are $K = 1$, $\Lambda = 1$, $\Gamma = 0.75$.

II: Less is more: Local behavior

Less is more: local behavior

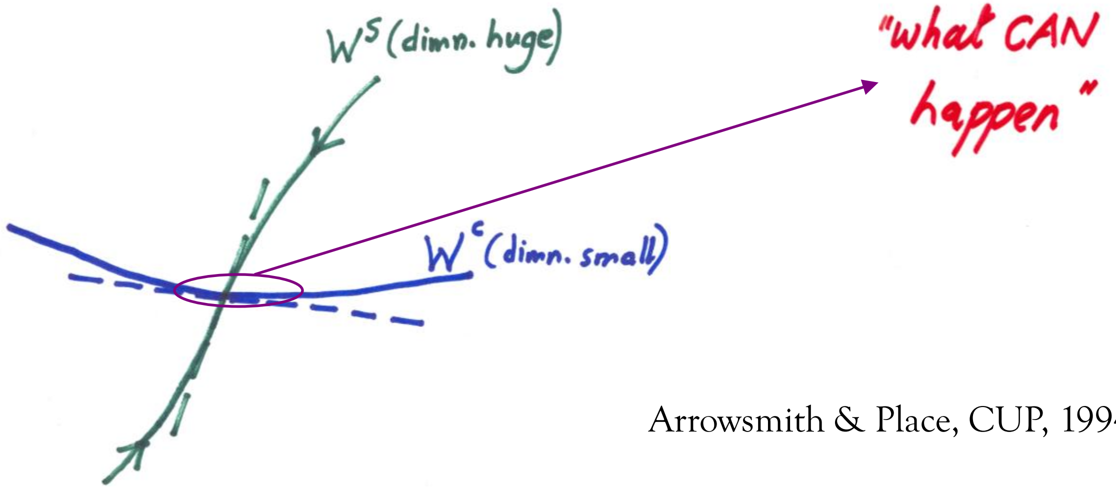
The early work tackled the hard problem of global behavior. Studies of behaviors near degenerate equilibria came later, starting with Andronov and Pontryagin's "coarse systems" in 1937 [*Dokl. Akad. Nauk*. SSSR 14, 247-251 The Gorkii (Nizhny-Novgorod) school + Moscow Mat Mech.].

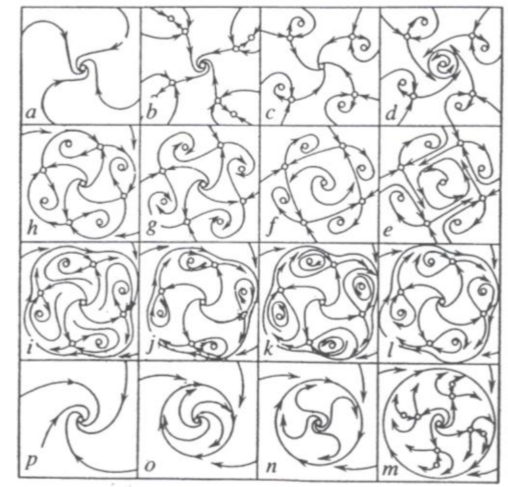
One takes a geometrical view of the infinite-dimensional space of all dynamical systems (perhaps with special structures or symmetries) and asks: Which ones survive small perturbations (***structural stability***) and Which ones are typically found (***generic properties***)?

If a system isn't structurally stable, then one asks: What wonders are lurking within it and how do I reveal them (***unfoldings***)?
This approach enormously extended, enriched and generalized the existing area of ***bifurcation theory***. It provides a taxonomy of beasts in the dynamical forest: a hunting license for nonlinear mechanics.

Center manifolds, normal forms, and unfolding

The center manifold theory of Pliss (USSR, 1964) and Kelley (USA, 1967) allows one to discard all the stable (and unstable) dimensions and focus on the bifurcating center directions:





Arrowsmith & Place, CUP, 1994.

