

# Two-Fluid Hydrodynamics in Trapped Bose Gases and in Superfluid Helium

Allan Griffin and Tetsuro Nikuni

*Department of Physics, University of Toronto  
Toronto, Ontario M5S 1A7, Canada*

*A review is given of recent theoretical work on the superfluid dynamics of trapped Bose gases at finite temperatures, where there is a significant fraction of non-condensate atoms. One can now reach large enough densities and collision cross-sections needed to probe the collective modes in the collision-dominated hydrodynamic region where the gas exhibits characteristic superfluid behavior involving the relative motions of the condensate and non-condensate components. The precise analogue of the Landau-Khalatnikov two-fluid hydrodynamic equations was recently derived from trapped Bose gases, starting from a generalized Gross-Pitaevskii equation for the condensate macroscopic wavefunction and a kinetic equation for the non-condensate atoms.*

*PACS numbers: 03.75.Fi, 05.30.Jp, 67.40.Bz, 67.40-w.*

## 1. INTRODUCTION

Superfluid behavior is the most striking property of liquid  $^4\text{He}$ . We recall that spectacular experiments of Kapitza as well as Allen and Misener in 1938 first showed that liquid  $^4\text{He}$  below the transition temperature of 2.19K could exhibit flow through thin channels without any viscosity. Since then, a dominant theme of research on liquid  $^4\text{He}$  has been to understand the origin of this superfluidity and to work out how it modifies the dynamical response functions and excitations of liquid  $^4\text{He}$ .

By the early 1960's, a successful field theoretic formalism was developed,<sup>1,2</sup> with the broken-symmetry expectation value of the quantum field operator  $\Phi(\mathbf{r}, t) = \langle \hat{\psi}(\mathbf{r}, t) \rangle$  playing the role of the order parameter of the

Bose superfluid. In this theory, Bose-condensation was the underlying source of *all* the unique properties of superfluid  $^4\text{He}$  (equivalence of density fluctuations with elementary excitations, two-fluid hydrodynamical behavior at low frequencies, superfluid flow, quantized vortices, etc). In striking contrast to the atomic Bose gases, the explicit role of the Bose condensate is somewhat hidden in superfluid  $^4\text{He}$ . Indeed, this fact has encouraged the development of theories for superfluid  $^4\text{He}$  based on variational groundstate wavefunctions (beginning with Feynman in the mid-1950's) without any explicit reference to the underlying role of the macroscopic wavefunction describing the condensate. While very useful for computational purposes, these approaches have given little insight into the fact that there is a new phase in liquid  $^4\text{He}$  which exhibits superfluidity. For this reason, they do not appear to give much hope for developing a general theory of the superfluid dynamics of Bose fluids, going from dilute gases to liquid  $^4\text{He}$ .

Such a unified description of this kind has taken on a new importance as a result of spectacular advances that have occurred in BEC research in the last year. These discoveries have started to focus research interest in atomic gases on issues related to superfluidity. The recent creation of quantized vortices (at JILA in Boulder<sup>3</sup> and the ENS in Paris<sup>4</sup>) have suddenly made the BEC community aware of rotating traps, mutual friction, vortex arrays, etc, ideas long familiar to the superfluid  $^4\text{He}$  community. In addition, the successful creation of a BEC in  $^{85}\text{Rb}$  gas at JILA<sup>5</sup> by working close to a Feshbach resonance has given us a superfluid Bose gas where the s-wave scattering length  $a$  can be enormous ( $a > 10^3 \text{ \AA}$ ). Already this has moved the value of the gas parameter ( $na^3$ ) from the range  $\sim 10^{-5}$  (typical for recent BEC experiments<sup>6</sup> using  $^{87}\text{Rb}$  and  $^{23}\text{Na}$  atoms) to a respectable  $10^{-2}$ . Thus atomic condensates are "starting" to overlap on a strongly interacting Bose liquid like  $^4\text{He}$  (where  $na^3 \sim 1$ , with  $a$  now being the hard core diameter).

