Non-Hermitian Quantum Mechanics

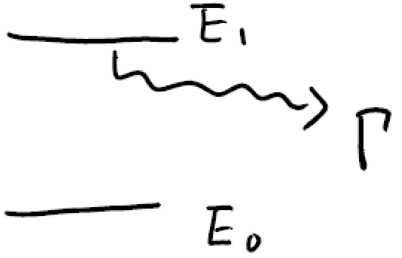
Chao Zhuang
QO Group meeting
Oct. 5th, 2011

Outline

- Simple non-Hermitian Hamiltonian
  - Exceptional Point
- Origins for non-Hermitian operator
  - What kind of problems are considered?
- Wannier-Stark Resonance*
- Hermitian property depends on the basis set?

Simple non-Hermitian Hamiltonian

$$H = \begin{bmatrix} E_1 - i\Gamma & 0 \\ 0 & E_0 \end{bmatrix}$$



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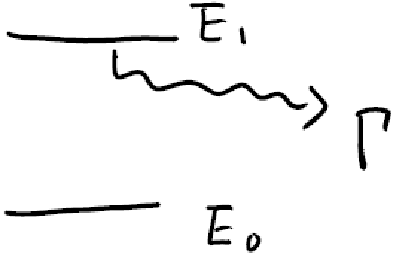
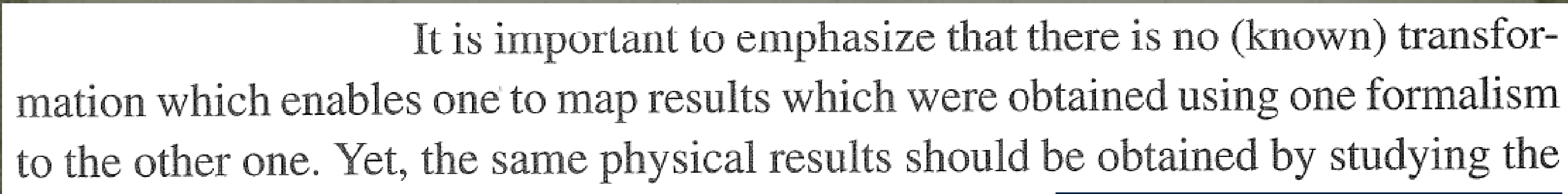


Master equation $$\partial_t \tilde{\rho} = -\frac{i}{\hbar}\left[\tilde{H}, \tilde{\rho}\right] + \Gamma \mathcal{D}[\sigma]\tilde{\rho}.$$

Superoperator

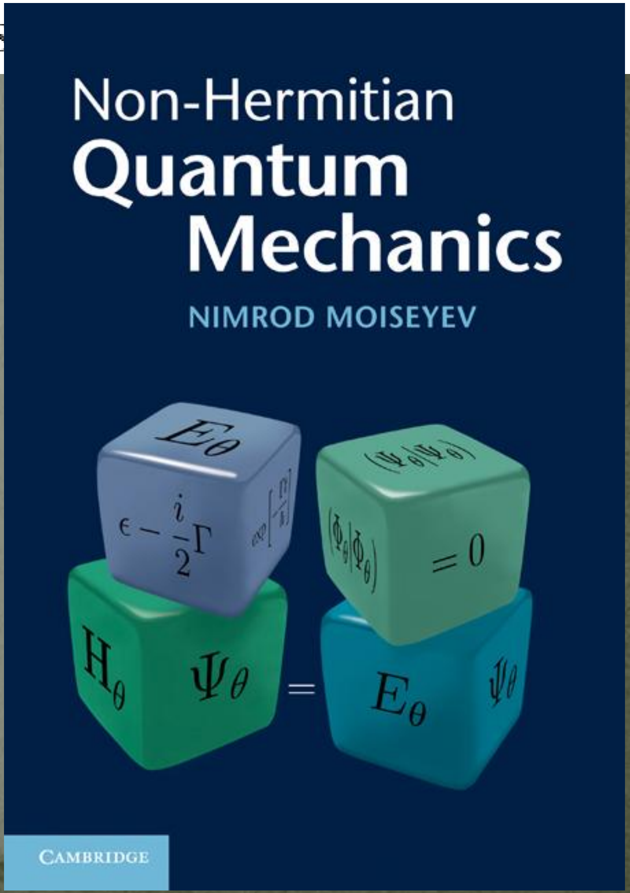
An alternative formalism



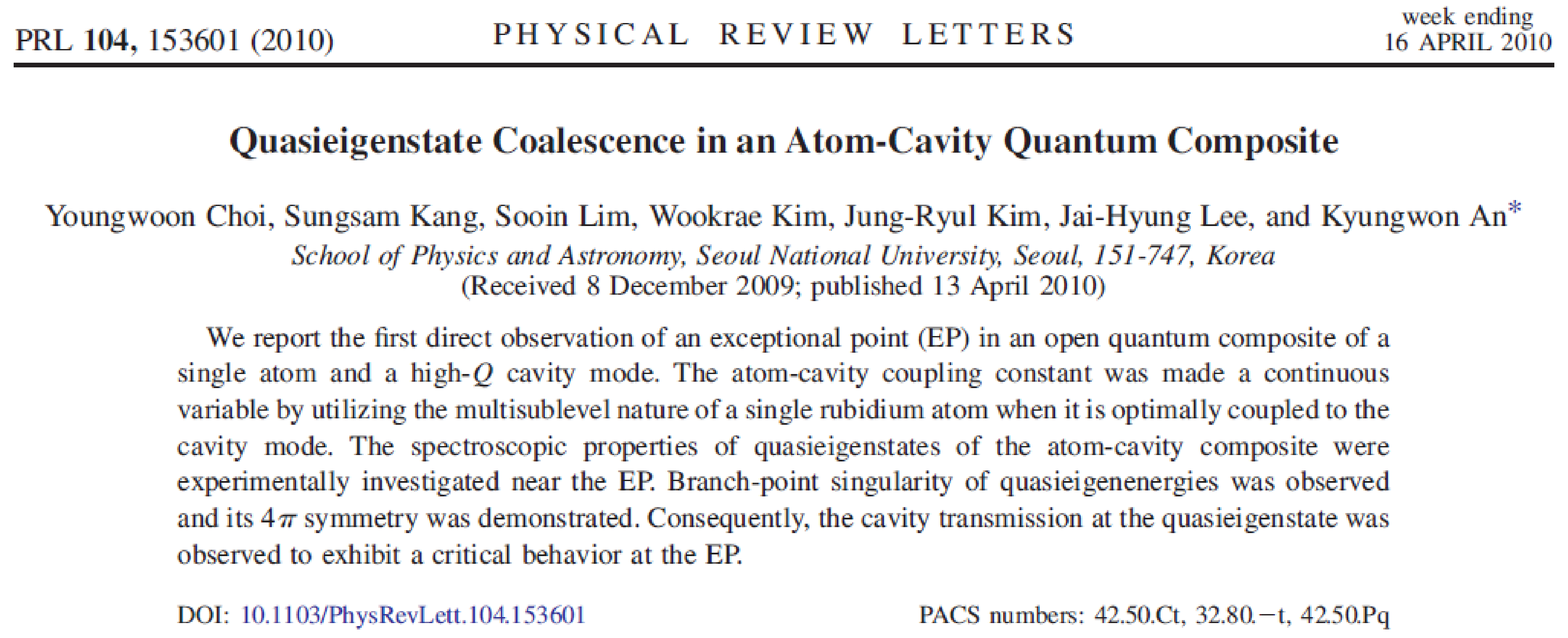
It is important to emphasize that there is no (known) transformation which enables one to map results which were obtained using one formalism to the other one. Yet, the same physical results should be obtained by studying the same phenomenon using the two formalisms. If this is