Nonlinear coordinates changes, giving normal forms , simplify the system and allow one to analyze it with a minimal parameter set (codimension):

$$\begin{aligned} \dot{x} &= y + ax^2 + bxy + cy^2 + \dots \\ \dot{y} &= dx^2 + exy + fy^2 + \dots \end{aligned} \quad \Longrightarrow \quad \begin{aligned} \dot{x} &= y \pm x^2 + \dots \\ \dot{y} &= \mu_1 + \mu_2 y - x^2 + \dots \end{aligned}$$

[Takens-Bogdanov codimension 2 normal form]

Thom and Zeeman's Catastrophe Theory (1960-75) achieved this for gradient systems, whose orbits go downhill with no recurrence, periodic orbits or chaos.

Unfolding fluid instabilities: Taylor-Couette flow

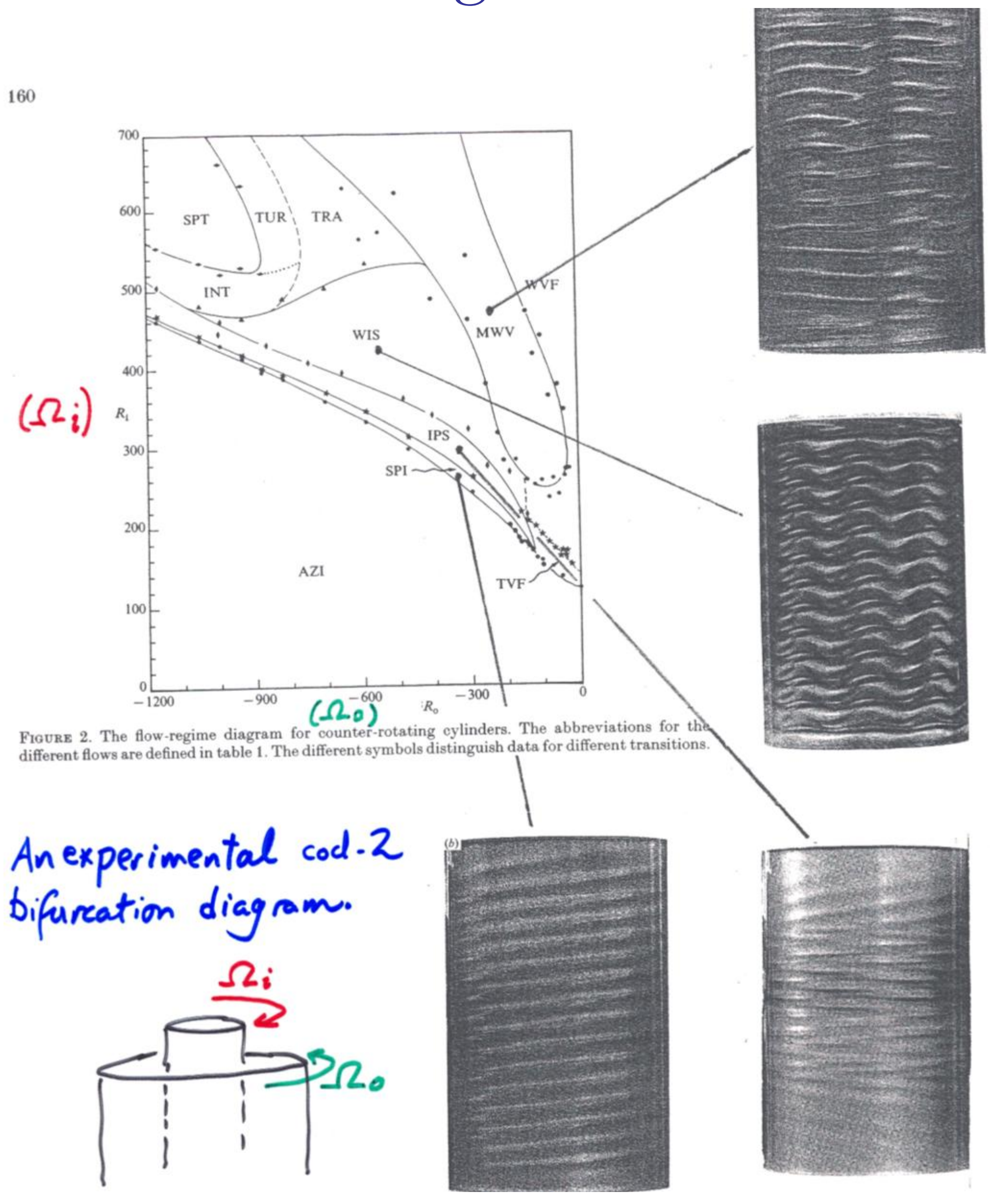
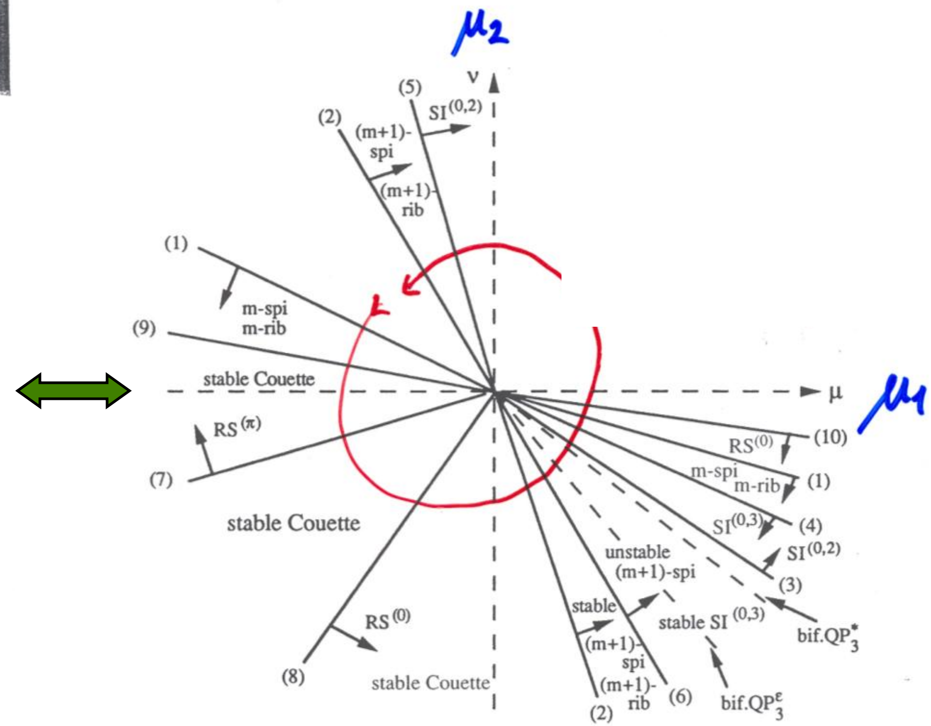


