The Wigner Distribution Function and its Optical Production

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An optical signal (image etc.) can be described by its complex amplitude $u(x, y)$, or by its spatial frequency spectrum. Both descriptions are complete and also equivalent, because one can be derived from the other by a Fourier transformation. Neither the complex amplitude nor the spatial frequency spectrum is suitable for answering a question like "what is the spatial frequency in a certain part of the image?". Here the term "local spectrum" is adequate. A rigorous definition of the "local spectrum" can be based on the Wigner distribution function. We developed optical methods for producing this "local spectrum" and we applied these methods to the investigation of sound patterns.

Asma Al-Qasimi

QO Group Meeting

Wednesday March 11 2009

5pm, MP307

Outline

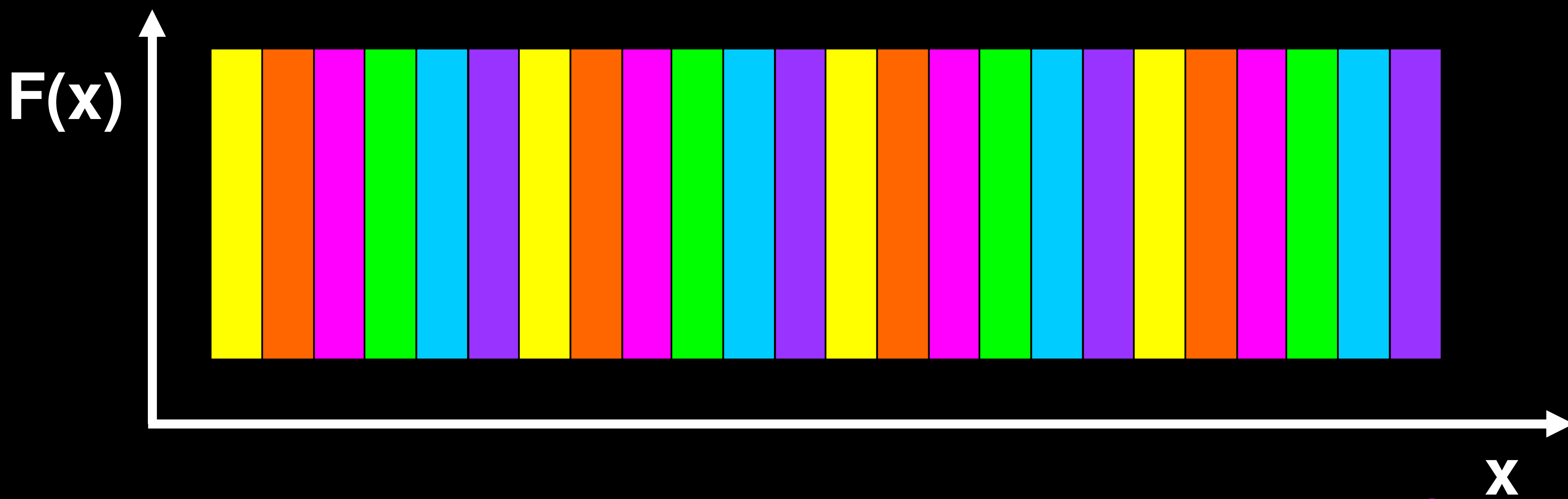
- Mathematical Descriptions
  → Fourier Transforms
  → Convolutions

- Heisenberg's Uncertainty Relations and Limitations
  → Problem

- Wigner Distributions Function
  → Definition and Properties
  → Solution

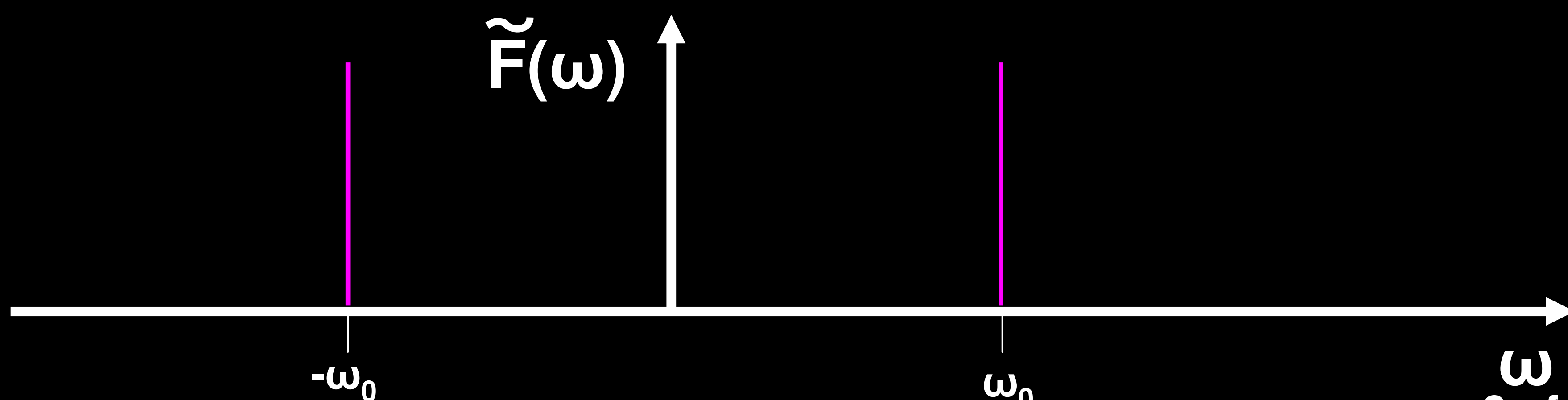
- Experimental Realization of the Solution

Fourier Transforms



What is the frequency of pattern repetition?