An alternative way
to do the calculation,
probably easier

Other advantages…



Exceptional point



PRL **104,** 153601 (2010) PHYSICAL REVIEW LETTERS week ending 16 APRIL 2010

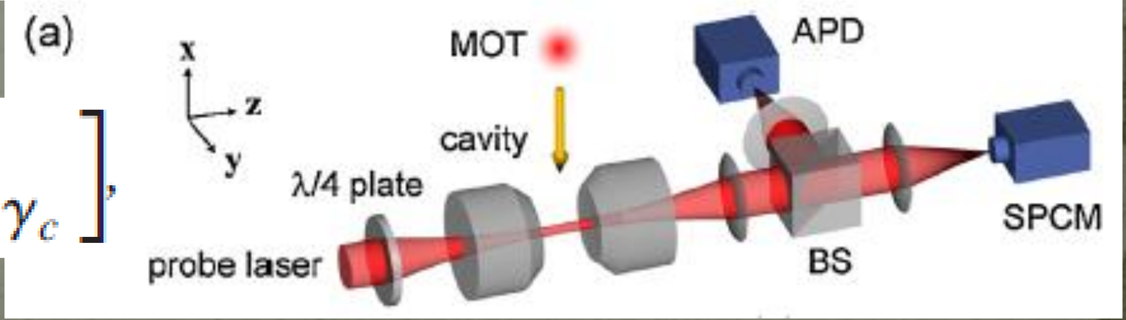
Quasieigenstate Coalescence in an Atom-Cavity Quantum Composite

Youngwoon Choi, Sungsam Kang, Sooin Lim, Wookrae Kim, Jung-Ryul Kim, Jai-Hyung Lee, and Kyungwon An[*]
School of Physics and Astronomy, Seoul National University, Seoul, 151-747, Korea
(Received 8 December 2009; published 13 April 2010)

We report the first direct observation of an exceptional point (EP) in an open quantum composite of a single atom and a high-Q cavity mode. The atom-cavity coupling constant was made a continuous variable by utilizing the multisublevel nature of a single rubidium atom when it is optimally coupled to the cavity mode. The spectroscopic properties of quasieigenstates of the atom-cavity composite were experimentally investigated near the EP. Branch-point singularity of quasieigenenergies was observed and its 4π symmetry was demonstrated. Consequently, the cavity transmission at the quasieigenstate was observed to exhibit a critical behavior at the EP.

DOI: 10.1103/PhysRevLett.104.153601 PACS numbers: 42.50.Ct, 32.80.−t, 42.50.Pq

$$H' = \hbar \begin{bmatrix} \omega_a - i\gamma_a & g \\ g & \omega_c - i\gamma_c \end{bmatrix},$$



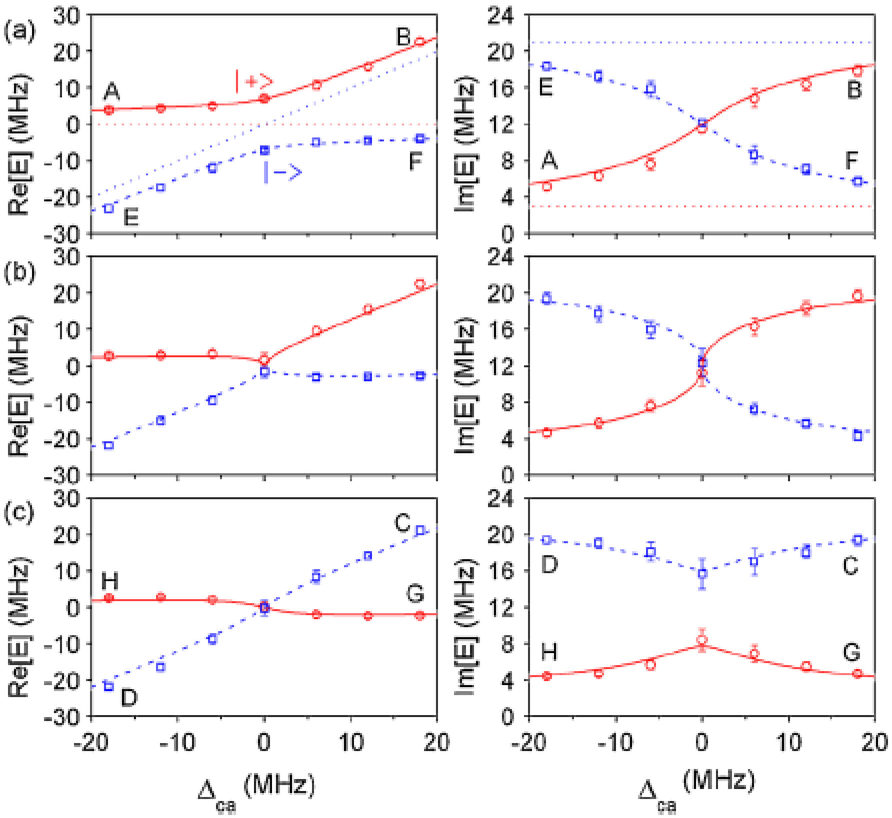


FIG. 2 (color online). Real and imaginary parts of the observed quasieigenenergies. (a) $g/\gamma_- = 1.23 \pm 0.04$ (strong coupling), (b) $g/\gamma_- = 1.01 \pm 0.04$ (EP condition), and (c) $g/\gamma_- = 0.90 \pm 0.03$ (weak coupling). Solid and dashed lines are theoretical predictions, not fits, for the experiment. Dotted lines in (a) represent the case of no coupling.