More generally, the BEC community is increasingly using the language and ideas developed in condensed matter and quantum liquid research. Arguments based on topological arguments are used (untwisting of order parameters,<sup>7</sup> for example). Recent Bragg experiments using second-order Raman scattering of light have allowed one to measure<sup>8</sup> the dynamic structure factor  $S(\mathbf{q}, \omega)$  of trapped Bose gases, a quantity which has played a central role in our understanding of superfluid  $^4\text{He}$  for almost 50 years.<sup>2</sup> In the last few years, much of the theoretical analysis of atomic condensates has come from condensed matter theorists who have worked on superfluid liquids (such as Baym, Fetter, Ho, Leggett, Pethick, Pitaevskii and Stringari). We hope that the superfluid  $^4\text{He}$  experimental community will be stimulated by the new BEC child in their midst and, in particular, come up with new ideas about how to probe the Bose-condensed nature of superfluid  $^4\text{He}$ .

As a contribution to this emerging synthesis, this article reviews recent work<sup>9,10</sup> on deriving (from an approximate but still microscopic model) the two-fluid hydrodynamic equations for a trapped Bose gas at finite temperatures. The equivalent two-fluid equations were first derived phenomenologically by Landau in 1941, initially without any reference to a Bose condensate. Generalized to include damping from transport coefficients, these coupled equations for the superfluid and normal fluid components form the basis of our understanding of all “superfluid” behavior in liquid  $^4\text{He}$ .<sup>11</sup> This two-fluid description was later proven to be a consequence of a Bose broken-symmetry by Bogoliubov<sup>12</sup> in 1963, with the superfluid velocity  $\mathbf{v}_s(\mathbf{r}, t)$  always being proportional to the gradient of the phase of  $\Phi(\mathbf{r}, t)$ .

Of course, the two-fluid hydrodynamic equations describe local equilibrium and require that there be enough collisions to achieve this ( $\omega\tau \ll 1$ , where  $\tau$  is the relaxation time needed to reach local equilibrium). This is easy to achieve in a liquid (including liquid  $^4\text{He}$ ) and that is why, historically, classical fluid hydrodynamics was understood a long time before a microscopic treatment was available. Using a kinetic equation for interacting atoms, it was Boltzmann who first ( $\sim 1885$ ) discussed the precise set of conditions required for this hydrodynamic description to be valid. Up to the present, most experiments on the collective modes of Bose gases have probed the opposite “collisionless region” ( $\omega\tau \gg 1$ ). However one expects that the two-fluid hydrodynamic region will be increasingly studied in the next few years, taking advantage of the fact that one can now produce high densities of atoms (currently,  $N \sim 10^6 - 10^7$  atoms) and also we can enormously increase the atomic collision cross-section ( $\sigma = 8\pi a^2$ ) by working near a Feshbach resonance (as in recent work<sup>5</sup> on  $^{85}\text{Rb}$ ).

As references on BEC research, we recommend the 1998 Varenna Summer School Lectures<sup>6</sup> as well as the excellent review<sup>13</sup> by the Trento theory group. For a detailed treatment of topics discussed here, see Refs. 14,9.

## 2. DYNAMICS OF A PURE CONDENSATE

The crucial idea behind the field-theoretic description of Bose condensation developed 40 years ago<sup>1,2</sup> is to isolate the condensate “degree of freedom.” Thus the quantum field operator is decomposed as  $\hat{\psi}(\mathbf{r}) = \langle \hat{\psi}(\mathbf{r}) \rangle + \tilde{\psi}(\mathbf{r})$ , where  $\Phi(\mathbf{r}) \equiv \langle \hat{\psi}(\mathbf{r}) \rangle$  describes the Bose condensate (which is treated as a classical field). Here  $\tilde{\psi}(\mathbf{r})$  is the non-condensate component of the quantum field operator and satisfies Bose commutation relations. This formalism (first developed in a systematic way by Beliaev in 1958) allows

one to neatly isolate the condensate. The many-body theory is then developed to study the dynamics of the non-condensate fields  $\tilde{\psi}$  and  $\tilde{\psi}^\dagger$ , with the condensate playing the role of the “vacuum.” A crucial feature is that  $\Phi(\mathbf{r}, t)$  is complex,

$$\Phi(\mathbf{r}, t) = \sqrt{n_c(\mathbf{r}, t)} e^{i\theta(\mathbf{r}, t)}, \quad (1)$$

with the identification (this is justified later) of the superfluid velocity  $m\mathbf{v}_s(\mathbf{r}) = \hbar\nabla\theta(\mathbf{r}, t)$ . At  $T = 0$  in a dilute Bose gas, one can assume that all the atoms are in the condensate  $\Phi(\mathbf{r}, t)$ , which satisfies the famous Gross-Pitaevskii time-dependent equation of motion,<sup>13</sup>

$$i\hbar \frac{\partial\Phi(\mathbf{r}, t)}{\partial t} = \left[ -\frac{\hbar^2\nabla^2}{2m} + V_{ex}(\mathbf{r}) + gn_c(\mathbf{r}, t) \right] \Phi(\mathbf{r}, t). \quad (2)$$

