THE WIGNER DISTRIBUTION FUNCTION AND ITS OPTICAL PRODUCTION

H.O. BARTELT, K.-H. BRENNER and A.W. LOHMANN
Physikalisches Institut der Universität, 8520 Erlangen, Fed. Rep. of Germany
8520 Erlangen, Fed. Rep. of Germany

Received 22 October 1979

An optical signal (image etc.) can be described by its complex amplitude $u(x,y)$, or by its spatial frequency spectrum. Both descriptions are complete and also equivalent, because one can be derived from the other by a Fourier transformation. Neither the complex amplitude nor the spatial frequency spectrum is suitable for answering a question like "what is the spatial frequency in a certain part of the image?". Here the term "local spectrum" is adequate. A rigorous definition of the "local spectrum" can be based on the Wigner distribution function. We developed optical methods for producing this "local spectrum" and we applied these methods to the investigation of sound patterns.

1. The philosophy behind the Wigner distribution function

The term "Wigner distribution function" (WDF) is not very well known in optics. Therefore we will begin by stating in philosophical terms the importance and the usefulness of the WDF. This philosophy can be explained best by thinking of music and how it is documented in print. Every physical event such as a piece of music can be described in various ways. For example, we may plot the air pressure $u(t)$ as a function of time t (fig. 1), or its temporal Fourier transform $\tilde{u}(\nu)$:

$$\tilde{u}(\nu) = \int u(t) \exp(-2\pi i \nu t)\, dt$$

Both descriptions $u(t)$ and $\tilde{u}(\nu)$ are complete, but nevertheless unsuitable for the performing musician. He needs to know which frequency ν he has to produce at time t. The "musical score" (fig. 1) satisfies the musician but not the mathematician. It is illegal to specify a monochromatic frequency at a given point in time. Such a specification would violate the uncertainty principle. We need a finite duration, at least one full period, for measuring the frequency. A longer duration would improve the accuracy of the frequency measurement but would destroy the sharpness of the time measurement. The conflict between the musician and the mathematician can be avoided, if the WDF is used for describing the acoustical signal. The WDF was thoroughly investigated by de Bruijn [2] and recently introduced into optics by Bastiaans [3].

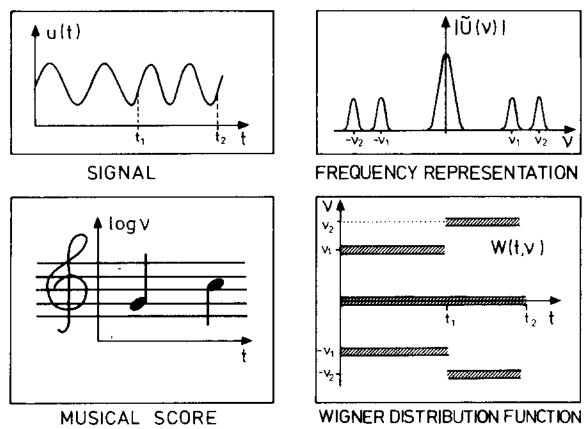

Fig. 1. Different descriptions of the same signal.

2. The goals of this study

First we will briefly show how the WDF relates to better known optical quantities such as the complex amplitude $u(x)$, the intensity $|u(x)|^2$, the spatial frequency spectrum $\tilde{u}(\nu)$, the power spectrum $|\tilde{u}(\nu)|^2$ and the ambiguity function $A(\mu, y)$. The WDF can be considered as a "masterform signal" from which all other forms of signal description can be derived, not only the ones mentioned above. As we will see, the local frequency spectrum $\hat{W}(x,\nu;\delta x)$, which contains information about the signal u in space and frequency, emerges from the WDF by convolution with a narrow Gauss function.

The main contribution of this paper are two optical methods for measuring the local spectrum $\hat{W}$. First we apply these methods to simple test patterns like gratings and zone plates. Later we use these methods to investigate acoustical signals, derived from spoken words.

3. Properties of the Wigner distribution function

The Wigner distribution function (WDF) can be defined in two equivalent forms:

$$W(x,\nu) = \int u(x+x'/2)\, u^*(x-x'/2) \times \exp(-2\pi i\nu x')\,dx', \qquad (1)$$

$$W(x,\nu) = \int \tilde{u}(\nu+\nu'/2)\, \tilde{u}^*(\nu-\nu'/2) \times \exp(2\pi i\nu' x)\,d\nu', \qquad (2)$$

$$u(x) = \int \tilde{u}(\nu) \exp(2\pi i\nu x)\,dx.$$

The complete symmetry between x and ν indicates that space and frequency have equal weight in this description.