Look at the Fourier Transform of F(x)

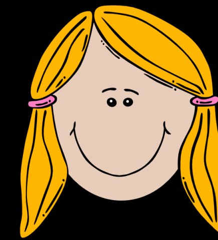


Fourier Transform Pair

$$F(x) = \int_{-\infty}^{\infty} \tilde{F}(\omega)\, e^{-ix\omega}\, d\omega$$

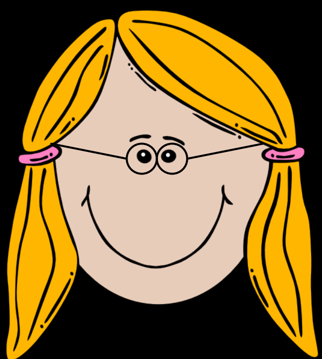
$$\tilde{F}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(x)\, e^{ix\omega}\, dx$$

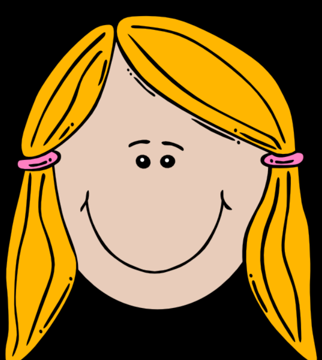
Convolution

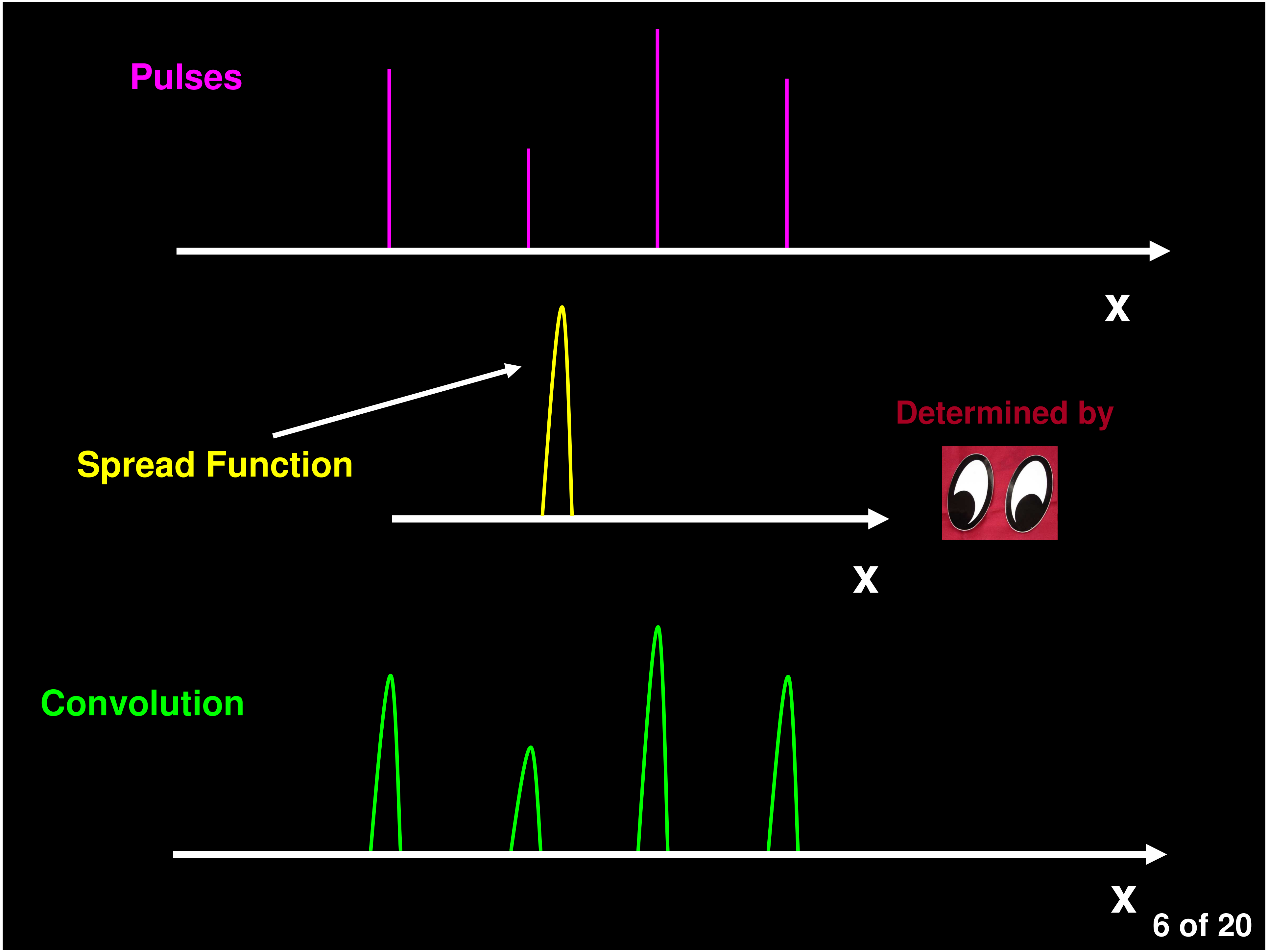


short-sighted

Stars



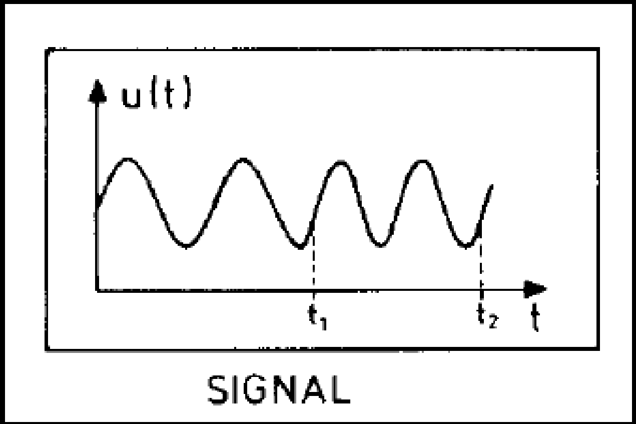



Pulses
x
Spread Function
Determined by
x
Convolution
x

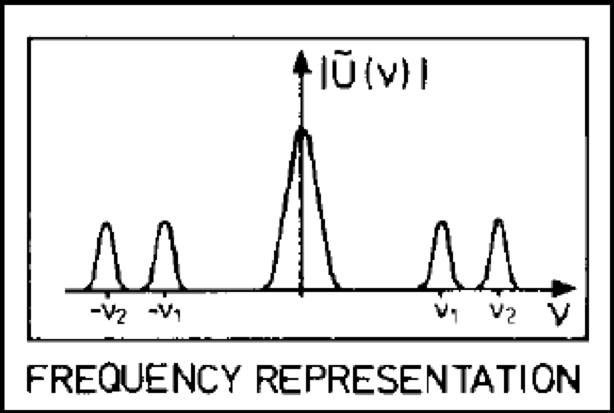