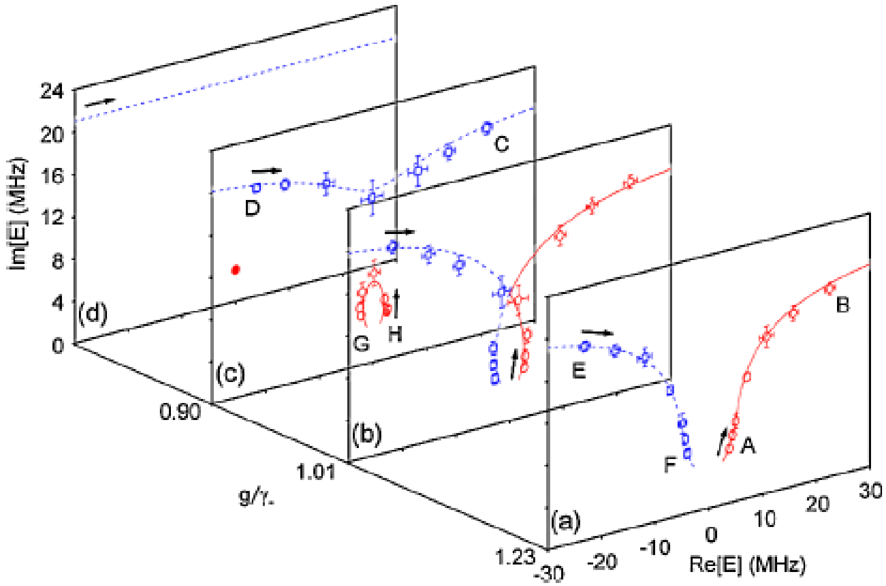


FIG. 3 (color online). Complex eigenenergy trajectories in the strong coupling (a) and in the weak coupling regimes (c). The arrows indicate the direction of progression of the eigenenergies when Δ_{ca} is scanned from -18 to $+18$ MHz. (b) Under the EP condition, the two trajectories are joined at one point, indicating coalescence of two eigenstates into one. (d) The case of no coupling.

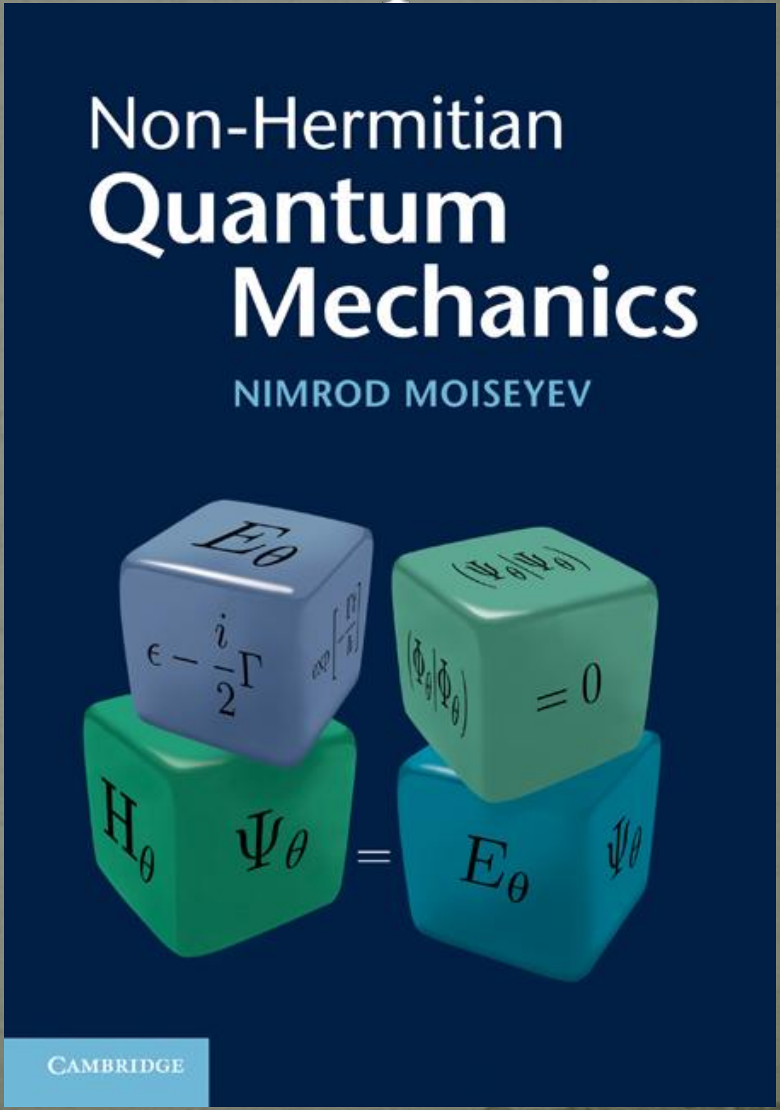
Goodies

- Quantum computer
  - Topological gates
- Different ways to do adiabatic passage
- PT-symmetry
- Time reversal symmetry

But why another formalism?

Origins for non-Hermitian operator

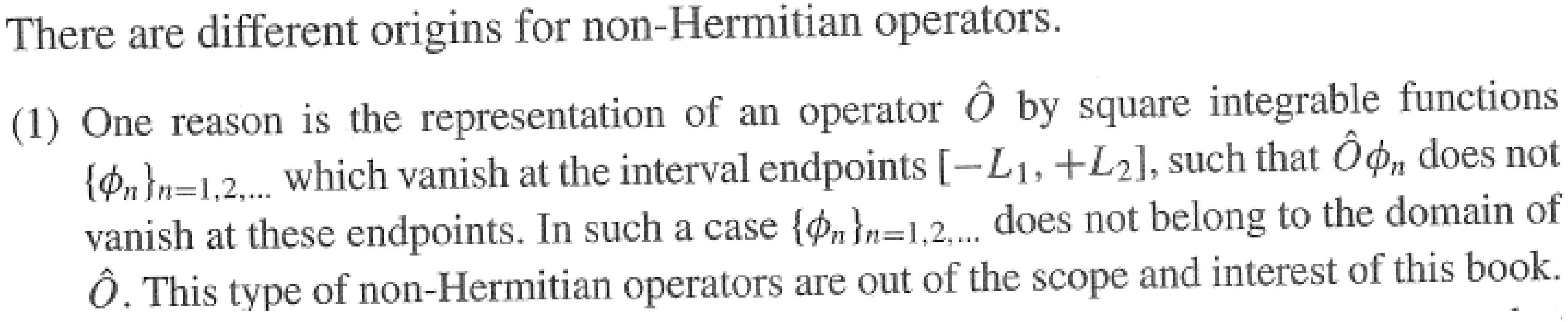
- What kind of problems?
  - 1…
  - 2…
  - 3…