For deriving further properties it is useful to define the function $F(x,y)$:

$$F(x,y) = u(x+y/2)\, u^*(x-y/2) = \int W(x,\nu) \exp(2\pi i\nu y)\,d\nu. \qquad (3)$$

Apparently, F is hermitian: $F(x,y) = F^*(x,-y)$. Hence the WDF is real:

$$W(x,y) = \int F(x,y) \exp(-2\pi i\nu y)\,dy, \qquad (4)$$

$$W(x,\nu) = W^*(x,\nu). \qquad (5)$$

The WDF is the double Fourier transform of the ambiguity function $A(\mu,y)$, that describes signal correlation in the presence of Doppler shifts:

$$A(\mu,y) = \int u(x+y/2)\, u^*(x-y/2) \exp(-2\pi i\mu x)\,dx, \qquad (6)$$

$$W(x,\nu) = \iint A(\mu,y) \exp[2\pi i(\mu x-\nu y)]\,d\mu\,dy. \qquad (7)$$

Important special cases of W and A are auto convolution and auto correlation:

$$W(x,0) = \int u(x+x'/2)\, u^*(x-x'/2)\,dx',$$

$$A(0,y) = \int u(x+y/2)\, u^*(x-y/2)\,dx. \qquad (8)$$

The intensity, the power spectrum and the total energy of the signal can be retrieved from the WDF by the following operations (projections):

$$|u(x)|^2 = \int W(x,\nu)\,d\nu, \qquad |\tilde{u}(\nu)|^2 = \int W(x,\nu)\,dx, \qquad (9)$$

$$E_{\text{total}} = \int W(x,\nu)\,dx\,d\nu.$$

The complex amplitude can be recovered apart from a constant phase factor. We use the function F, which can be derived from the WDF by eq. (3), with a special argument:

$$u(x) = \exp(i\varphi_0)\, F(x/2,x)/\sqrt{F(0,0)}, \qquad (10)$$

$$\exp(i\varphi_0) = u(0)/|u(0)|.$$

The frequency spectrum $\tilde{u}(\nu)$ can be obtained similarly from $\tilde{F}(\mu,\nu) = \tilde{u}(\nu+\mu/2)\, \tilde{u}^*(\nu-\mu/2)$. Further properties of the WDF can be found in the papers by de Bruijn [2] and Bastiaans [3]. According to [3], the Fraunhofer diffraction is represented by a 90° rotation of the WDF. Fresnel diffraction corresponds to a shearing of the WDF.

Three simple examples illustrate the meaning of the WDF:

1) Monofrequency (fig. 2a)

$$u(x) = \exp(2\pi i\nu_0 x) \rightarrow W(x,\nu) = \delta(\nu-\nu_0)$$

2) Pulse (fig. 2b)

$$u(x) = \delta(x-x_0) \rightarrow W(x,\nu) = \delta(x-x_0)$$

3) Linearly increasing frequency (fig. 2c)

$$u(x) = \exp[2\pi i(\tfrac{1}{2}a\,x^2 + bx + c)] \rightarrow W(x,\nu) = \delta(ax+b-\nu).$$

Here $\delta(x)$ denotes the Dirac delta function.

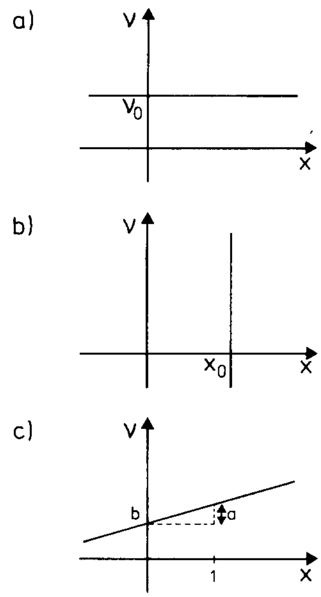

Fig. 2. The Wigner distribution function of some simple signals. a) Monofrequency. b) Pulse. c) Linearly increasing frequency.

These examples suggest to interpret the WDF as a function, which is able to give information about an infinitely small space-frequency spot. Of course this interpretation cannot be legal because it would be in conflict with the uncertainty principle. It is impossible to determine the frequency of a signal within an infinitely small space interval. For an interpretation of the WDF, one might compare it to a probability density function $p(z)$. The value p itself has no significant meaning, but

$$P(z) = \int_{z-\Delta z}^{z+\Delta z} p(z')\,\mathrm{d}z'$$

is the probability that the stochastic variable z' lies between $z-\Delta z$ and $z+\Delta z$.