Given F(x) and G(x), their convolution is given by:

$$C(x) = \int_{-\infty}^{\infty} F(x')G(x - x')dx'$$

air pressure vs time



Using Fourier Transforms, we obtain



At time t, what is the frequency ν ?

ν and t <u>cannot</u> be specified simultaneously

Heisenberg Uncertainty Principle

Can problem be solved by finding a different mathematical representation?

Wigner Distribution Function

↓

Local Spectrum

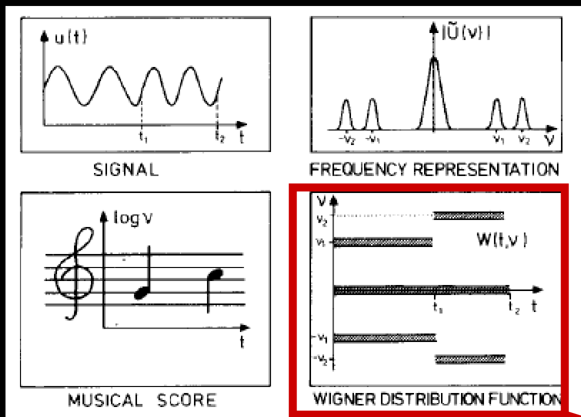
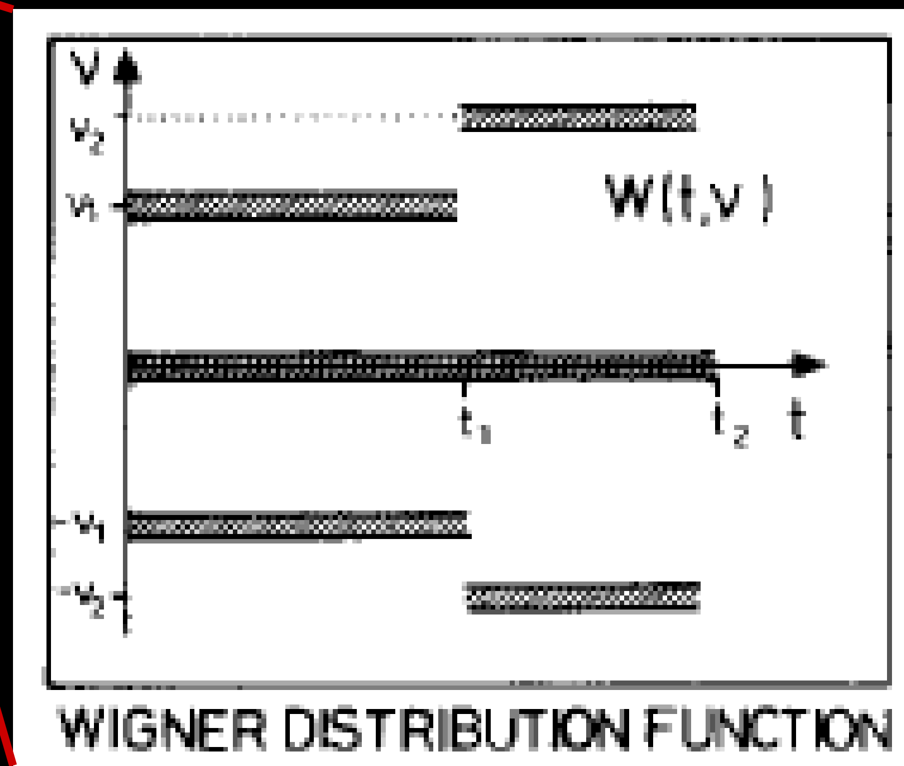


Fig. 1. Different descriptions of the same signal.