FIGURE 2. The flow-regime diagram for counter-rotating cylinders. The abbreviations for the different flows are defined in table 1. The different symbols distinguish data for different transitions.



Andereck, Liu & Swinney, JFM, 164, 155-183, 1986.

Chossat & Iooss, Springer, 1994.

More is different: complex systems ...

… well, not **those** complex systems (Santa Fe Inst & all), but many important problems don’t belong to the nice classes of smooth, structurally stable, hyperbolic, dynamical systems for which we have nice theories. Some examples are:

Differential-delay dynamical systems

Hybrid dynamical systems *

Piecewise smooth dynamical systems

Stochastic dynamical systems *

Hybrid chaos: milling cutters

CHAOS VOLUME 14, NUMBER 4

Global dynamics of low immersion high-speed milling

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S. John Hogan[c]
Department of Engineering Mathematics, University of Bristol, BS8 1TR, Bristol, United Kingdom

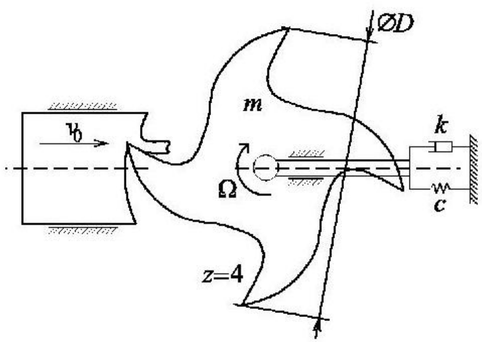


FIG. 1. Scheme of high-speed milling. Feed is provided by the workpiece velocity v_0, cutting speed is provided by the (rotating) tool.

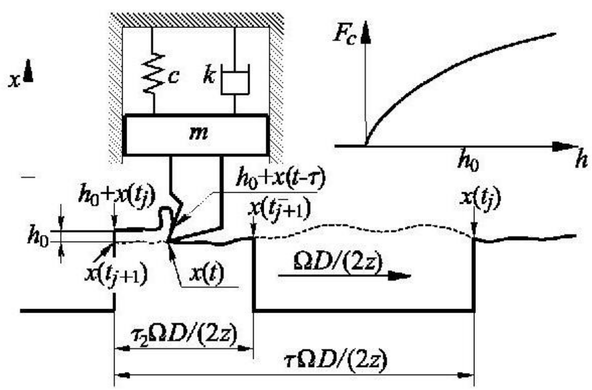
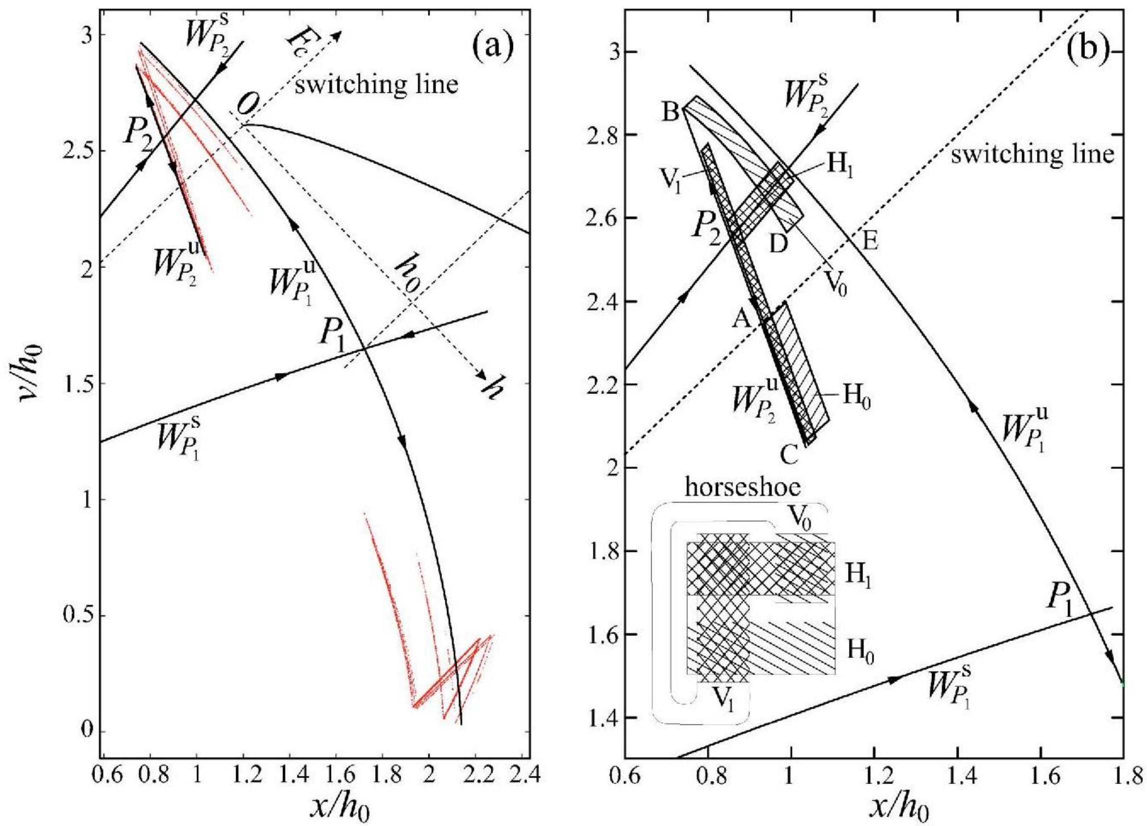