Here  $V_{ex}(\mathbf{r})$  is the parabolic trap potential (a magnetic trap acting on the spin of an hyperfine atomic level) and  $gn_c(\mathbf{r}, t)$  is the Hartree field produced by the condensate atoms. At the very low temperatures of interest in BEC studies, the atoms have very low energy and one can use the  $s$ -wave scattering length approximation. In this case, the effective interatomic interaction is given by  $v(\mathbf{r} - \mathbf{r}') = g\delta(\mathbf{r} - \mathbf{r}')$ , where  $g = 4\pi a\hbar^2/m$ . One should not think of (2) as simply a Schrodinger equation for a “single-particle” wavefunction, since  $\Phi(\mathbf{r}, t)$  is an order-parameter which is well defined even at finite temperatures (see later). As we all have learned over the last 5 years, the  $T = 0$  GP equation in (2) contains a huge amount of physics and its solutions have dominated BEC research up to now.<sup>6,13</sup> Many examples are discussed in the invited and contributed papers at this QFS Conference.

The stationary solutions that the GP equation  $\Phi_0(\mathbf{r}, t) = \Phi_0(\mathbf{r})e^{-i\mu_0 t/\hbar}$  satisfy

$$\left[ -\frac{\hbar^2\nabla^2}{2m} + V_{ex}(\mathbf{r}) + g|\Phi_0(\mathbf{r})|^2 \right] \Phi_0(\mathbf{r}) = \mu_0\Phi_0(\mathbf{r}). \quad (3)$$

This can exhibit ground state solutions corresponding to vortices, in which there is a time-independent condensate current given by  $m\mathbf{v}_{s0}(\mathbf{r}) = \hbar\nabla\theta_0(\mathbf{r})$ . In the absence of vortices (irrotational flow), the Thomas-Fermi (TF) approximate solution of (3) is very simple and leads to the famous parabolic condensate profile

$$n_{c0}(\mathbf{r}) \equiv |\Phi_0(\mathbf{r})|^2 = \frac{1}{g} \left[ \mu_0 - \frac{1}{2}m\omega_0^2 r^2 \right]. \quad (4)$$

This profile is considerably wider than the ground state Gaussian wavefunction prediction for a non-interacting trapped gas.

It is very convenient to rewrite the GP equation (2) in terms of the condensate density  $n_c(\mathbf{r}, t)$  and velocity  $\mathbf{v}_s(\mathbf{r}, t)$  local variables. One finds

$$\begin{aligned} \frac{\partial n_c(\mathbf{r}, t)}{\partial t} &= -\nabla \cdot n_c(\mathbf{r}, t) \mathbf{v}_s(\mathbf{r}, t) \\ m \left( \frac{\partial \mathbf{v}_s}{\partial t} + \frac{1}{2} \nabla \mathbf{v}_s^2 \right) &= -\nabla \mu_c(\mathbf{r}, t), \end{aligned} \quad (5)$$

where the condensate chemical potential is

$$\mu_c(\mathbf{r}, t) \equiv -\frac{\hbar^2 \nabla^2 \sqrt{n_c}}{2m\sqrt{n_c}} + V_{ex}(\mathbf{r}) + gn_c(\mathbf{r}, t). \quad (6)$$

Written in this form, the condensate dynamics is “hydrodynamic” looking, even in the absence of collisions. This is because the GP equation (2) describes a large number of atoms in the *same* single-particle quantum state.