In this way the local spectrum, which describes the frequency content of a signal depending on position, can be derived from the WDF by integration over an area $\delta x \delta \nu$. Instead of sharp boarders we take a weighting function

$$G(x;\delta x)\, G(\nu;\delta\nu)$$

with

$$G(x;\delta x) = 1/\sqrt{\delta x}\, \exp[-2\pi(x/\delta x)^2],$$

$$G(\nu;\delta\nu) = 1/\sqrt{\delta\nu}\, \exp[-2\pi(\nu/\delta\nu)^2]. \qquad (11)$$

If we choose the product $\delta x \delta\nu \geqslant 1$, the uncertainty principle is satisfied. In the continuing we will use the minimal uncertainty $\delta x \delta \nu = 1$. We will see that the resulting quantity:

$$\hat{W}(x,\nu;\delta x) = \iint W(x',\nu') \times G(x-x';\delta x)\, G(\nu-\nu';1/\delta x)\,\mathrm{d}x'\mathrm{d}\nu', \qquad (12)$$

is positive and has the wanted physical meaning. We insert the former WDF definition eq. (1), (2) into the formula of the modified WDF (eq. (12)) and obtain:

$$\hat{W}(x,\nu;\delta x) = \left| \int u(x') \times G(x-x';\sqrt{2}\delta x)\, \exp(-2\pi \mathrm{i}\nu x')\,\mathrm{d}x' \right|^2, \qquad (13)$$

$$\hat{W}(x,\nu;\delta x) = \left| \int \tilde{u}(\nu') \times G(\nu-\nu';\sqrt{2}/\delta x)\, \exp(2\pi \mathrm{i}\nu' x)\,\mathrm{d}\nu' \right|^2. \qquad (14)$$

Fig. 3 illustrates qualitatively the effect of different sizes of δx in the space-frequency domain (top) (eq. 12) and in the space domain (bottom) (eq. (13)).

The two extreme cases are: good spatial, bad frequency resolution (left); good frequency, bad spatial resolution (right).

For clarification we assume the complex amplitude $u(x)$ to be essentially zero outside of $-\Delta x_0/2 < x <$

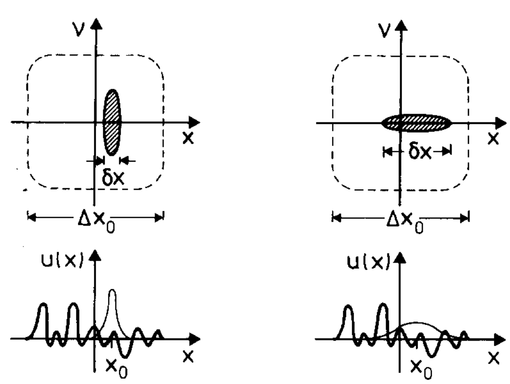

Fig. 3. Illustration of the relationship between u, W and $\hat{W}$ according to eqs. (12), (13). The dotted lines (upper half) are the boundaries of W.

$\Delta x_0/2$. The resolution length δx_0 is the inverse of the bandwidth $\Delta\nu_0 = 1/\delta x_0$. Conversely the frequency resolution $\delta\nu_0$ of $\tilde{u}(\nu)$ is $1/\Delta x_0$. If the question to be asked requires spatial sharpness as good as possible we set $\delta x \ll \delta x_0$. The gaussian in eq. (13) is almost like a delta function. Hence we get:

$$\hat{W}(x,\nu;\delta x) \approx |u(x)|^2. \tag{15}$$

The other extreme case is $\delta x > \Delta x_0$, yielding optimal sharpness in the frequency domain. Now eq. (14) produces:

$$\hat{W}(x,\nu;\delta x) \approx |\tilde{u}(\nu)|^2. \tag{16}$$

The in-between-case $\delta x_0 < \delta x < \Delta x_0$ can be understood as "local spatial frequency spectrum".