Non-Hermitian
Quantum
Mechanics

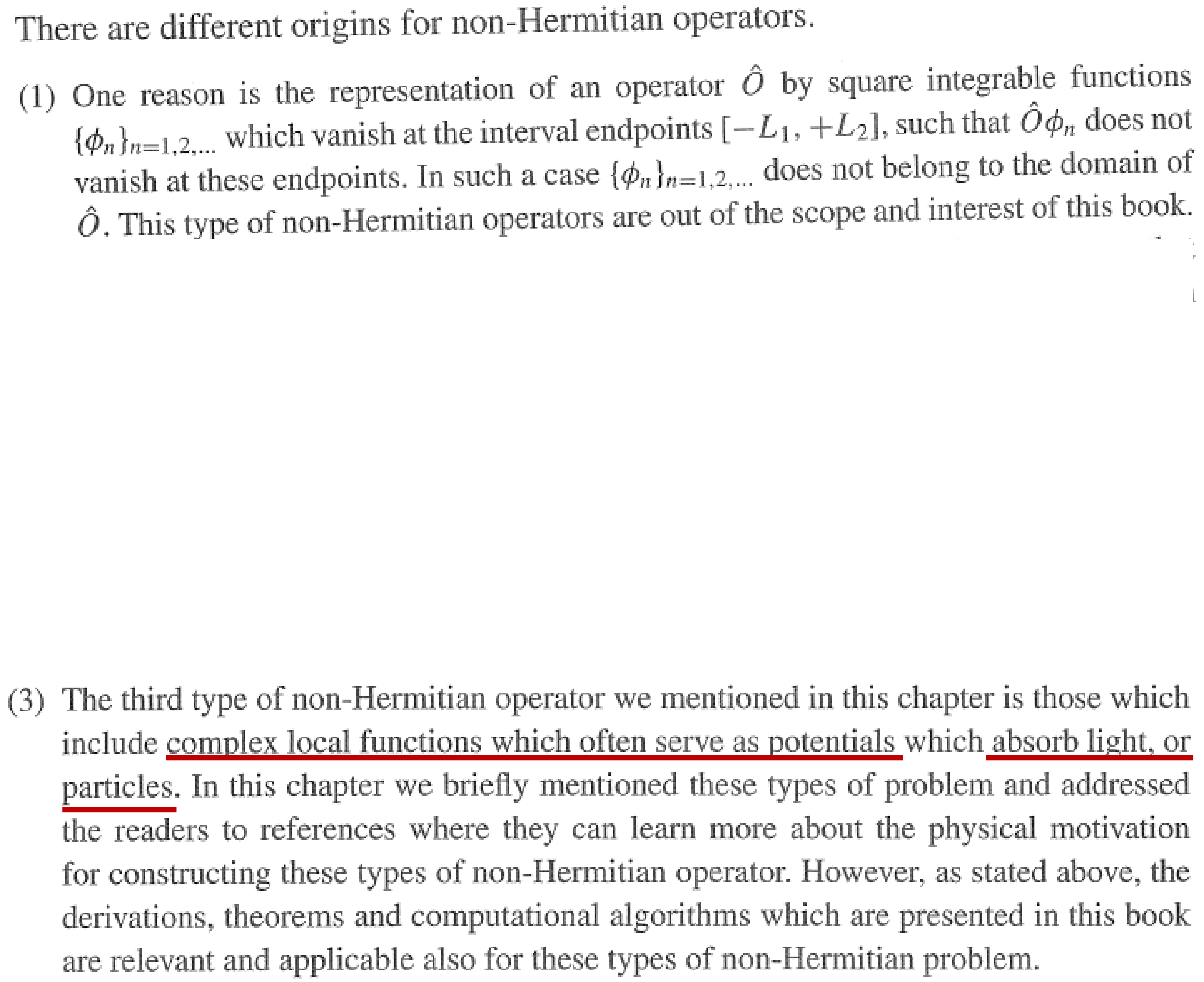
NIMROD MOISEYEV

CAMBRIDGE



There are different origins for non-Hermitian operators.

(1) One reason is the representation of an operator $\hat{O}$ by square integrable functions $\{\phi_n\}_{n=1,2,...}$ which vanish at the interval endpoints $[-L_1, +L_2]$, such that $\hat{O}\phi_n$ does not vanish at these endpoints. In such a case $\{\phi_n\}_{n=1,2,...}$ does not belong to the domain of $\hat{O}$. This type of non-Hermitian operators are out of the scope and interest of this book.



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(3) The third type of non-Hermitian operator we mentioned in this chapter is those which include complex local functions which often serve as potentials which absorb light, or particles. In this chapter we briefly mentioned these types of problem and addressed the readers to references where they can learn more about the physical motivation for constructing these types of non-Hermitian operator. However, as stated above, the derivations, theorems and computational algorithms which are presented in this book are relevant and applicable also for these types of non-Hermitian problem.

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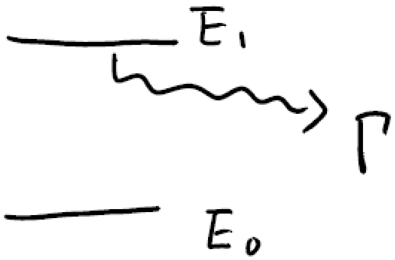

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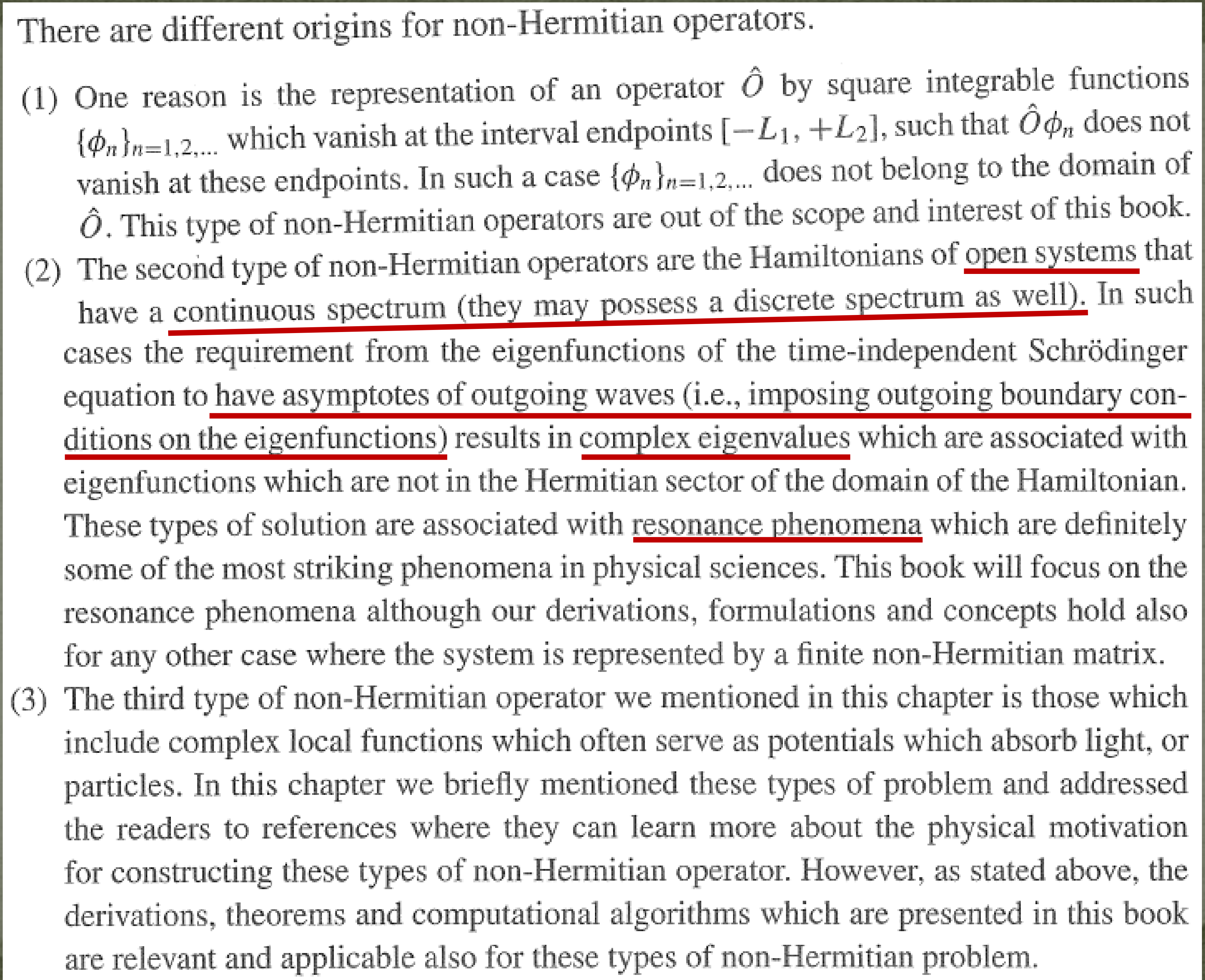