Linearizing around  $\Phi_0(\mathbf{r})$ , the two coupled equations in (5) can be solved for the collective oscillations of the condensate at  $T = 0$ . A standard simplification (first introduced by Stringari<sup>15</sup>) is to neglect the “quantum pressure”, the first term in (6). Within this TF approximation, one obtains the Stringari “wave equation” for fluctuations in  $\delta n_c(\mathbf{r}, t)$

$$\frac{\partial^2 \delta n_c}{\partial t^2} = -\nabla \cdot \left( n_{c0} \frac{\partial \delta \mathbf{v}_s}{\partial t} \right) = \frac{g}{m} \nabla \cdot (n_{c0}(\mathbf{r}) \nabla \delta n_c). \quad (7)$$

We note that this wave equation could be equally well written in terms of fluctuations of the phase of  $\Phi(\mathbf{r}, t)$ , since  $\partial \delta \theta / \partial t = -g \delta n_c$ . As an example of such condensate oscillations, we consider a uniform Bose gas, where  $\delta n_c(\mathbf{r}, t) \propto e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)}$ . This gives the famous Bogoliubov phonon excitations of the condensate (first derived in 1947 by a different method)

$$\omega = c_0 k, \quad c_0 = \sqrt{\frac{gn_{c0}}{m}}. \quad (8)$$

These phonons are physically unrelated to ordinary (hydrodynamic) sound waves in a normal fluid. Solutions of (7) corresponding to breathing, dipole, quadrupole and surface oscillations of an axially symmetric trap are in excellent agreement with experiment (when  $N \gtrsim 10^4$  atoms).<sup>13,15</sup>

### 3. TWO-FLUID SUPERFLUID HYDRODYNAMICS

It is straightforward to extend the preceding  $T = 0$  analysis to finite temperatures where there is a large fraction of atoms outside of the condensate described by  $\Phi(\mathbf{r}, t)$ . One then needs *two* equations of motion:<sup>9</sup> (a)

A generalized GP equation for  $\Phi(\mathbf{r}, t)$  which includes coupling to the non-condensate atoms. (b) A Boltzmann kinetic equation for the single-particle distribution function  $f(\mathbf{p}, \mathbf{r}, t)$  describing the non-condensate atoms.

A simple microscopic model has been analyzed in some detail in Ref. 9 which appears to contain the correct physics. By restricting oneself to finite temperatures, the important non-condensate atoms can be described by the simple particle-like spectrum

$$\begin{aligned}\tilde{\epsilon}_p(\mathbf{r}, t) &= \frac{p^2}{2m} + V_{ex}(\mathbf{r}) + 2g[n_c(\mathbf{r}, t) + \tilde{n}(\mathbf{r}, t)] \\ &\equiv \frac{p^2}{2m} + U(\mathbf{r}, t).\end{aligned}\quad (9)$$

This assumes that  $\hbar\omega_0 \ll k_B T$ ;  $gn_c(\mathbf{r}, t) \ll k_B T$ , namely both the energy level spacing in the harmonic well trap and the average mean field should be less than the average kinetic energy of the atoms. Under these conditions, the high-energy thermal atoms can be described by the single-particle distribution function  $f(\mathbf{p}, \mathbf{r}, t)$  obeying a Boltzmann equation

$$\begin{aligned}\frac{\partial f(\mathbf{p}, \mathbf{r}, t)}{\partial t} + \frac{\mathbf{p}}{m} \cdot \nabla_{\mathbf{r}} f(\mathbf{p}, \mathbf{r}, t) - \nabla_{\mathbf{r}} U(\mathbf{r}, t) \cdot \nabla_{\mathbf{p}} f(\mathbf{p}, \mathbf{r}, t) \\ = C_{22}[f] + C_{12}[f, \Phi].\end{aligned}\quad (10)$$

This involves two types of collision integrals. Collisions between non-condensate atoms are described by

$$\begin{aligned}C_{22}[f] &= \frac{2g^2}{(2\pi)^5} \int d\mathbf{p}_2 \int d\mathbf{p}_3 \int d\mathbf{p}_4 \delta(\mathbf{p} + \mathbf{p}_2 - \mathbf{p}_3 - \mathbf{p}_4) \\ &\times \delta(\tilde{\epsilon}_p + \tilde{\epsilon}_{p_2} - \tilde{\epsilon}_{p_3} - \tilde{\epsilon}_{p_4}) \\ &\times [(1+f)(1+f_2)f_3f_4 - ff_2(1+f_3)(1+f_4)].\end{aligned}\quad (11)$$