3. Experimental setups for producing the local spectrum

The definition of the local spectrum (eq. (13)) leads directly to the first method for producing $\hat{W}(x,\nu;\delta x)$. As shown in fig. 4 the signal is multiplied with the gaussian window. The product is Fourier transformed in a coherent 2-f setup. The intensity can be recorded on photographic film. Moving the signal transparency and the recording film simultaneously provides the variation of the position coordinate x. In our experiment the movements were synchronized by two coupled stepmotors. After a scan over the whole signal transparency, $\hat{W}(x,\nu;\delta x)$ is recorded on the film. This method is convenient if the signal $u(x)$ is traveling by its own nature like $u(x-vt)$. For realtime applications we developed a second setup (fig. 5). Here the gaussian transmittance G is rotated by an angle of 45°.

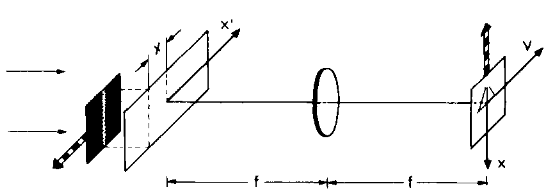

Fig. 4. First setup for producing the local spectrum $\hat{W}$. The gaussian slit (left) moves horizontally, the recording plate (right) vertically.

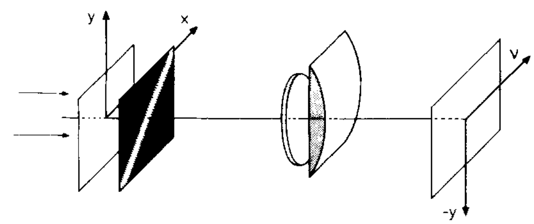

Fig. 5. Second setup for producing the local spectrum $\hat{W}$. No part is moving.

A one-dimensional Fourier transformation together with the modulus square by recording the intensity again yields the local spectrum:

$$\left|\int u(x)\, G(x-y;\sqrt{2}\delta x) \times \exp(-2\pi i \nu x)\,\mathrm{d}x\right|^2 = \hat{W}(y,\nu;\delta x). \tag{17}$$

$(1+\cos(\omega_0 x))\,\mathrm{rect}(x/\Delta x)$

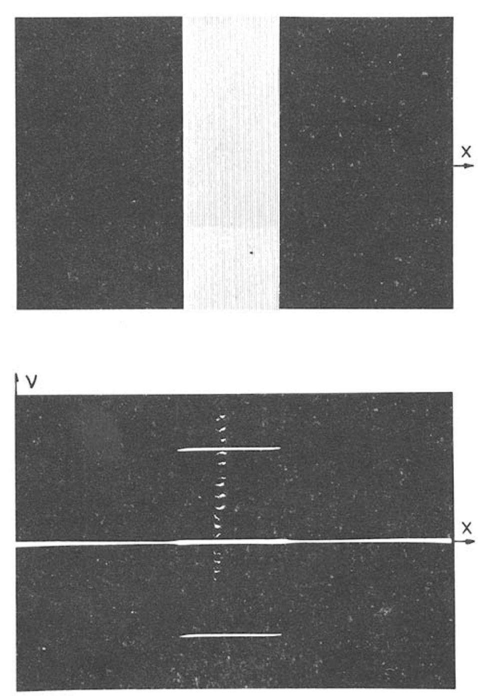

Fig. 6. Test signal (monofrequency of finite duration) and local spectrum $\hat{W}$.

A variation of the effective gaussian width δx can be achieved simply by changing the angular orientation of the gaussian slit in fig. 5.

4. Experimental results

With the setups discussed in the last chapter we produced the local spectra of some test signals and as a practical example the local spectra of speech signals. For this reason the analytic signal has been transformed into an optical transmittance, where transmittance $T = 1$ corresponds to maximum signal and $T = 0$ to minimum signal. In fig. 6–10 the signal is always shown above the associated local spectrum. Fig. 6 represents a sinusoidal signal of finite duration. The space-frequency representation is able to answer questions concerning the spatial behavior as well as the frequency content. On fig. 7 we used a sinusoidal grating with linear-