Wigner Distribution Function

$$W(x,\nu) = \int u(x+x'/2)\, u^*(x-x'/2)$$

$$\times \exp(-2\pi i \nu x')\, dx',$$

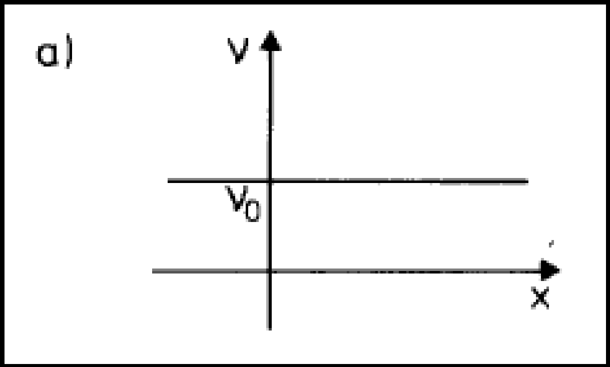
$$W(x,\nu) = \int \tilde{u}(\nu+\nu'/2)\, \tilde{u}^*(\nu-\nu'/2)$$

$$\times \exp(2\pi i \nu' x)\, d\nu',$$

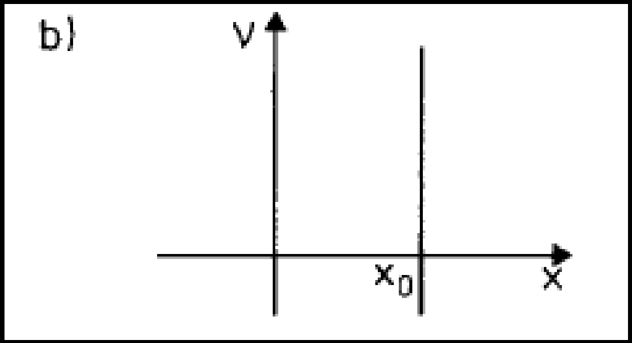
$$|u(x)|^2 = \int W(x,\nu)\, d\nu, \qquad |\tilde{u}(\nu)|^2 = \int W(x,\nu)\, dx,$$

$$E_{\text{total}} = \int W(x,\nu)\, dx\, d\nu.$$

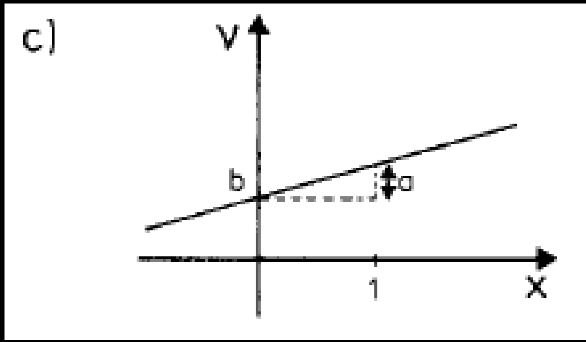
Examples of the Wigner distribution function of some signals



Monofrequency



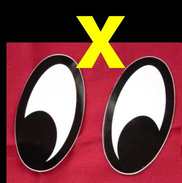
Pulse



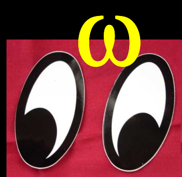
Linearly Increasing Frequency

Define a "local spectrum" to give us the frequency content of a position interval.

Define the x and ν "eyes"