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(2) The second type of non-Hermitian operators are the Hamiltonians of open systems that have a continuous spectrum (they may possess a discrete spectrum as well). In such cases the requirement from the eigenfunctions of the time-independent Schrödinger equation to have asymptotes of outgoing waves (i.e., imposing outgoing boundary conditions on the eigenfunctions) results in complex eigenvalues which are associated with eigenfunctions which are not in the Hermitian sector of the domain of the Hamiltonian. These types of solution are associated with resonance phenomena which are definitely some of the most striking phenomena in physical sciences. This book will focus on the resonance phenomena although our derivations, formulations and concepts hold also for any other case where the system is represented by a finite non-Hermitian matrix.

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Resonance & Quasi-bound state?

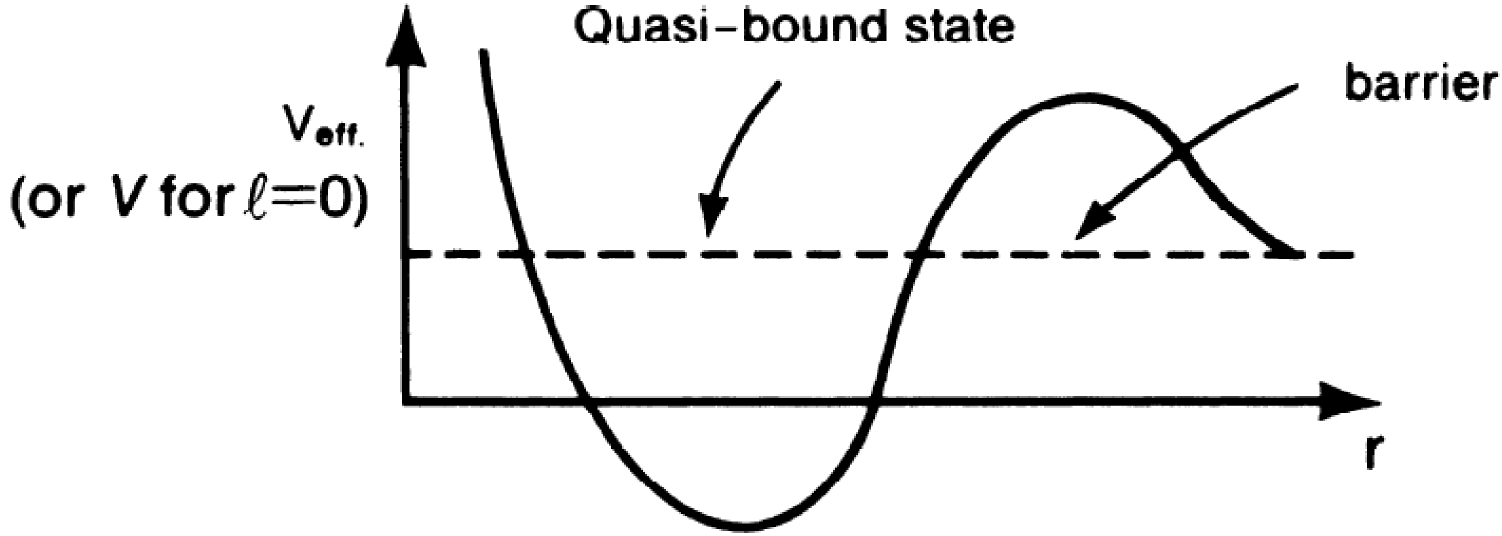
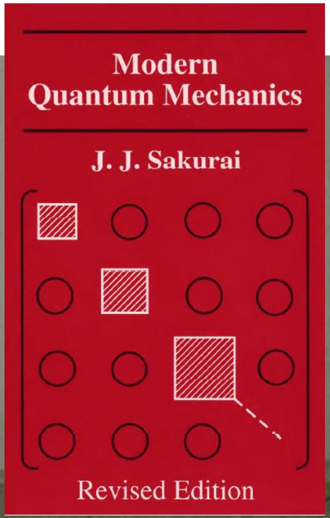


FIGURE 7.11. $V_{\text{eff}} = V(r) + (\hbar^2/2m)[l(l+1)/r^2]$ versus r. For $l \neq 0$ the barrier can be due to $(\hbar^2/2m)[l(l+1)/r^2]$; for $l = 0$ barrier must be due to V itself.

Particle cannot be trapped forever



Bound state V.S. Quasi-bound state

- Physically: forever and ever … ever …

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Bound States as Poles of $S_l(k)$

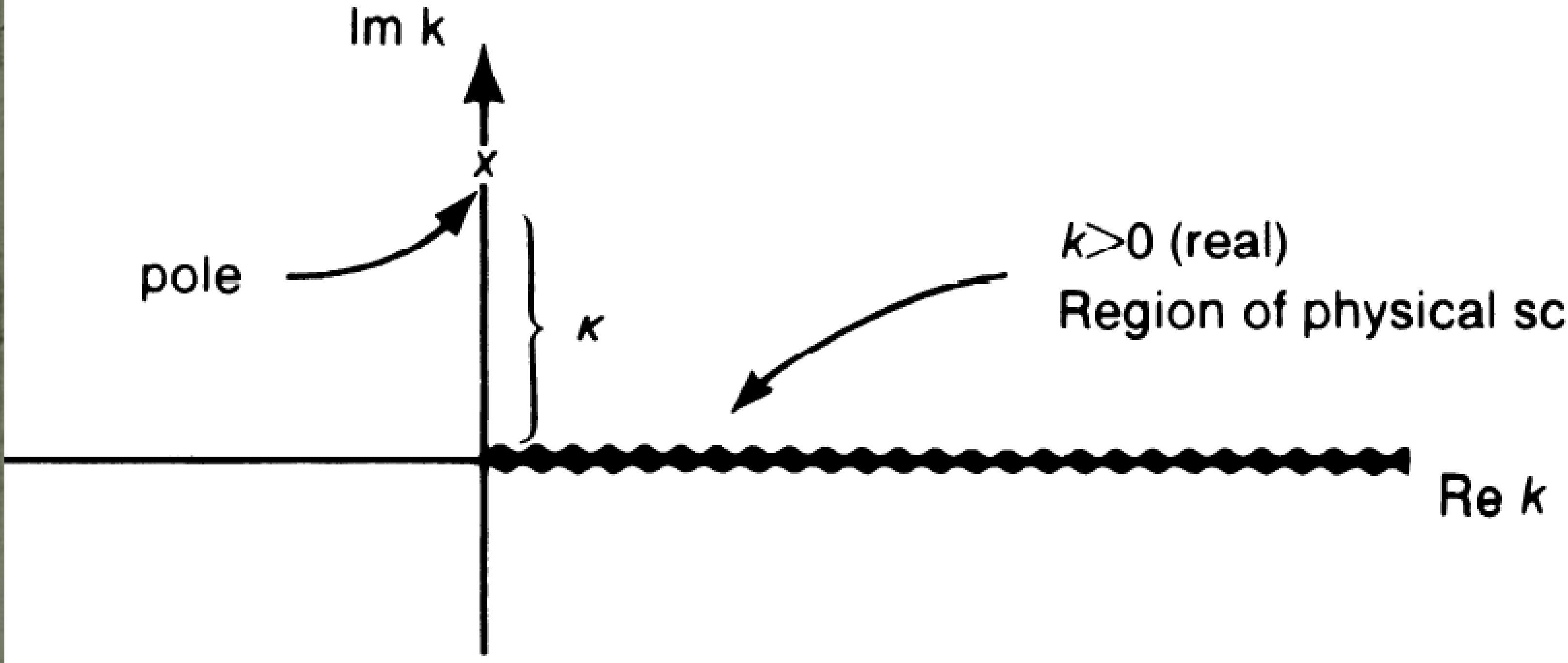
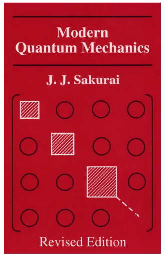