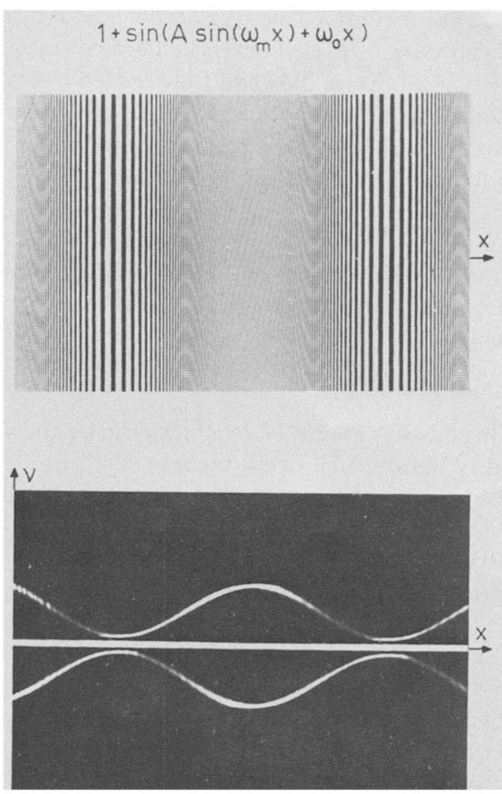

Fig. 7. Test signal (linearly increasing frequency) and local spectrum $\hat{W}$.

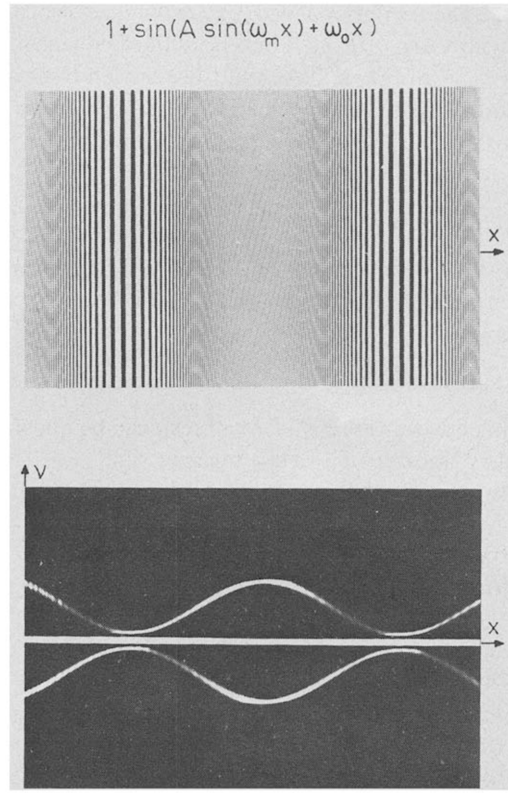

Fig. 8. Test signal (sinusoidal frequency modulation) and local spectrum $\hat{W}$.

ly increasing frequency. In this representation the frequency can be determined for every position.

The last test signal in fig. 8 is again a cosine grating, but the frequency of this grating now varies sinusoidally. The Wigner representation again shows very clearly the frequency variation depending on position. The next three experiments (figs. 9, 10, 11) were performed with sound patterns $u(t)$. These time signals have been transformed into spatial signals $u(x)$. For this purpose they were biased and transferred onto photographic film. These transparencies served as input signal for our optical transformation. The local spectra are distinctly different if the same speaker utters different words (figs. 9, 10) or if different speakers pronounce the same word (fig. 11). Hence the observed local spectra should be suitable as inputs both for speech recognition and for speaker identification.

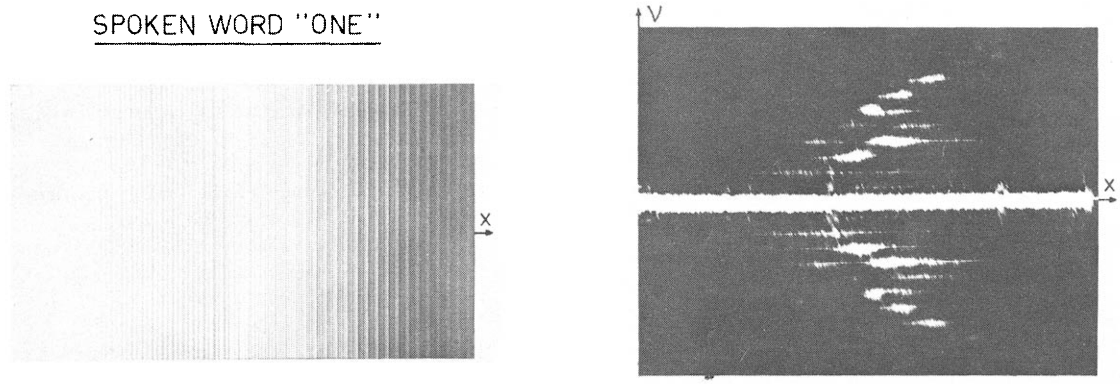

Fig. 9. Signal $u(x)$ of spoken word "one" and $\hat{W}(x,\nu;\delta x)$.