FIG. 2. Mechanical model. Note the difference from the model in Fig. 1: the feed is provided by the tool while the cutting speed is provided by the motion of the (rotating) workpiece.



. 8. The chaotic map ($z\hat{\Omega}=0.459$, $\hat{w}=1.64$); (a) simulation with invariant manifolds and how cutting force varies perpendicular to the switching line and horizontal and vertical slabs (see in the text).

The morals of the story

- The ivory tower of abstraction is good, but you'd better have some friends with their feet on the ground.
- Simple (canonical) models are really useful.
- Central themes: ***homoclinic orbits, judicious linearization dimension reduction, normal forms, unfolding.***
- Less is more: ***reduce, transform and simplify!***
- More is different: ***parameters, dimensions, components, impacts, noises, ….. !***

-- The End *--*

& thank you for your attention.

http://tutorials.siam.org/dsweb/enoc/

Early attempts to unfold a codimension two singularity:

again the van der Pol oscillator played a role (this time in the near-harmonic oscillator limit), and again Mary Cartwright was involved:

J. Inst. E.E. 95, iii, 88, 1948.

FORCED OSCILLATIONS IN NEARLY SINUSOIDAL SYSTEMS

By MARY L. CARTWRIGHT, D.Phil., F.R.S.

(The paper was first received 5th March, and in revised form 10th September, 1947.)

SUMMARY

A large class of radio circuits which are analytically equivalent to an oscillatory network in parallel with a non-linear negative resistance, are represented fairly accurately by the differential equation

$$\ddot{v} - (\alpha + \beta v - \gamma v^2)\dot{v} + \omega^2 v = E\omega_1^2 \sin \omega_1 t$$

where α/ω, β/ω, γ/ω are small. The behaviour of the solutions of this equation near resonance has been discussed by Appleton, van der Pol and others.

The paper contains a more complete discussion of the synchronized and quasi-periodic solutions near resonance, their phases, amplitudes and energy, and also the way in which one type of stable solution gives way to another as the parameters of the system vary, for instance as the electromotive force or detuning vary. It is shown that the phase and amplitude favourable to synchronization are prolonged just before synchronization. This agrees with Appleton's experimental results. It is also found that hysteresis occurs. The decrease in energy with the decrease in detuning is explained by the fact that the phase favourable to synchronization is that which opposes the motion and is prolonged.

(1) INTRODUCTION

There is a large class of radio circuits which are analytically equivalent to an oscillatory network in parallel with a non-linear negative resistance the current/voltage characteristic of which can be represented with sufficient accuracy for many purposes by a power series up to the third power of the voltage. The differential equations of such systems are in general reducible to the form

$$\ddot{v} - (\alpha + \beta v - \gamma v^2)\dot{v} + \omega^2 v = f(t).$$

a very large number of different cases to be considered within a small range of parameters on the border line between strong and weak signals.