$$G(x;\delta x) = 1/\sqrt{\delta}\, x \exp[-2\pi(x/\delta x)^2]$$



$$G(\nu;\delta\nu) = 1/\sqrt{\delta}\, \nu \exp[-2\pi(\nu/\delta\nu)^2]$$

$$\delta x \delta \nu = 1$$

Minimal Uncertainty

$$\hat{W}(x,\nu;\delta x) = \int\int W(x',\nu')$$

$$\times\, G(x-x';\delta x)\, G(\nu-\nu';1/\delta x)\, dx'\, d\nu'$$

$\hat{W}(x,\nu;\delta x) \approx |u(x)|^2$

$\delta x \ll \delta x_0$

$\hat{W}(x,\nu;\delta x) \approx |\tilde{u}(\nu)|^2$

$\delta x > \Delta x_0$

δx_0 is the resolution length

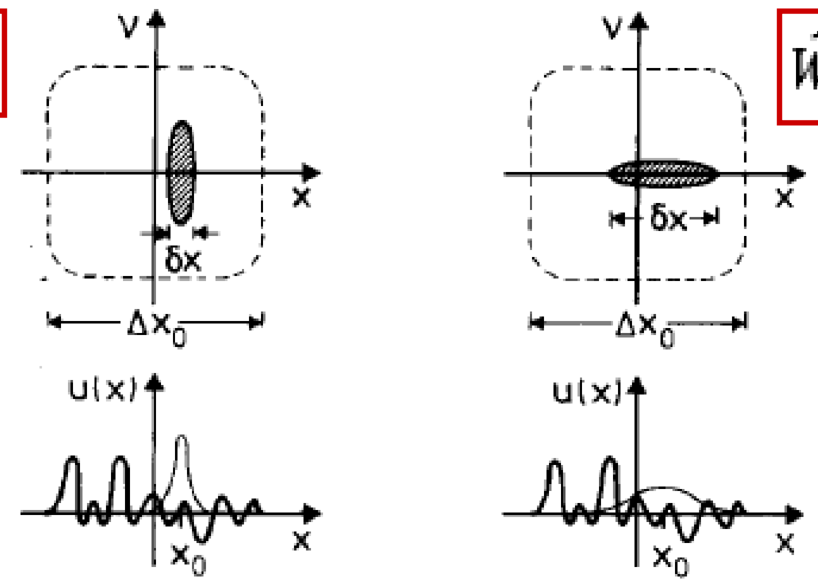
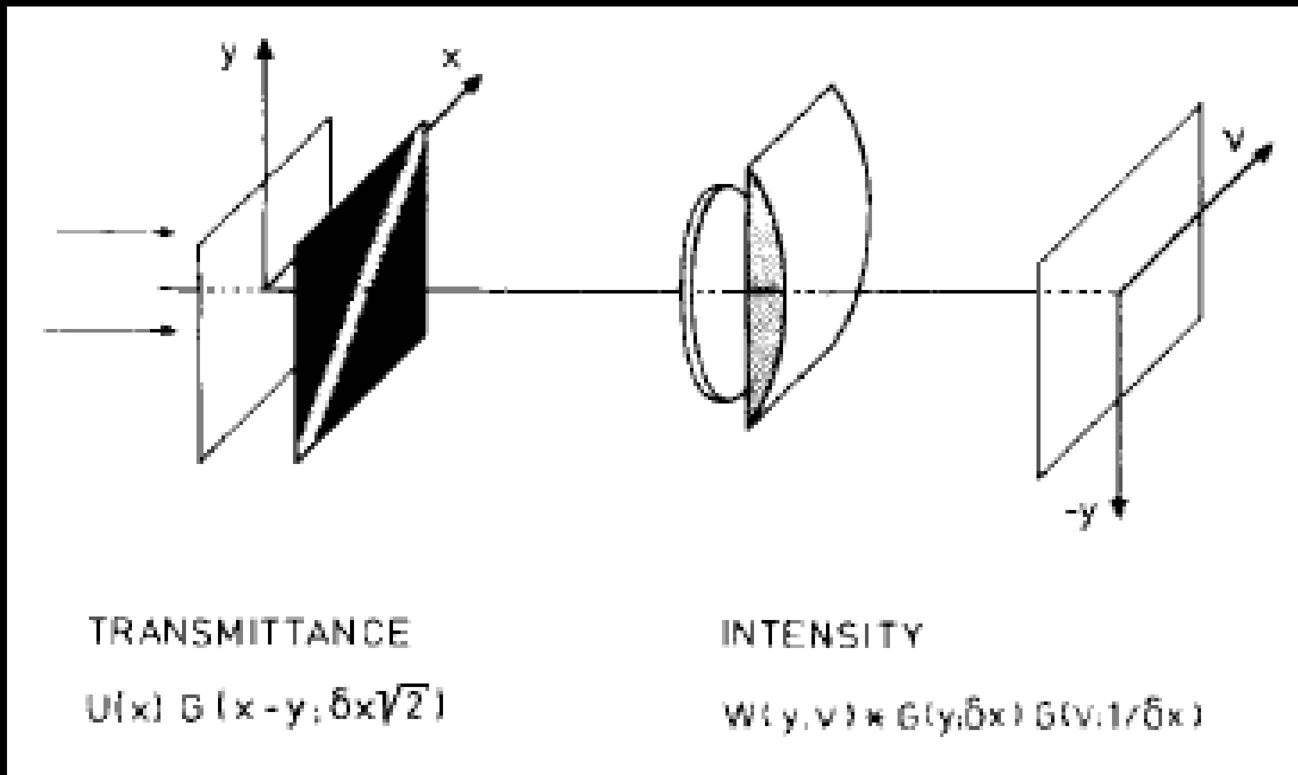


Fig. 3. Illustration of the relationship between u, W and $\hat{W}$ according to eqs. (12), (13). The dotted lines (upper half) are the boundaries of W.