FIGURE 7.10. The complex k-plane with bound-state pole at $k = +i\kappa$.


Modern
Quantum Mechanics
J. J. Sakurai
Revised Edition

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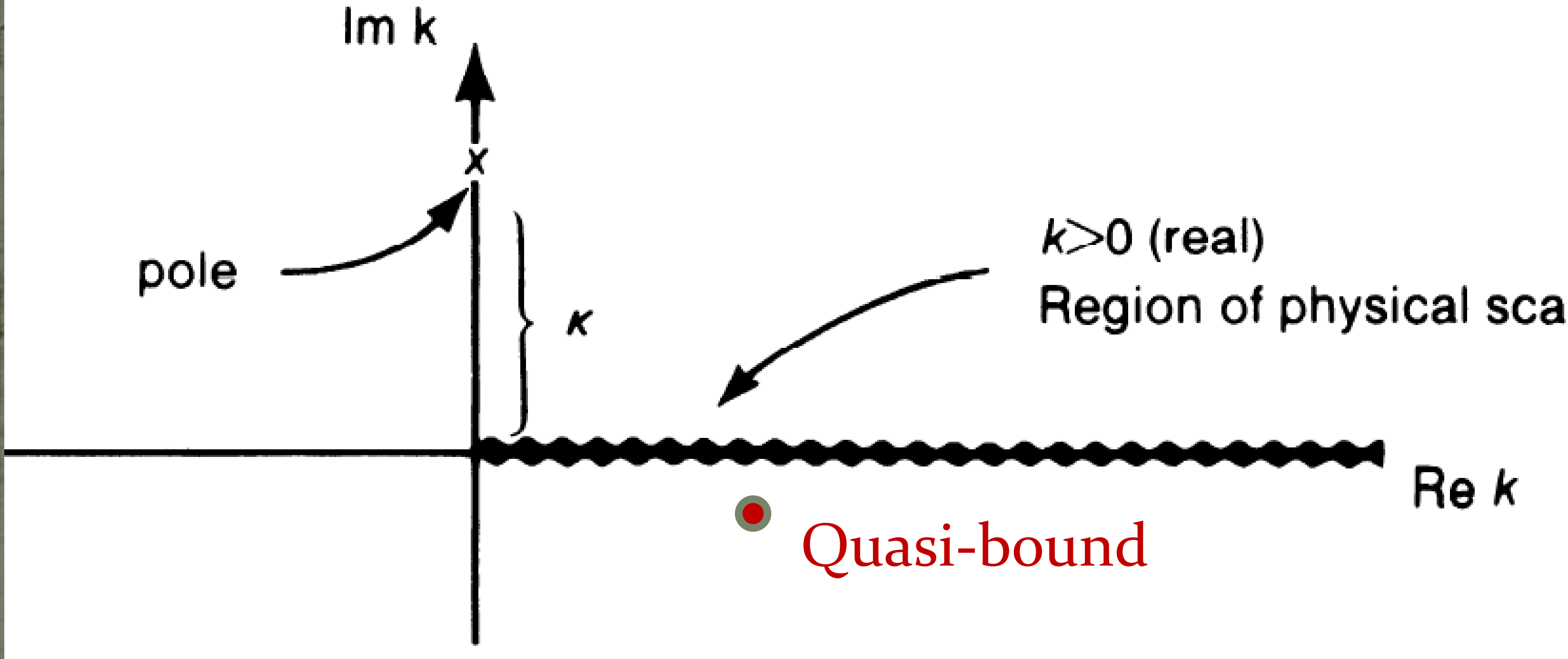