(2) RANGE OF VALIDITY

The condition α/ω small ensures that, for $E = 0$, eqn. (1) has a stable solution approximately of the form

$$v = a_0 \sin \omega t$$

where

$$a_0^2 = 4\gamma/\alpha$$

provided that β is not large compared with α and γ. It is sufficient to assume that

$$\beta < K \max(\alpha, \gamma)$$

where K is an absolute constant. We will assume that this holds but it should be observed that, if the equation is derived from one of the form[6]

$$\ddot{v} + \frac{1}{C}\left[\frac{1}{R} - \phi'(v)\right]\dot{v} + \omega^2 v = 0 \quad . \quad . \quad . \quad (2)$$

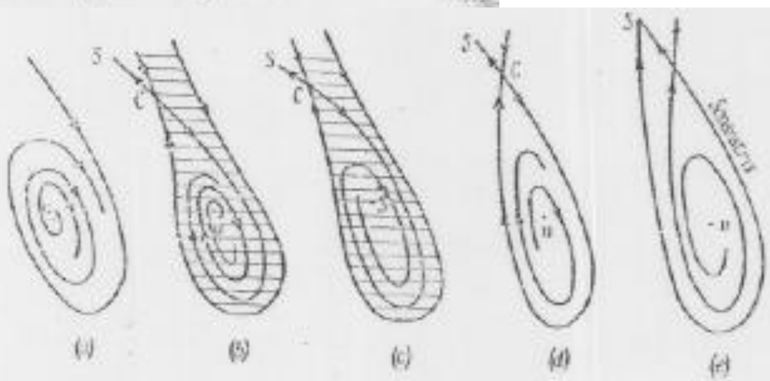


Fig. 12

From Cartwright to Gillies and Takens-Bogdanov

ON THE TRANSFORMATIONS OF SINGULARITIES AND LIMIT CYCLES OF THE VARIATIONAL EQUATIONS OF VAN DER POL

By A. W. GILLIES (*Northampton Polytechnic, London*)

[Received 6 August 1952]

SUMMARY

The investigation of the solution of van der Pol's equation with forcing term leads to equations for the amplitude, b, and phase of the oscillation, ϕ, of the form

$$\dot{b} = b(1-b^2) - F\cos\phi,$$
$$b\dot{\phi} = -bx + F\sin\phi.$$

The solution of this autonomous system of first-order equations has been discussed by Cartwright. By considering the isoclines on the plane with (b, ϕ) as polar coordinates, it is shown that Cartwright's solution is incorrect in one range of the parameters. The corrected solution is given. In consequence of this, it is shown that the hysteresis effects to be expected for a van der Pol oscillator with increasing and decreasing frequency are confined to a narrower frequency interval and are less varied in character than was suggested by Cartwright's solution.

Introduction

THE investigation of the solution of van der Pol's equation with forcing term (1), (2), (3) leads to the study of the variational equations of the form†

$$\boxed{\begin{aligned} \dot{b} &= b(1-b^2) - F\cos\phi, \\ b\dot{\phi} &= -bx + F\sin\phi \end{aligned}} \qquad \begin{aligned} &(1) \\ &(2) \end{aligned}$$