$\delta x_0 < \delta x < \Delta x_0$ **Local Spectrum**

Experiment



δ_x controlled by the transmittance angle

$$\left| \int u(x)\, G(x-y; \sqrt{2}\delta x) \times \exp(-2\pi i \nu x)\, \mathrm{d}x \right|^2 = \hat{W}(y, \nu; \delta x)$$

Experimental Results

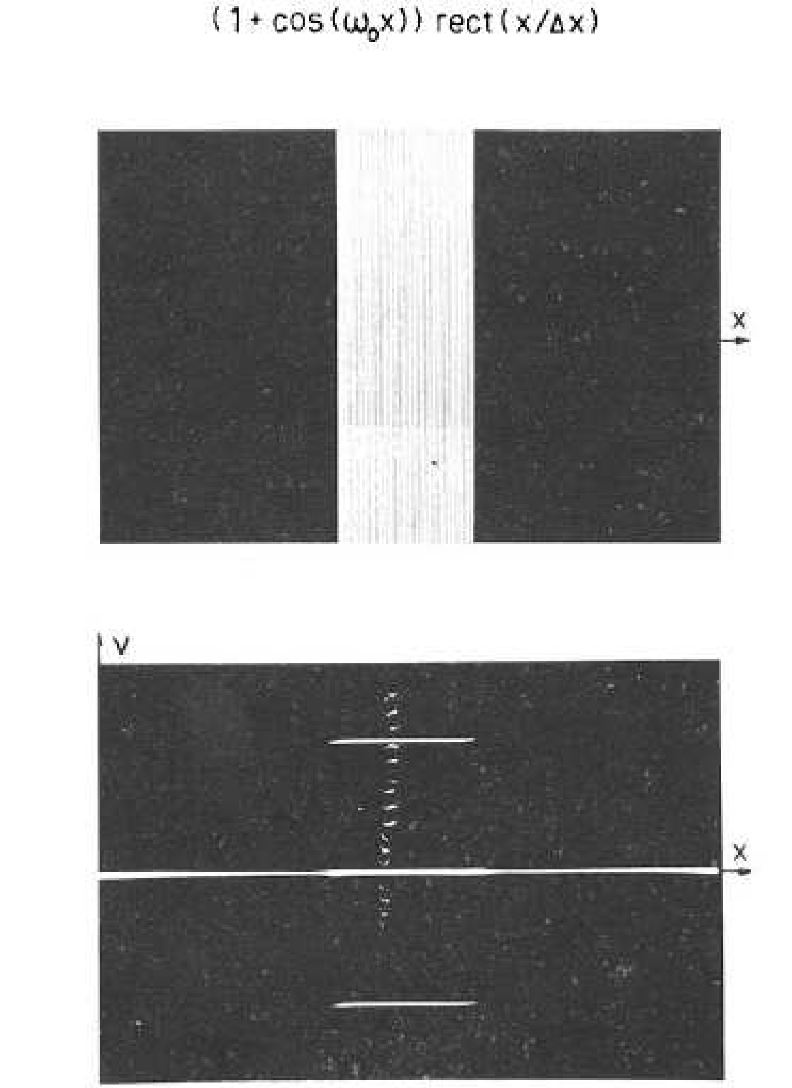


Fig. 6. Test signal (monofrequency of finite duration) and local spectrum $\hat{W}$.