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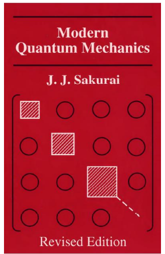
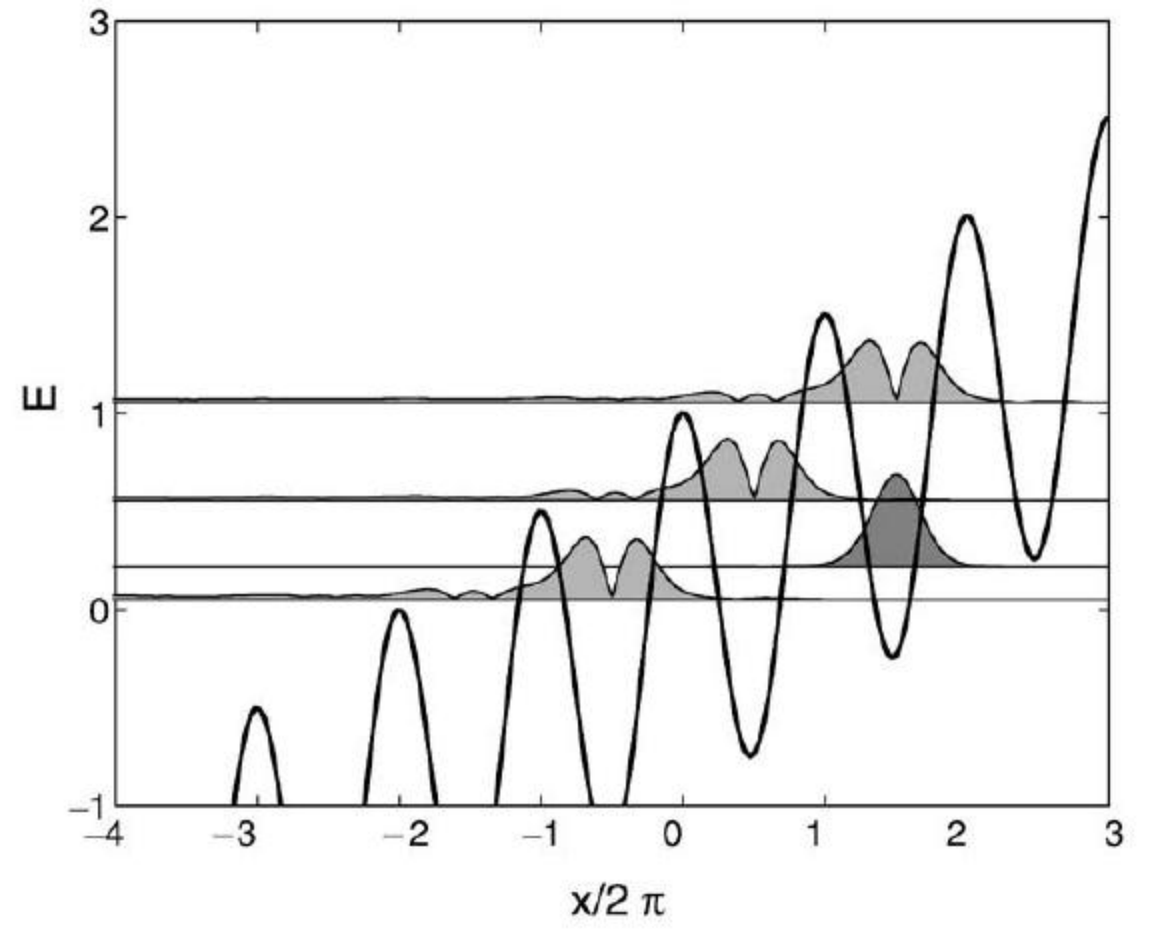


Wannier-Stark Resonance

$$H_{\mathrm{W}} = \frac{p^2}{2} + V(x) + Fx, \quad V(x + 2\pi) = V(x) \ .$$

Hermitian?

Non-Hermitian?



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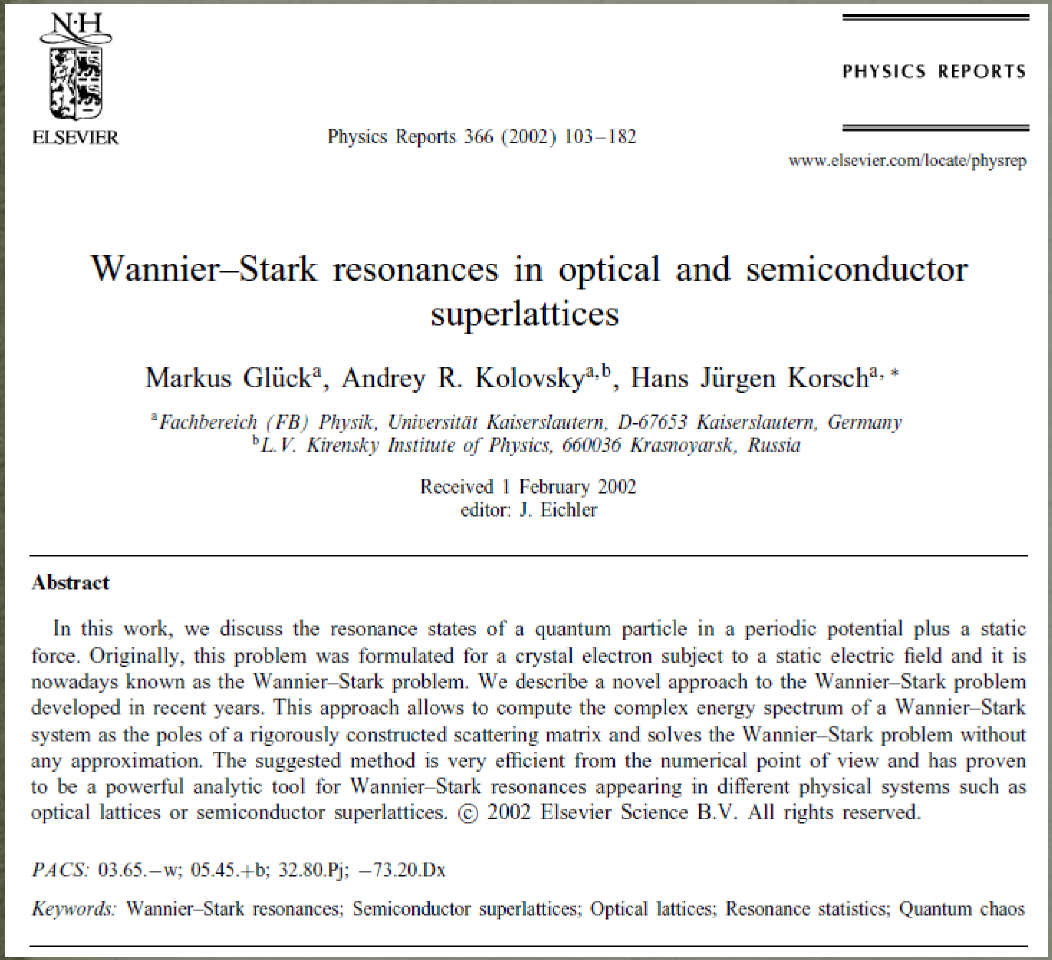

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Wannier–Stark resonances in optical and semiconductor superlattices

Markus Glück[a], Andrey R. Kolovsky[a,b], Hans Jürgen Korsch[a,*]

[a] *Fachbereich (FB) Physik, Universität Kaiserslautern, D-67653 Kaiserslautern, Germany*
[b] *L.V. Kirensky Institute of Physics, 660036 Krasnoyarsk, Russia*

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Abstract

In this work, we discuss the resonance states of a quantum particle in a periodic potential plus a static force. Originally, this problem was formulated for a crystal electron subject to a static electric field and it is nowadays known as the Wannier–Stark problem. We describe a novel approach to the Wannier–Stark problem developed in recent years. This approach allows to compute the complex energy spectrum of a Wannier–Stark system as the poles of a rigorously constructed scattering matrix and solves the Wannier–Stark problem without any approximation. The suggested method is very efficient from the numerical point of view and has proven to be a powerful analytic tool for Wannier–Stark resonances appearing in different physical systems such as optical lattices or semiconductor superlattices.

PACS: 03.65.−w; 05.45.+b; 32.80.Pj; −73.20.Dx

Keywords: Wannier–Stark resonances; Semiconductor superlattices; Optical lattices; Resonance statistics; Quantum chaos

$$S(E) = \lim_{k\to\infty} \frac{\Psi_S(-k;E)}{\Psi_0(-k;E)} \frac{\Psi_0(k;E)}{\Psi_S(k;E)}$$

$$H_W \Psi_S(E) = E\, \Psi_S(E)$$

If it’s non-Hermitian, E can be complex

Non-Hermitian?