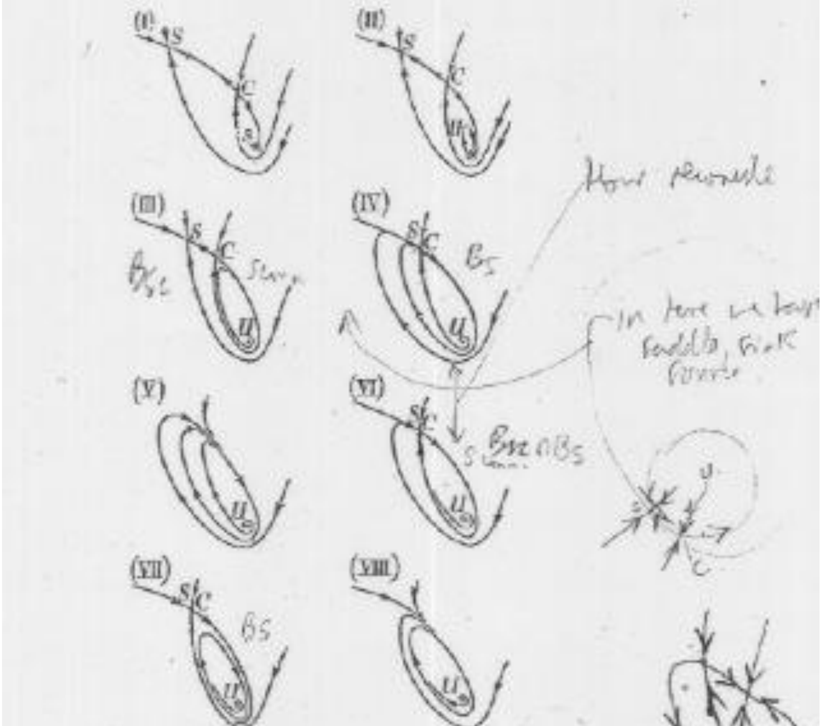
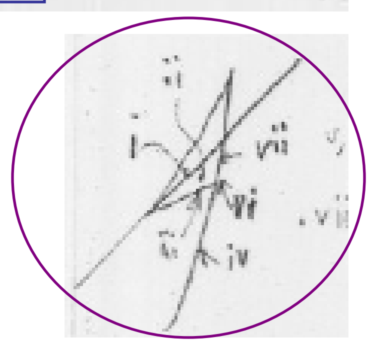


FIG. 5. Sequence of changes in integral curve pattern. The critical region is between $F = \frac{1}{2}$ and $F = \sqrt{(9/32)}$. For increasing x we have first of all the single stable node S. The col C and stable node s form on the lower boundary of $\mathscr{C}$, the node s changing very quickly to a focus to give Fig. (i). The focus moves inwards towards the origin and becomes unstable (U), throwing off a stable limit cycle giving (ii). For $\frac{1}{2} < F < F_1$ the limit cycle expands to pass through the col (iii). The curve leaving the col on the right takes a progressively wider sweep, passing to the other side of S before S and C move into coincidence in (iv). The double singularity then disappears leaving a new limit cycle surrounding the unstable focus U, (v).



Quart. J. Mech. Appl. Math. 7, 152-167, 1954.

Takens-Bogdanov double zero normal form:

H-Rand, Quart. J. Appl. Math. 35, 495-509, 1978

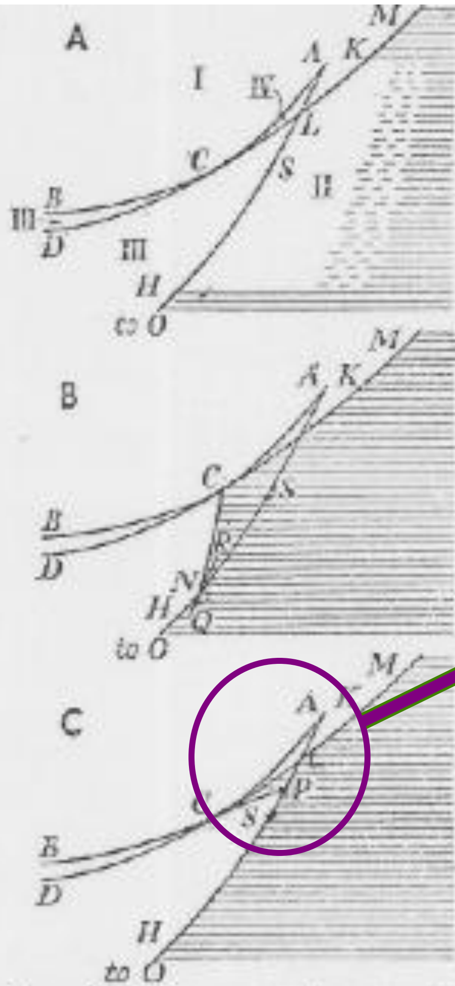


FIG. 2. Diagrammatic representation of the (x, F)-plane.