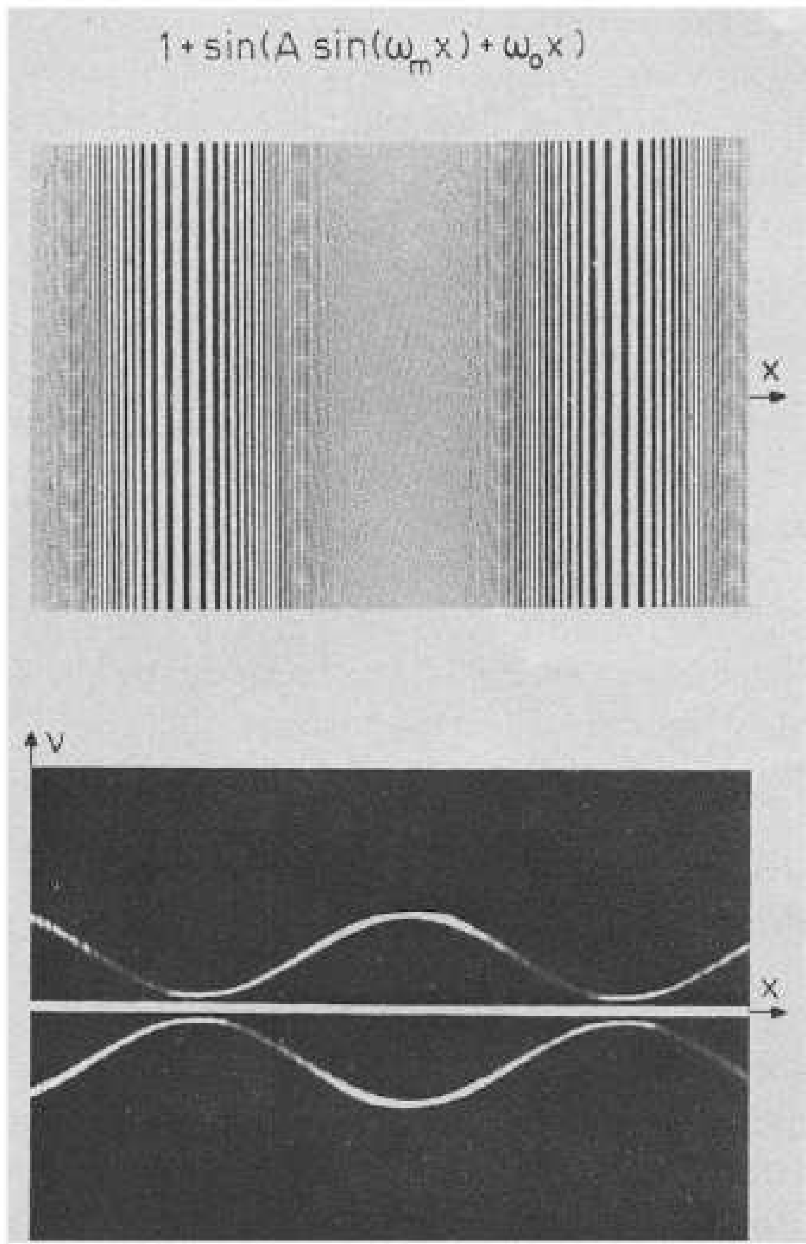


Fig. 8. Test signal (sinusoidal frequency modulation) and local spectrum $\hat{W}$.

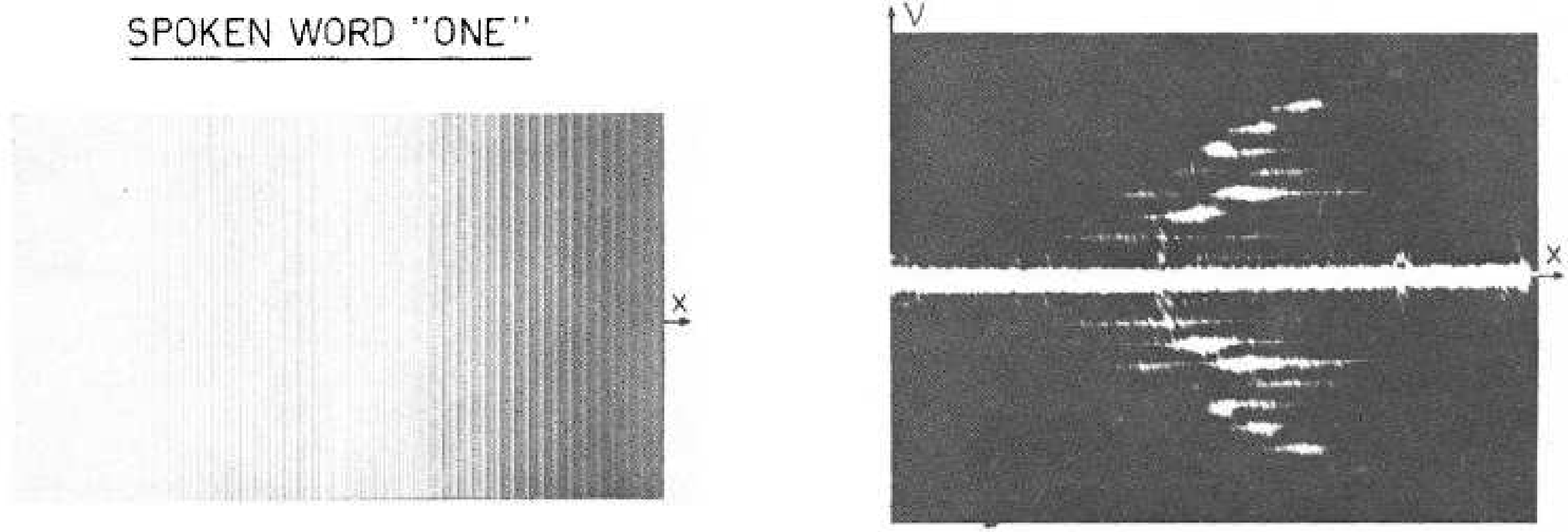


Fig. 9. Signal $u(x)$ of spoken word "one" and $\hat{W}(x,\nu;\delta x)$.