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x}) \qquad V(\hat{x}) \ \text{real}$$

Hermitian Hamiltonian

$$\hat{H}^{+} = \hat{H}$$

$$\Rightarrow \quad \forall\ |f\rangle, |g\rangle, \quad \langle f|\hat{H}^{+}|g\rangle = \langle f|\hat{H}|g\rangle$$

$$\Rightarrow \quad \forall\ |f\rangle, |g\rangle, \quad \langle f|\hat{H}|g\rangle = (\langle g|\hat{H}|f\rangle)^{*}$$

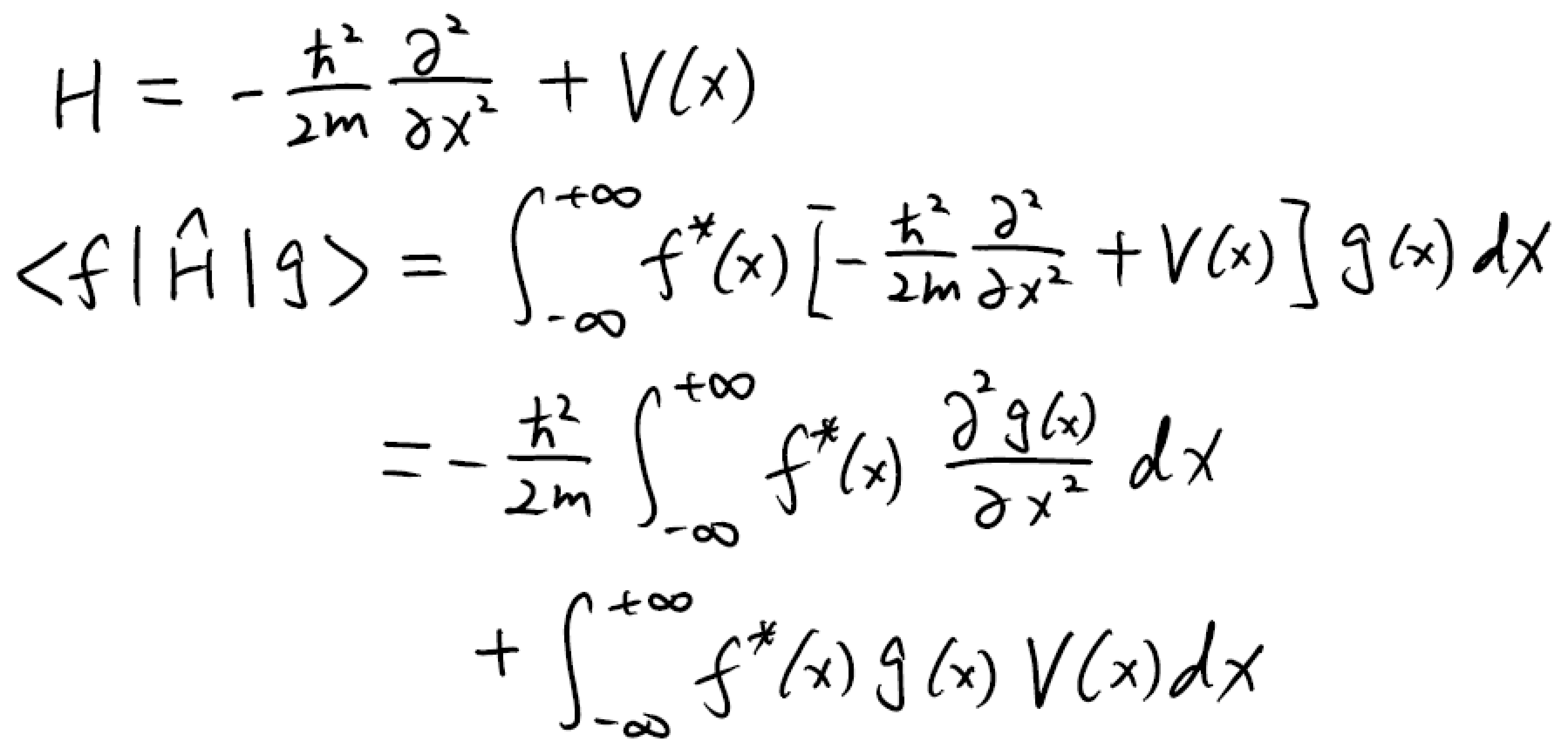
$$\hat{H} = \frac{\hat{p}^{2}}{2m} + V(\hat{x}) \qquad V(\hat{x}) \text{ real}$$

$$\langle f|\hat{H}|g\rangle - (\langle g|\hat{H}|f\rangle)^{*} = 0$$

Hermitian Property

- “the Hermitian property of an operator is heavily dependent on the basis set which is used to represent a dynamical variable by matrices of infinite order”
- “the Hermitian property of …… depends also on the boundary conditions of ……”
- And here’s an example ……

In x-representation



$$H = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)$$

$$\langle f|\hat{H}|g\rangle = \int_{-\infty}^{+\infty} f^*(x)\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)\right] g(x)\,dx$$

$$= -\frac{\hbar^2}{2m}\int_{-\infty}^{+\infty} f^*(x)\,\frac{\partial^2 g(x)}{\partial x^2}\,dx$$

$$+ \int_{-\infty}^{+\infty} f^*(x)\,g(x)\,V(x)\,dx$$

In x-representation

$$\langle g|\hat{H}|f\rangle = -\frac{\hbar^2}{2m}\int_{-\infty}^{+\infty} g^*(x)\,\frac{\partial^2 f(x)}{\partial x^2}\,dx$$

$$+\int_{-\infty}^{+\infty} f(x)\,g^*(x)\,V(x)\,dx$$

$$\left(\langle g|\hat{H}|f\rangle\right)^* = -\frac{\hbar^2}{2m}\int_{-\infty}^{+\infty} g(x)\,\frac{\partial^2 f^*(x)}{\partial x^2}\,dx$$

$$+\int_{-\infty}^{+\infty} f^*(x)\,g(x)\,V(x)\,dx$$

$$\langle f|\hat{H}|g\rangle - (\langle g|\hat{H}|f\rangle)^*$$

$$= \frac{\hbar^2}{2m}\int_{-\infty}^{+\infty}\left[g(x)\frac{\partial^2 f^*(x)}{\partial x^2} - f^*(x)\frac{\partial^2 g(x)}{\partial x^2}\right]dx$$

$$= \frac{\hbar^2}{2m}\int_{-\infty}^{+\infty}\frac{\partial}{\partial x}\left[g(x)\frac{\partial f^*(x)}{\partial x} - f^*(x)\frac{\partial g(x)}{\partial x}\right]dx$$

$$= \frac{\hbar^2}{2m}\left[g(x)\frac{\partial f^*(x)}{\partial x} - f^*(x)\frac{\partial g(x)}{\partial x}\right]\Bigg|_{-\infty}^{+\infty}$$

$$= 0$$

$$\Rightarrow \left[g(x)\frac{\partial f^*(x)}{\partial x} - f^*(x)\frac{\partial g(x)}{\partial x}\right]\Bigg|_{-\infty}^{+\infty} = 0$$

The end.