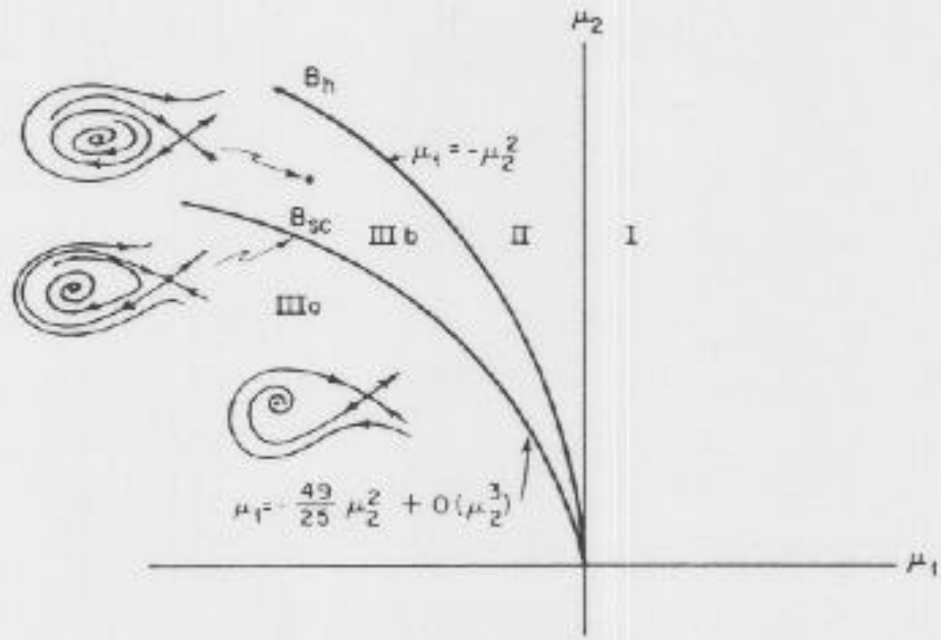


Figure 7.3.3. The global saddle connection bifurcation.

Guckenheimer-H, Springer, 1983.

If you think this was all too much, well …

I left a lot out:

A poll on important topics [SIAM@50, 2003]

[Applied] Dynamical Systems: past 40 years

- Ergodic theory and strange attractors
- Low dimensional paradigms: the logistic and Hénon maps; van der Pol oscillator, Lorenz ODE; 3 body problem
- Spatial systems: water waves, light waves, elastic buckling
- Ruelle-Eckmann conj. and fractal dimensions in dimn > 2
- Atmospheric dynamics: (non)predictability
- Weakly hyperbolic systems and applications in time series analysis
- DS diagnostics for experimental and numerical data
- Economics and market dynamics (speculative bubbles)
- Dimension reduction: invariant and center manifolds
- Normal forms; unfolding local and global bifurcations, with and without symmetry
- Complex networks: power grids; cell signaling and regulation; the brain
- Finite-dimnl dynamics in infinite-dimnl systems; inertial manifolds; global attractors
- Nonlinear optics
- KAM theory in infinite dimensions
- KAM theory for the FPU chain
- Dynamics of granular materials
- Exponentially small separatix splitting in maps, ODE and PDE; integrability; Arnol'd diffusion; cantori
- Chaotic mixing/Lagrangian transport theory
- Turbulence and transition; non-normal systems
- Geometric singular perturbation theory
- Homoclinic orbits and chaos; spatial chaos
- Neuroscience: bursting oscillations; multiple time scales
- Ecosystems and evolution models; virus evolution
- Deterministic and stochastic dynamical systems
- Numerics for simulation and rigor: shadowing, convergence of invariant sets
- Symplectic algorithms; variational integrators
- The B-Z reaction; patterns in active media; heartbeats; morphogenesis
- Pattern formation in unbounded domains
- Space mission design via invariant manifolds
- Good bases: POD; coarse graining; homogenization
- Infection, disease, HIV models
- Hybrid systems: tool vibrations; legged locomotion
- Nonholonomic mechanics
- Geometric mechanics and control theory