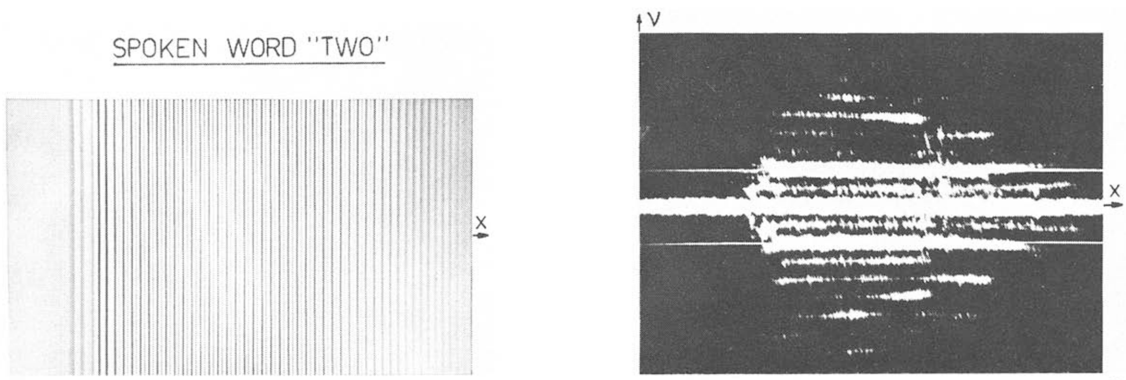

Fig. 10. Signal $u(x)$ of spoken word "two" and $\hat{W}(x,\nu;\delta x)$.

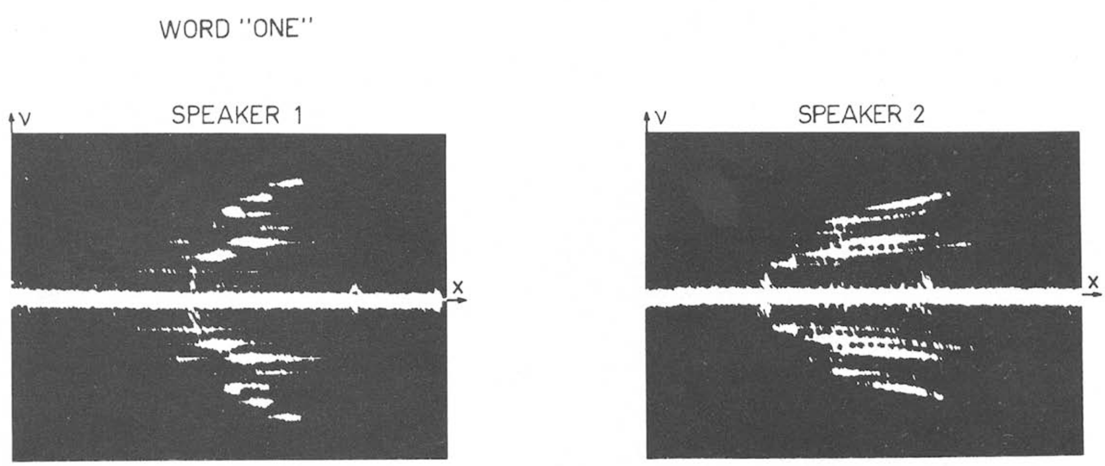

Fig. 11. The local spectra $\hat{W}(x,\nu;\delta x)$ of the word "one", spoken by different persons.

5. Conclusions

We showed the Wigner distribution function to be a very profound and useful form for describing physical signals. The "local spectrum" has been derived from the Wigner distribution function. This was the base for two new optical methods, that could be useful for speech recognition and speaker identification. Beyond this particular application the WDF is quite suitable for handling many optical signal processing problems, as has been shown recently by Bastiaans [4], whose valuable comments are gratefully acknowledged.

References

[1] E. Wigner, Phys. Rev. 40 (1932) 749.
[2] N.G. de Bruijn, in: Inequalities, ed. O. Shisha (Academ. Press, New York, 1967) p. 57–71.
[3] M.J. Bastiaans, Optics Comm. 25 (1978) 26.
[4] M.J. Bastiaans, Transport equations for the Wigner distribution function, to appear, Optica Acta;
M.J. Bastiaans, Transport equations for the Wigner distribution function in an inhomogeneous and dispersive medium, Optica Acta, to appear;
M.J. Bastiaans, Optics Comm. 30 (1979) 321;
M.J. Bastiaans, The Wigner distribution function and its application to first-order optics, Optics Comm., to be submitted.