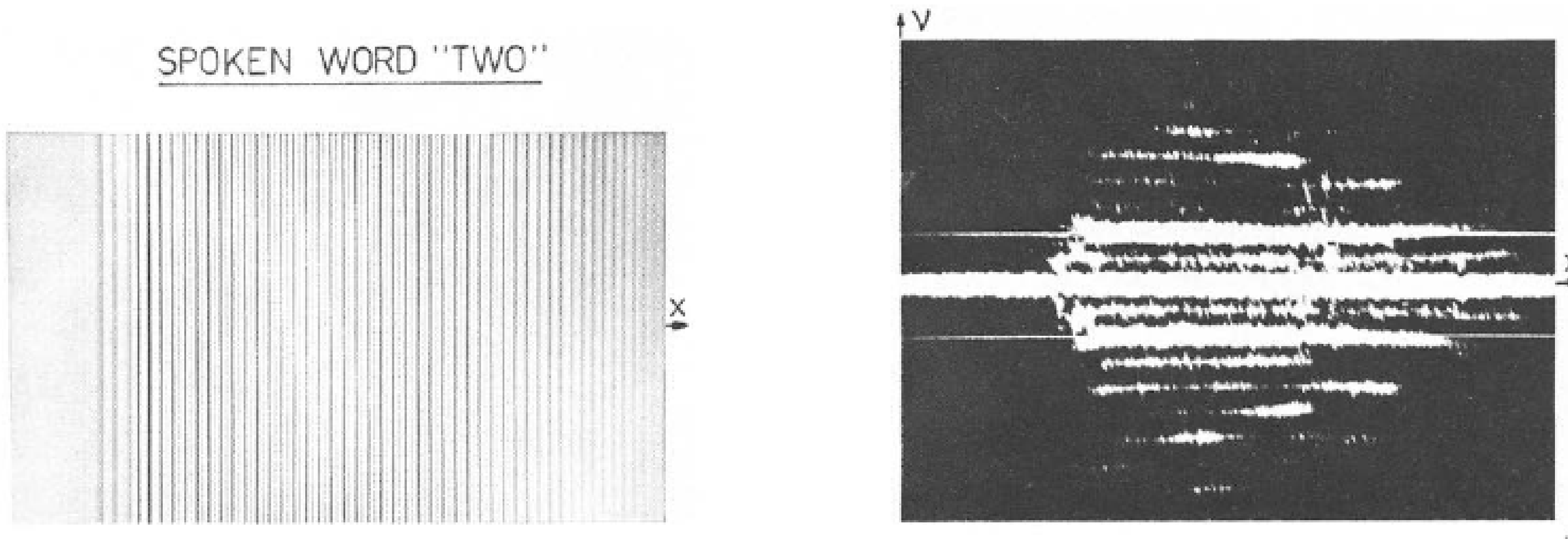


Fig. 10. Signal $u(x)$ of spoken word "two" and $\hat{W}(x,\nu;\delta x)$.

Word/Voice Recognition

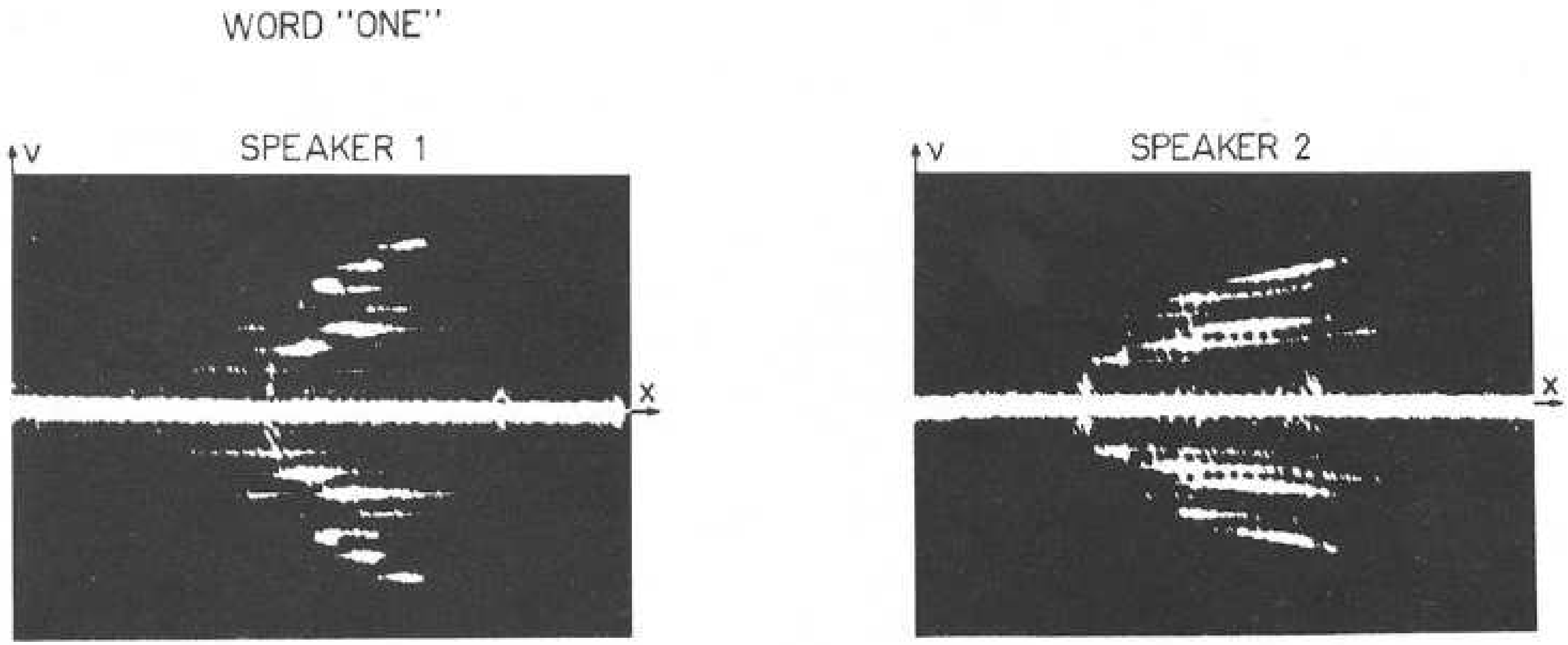


Fig. 11. The local spectra $\hat{W}(x,\nu;\delta x)$ of the word "one", spoken by different persons.

Conclusions

- Wigner distribution functions give the distribution of ν and x for optical signals.

- To find the ν associated with an x range and not violate the Heisenberg's uncertainty relation, the concept of "local spectrum" is developed.

- Applications of the "local spectrum" include technologies requiring voice recognition capabilities.