In contrast, collisions which transfer atoms between the condensate and non-condensate are described by

$$\begin{aligned}C_{12}[f, \Phi] &= \frac{2g^2}{(2\pi)^2} \int d\mathbf{p}_1 \int d\mathbf{p}_2 \int d\mathbf{p}_3 \delta(m\mathbf{v}_s + \mathbf{p}_1 - \mathbf{p}_2 - \mathbf{p}_3) \\ &\times \delta(\epsilon_c + \tilde{\epsilon}_{p_1} - \tilde{\epsilon}_{p_2} - \tilde{\epsilon}_{p_3}) [\delta(\mathbf{p} - \mathbf{p}_1) - \delta(\mathbf{p} - \mathbf{p}_2) - \delta(\mathbf{p} - \mathbf{p}_3)] \\ &\times [n_c(1+f_1)f_2f_3 - n_cf_1(1+f_2)(1+f_3)].\end{aligned}\quad (12)$$

Here  $\epsilon_c(\mathbf{r}, t) = \mu_c + \frac{1}{2}m\mathbf{v}_s^2$  is the condensate atom local energy ( $\mu_c$  is defined below in (16)) and  $m\mathbf{v}_s$  is the condensate atom momentum.

Clearly the  $C_{12}$  collisions do not conserve the number of atoms in the condensate. Thus they enter (in contrast to  $C_{22}$  collisions) in a direct way

in a “generalized” GP equation for  $\Phi(\mathbf{r}, t)$ , namely<sup>9,14</sup>

$$i\hbar \frac{\partial \Phi}{\partial t} = \left[ -\frac{\hbar^2 \nabla^2}{2m} + V_{ex}(\mathbf{r}) + gn_c(\mathbf{r}, t) + 2g\tilde{n}(\mathbf{r}, t) - i\hbar R(\mathbf{r}, t) \right] \Phi. \quad (13)$$

There is a new term ( $2g\tilde{n}$ ) associated with the Hartree-Fock mean field of the thermal cloud on the condensate. In addition, there is a dissipative term associated with  $C_{12}$  collisions,

$$R(\mathbf{r}, t) = \int \frac{d\mathbf{p}}{(2\pi)^3} \frac{C_{12}[f, \Phi]}{2n_c(\mathbf{r}, t)} \equiv \frac{\Gamma_{12}[f, \Phi]}{2n_c(\mathbf{r}, t)} \sim 0(g^2). \quad (14)$$

As with the  $T = 0$  GP equation of Section 2., one can rewrite (13) in terms of the  $n_c(\mathbf{r}, t)$  and  $\mathbf{v}_s(\mathbf{r}, t)$  variables to find (compare with (5) and (6))

$$\begin{aligned} \frac{\partial n_c}{\partial t} + \nabla \cdot n_c \mathbf{v}_c &= -\Gamma_{12}[f, \Phi] \\ m \left( \frac{\partial \mathbf{v}_c}{\partial t} + \frac{1}{2} \nabla \mathbf{v}_c^2 \right) &= -\nabla \mu_c, \end{aligned} \quad (15)$$

where now

$$\mu_c(\mathbf{r}, t) = -\frac{\nabla^2 \sqrt{n_c}}{2m\sqrt{n_c}} + V_{ex}(\mathbf{r}) + gn_c(\mathbf{r}, t) + 2g\tilde{n}(\mathbf{r}, t). \quad (16)$$

We note that even at finite  $T$ , the condensate equations of motion in (15) are “hydrodynamic looking” except that now in the continuity equation for the condensate, there is a source term  $\Gamma_{12}$  associated with  $C_{12}$  collisions.

One has to solve these coupled equations for  $\Phi(\mathbf{r}, t)$  and  $f(\mathbf{p}, \mathbf{r}, t)$  self-consistently - and clearly  $C_{12}$  will play a special role. While (3) and (13) give a sound basis for discussing the general non-equilibrium behavior of trapped gases at finite  $T$ , the collision-dominated hydrodynamic region is especially interesting since then the non-condensate can be described in terms of a few “coarse-grained” local variables. If the  $C_{22}$  collisions are rapid enough (relative to the frequency  $\omega$  of the collective mode of interest,  $\omega\tau_{22} \ll 1$ ), then  $f(\mathbf{p}, \mathbf{r}, t)$  will be driven to the *local* equilibrium Bose distribution

$$\tilde{f}(\mathbf{p}, \mathbf{r}, t) = \frac{1}{e^{\beta\left\{\frac{(\mathbf{p}-m\mathbf{v}_n)^2}{2m} + U(\mathbf{r}, t) - \tilde{\mu}(\mathbf{r}, t)\right\}} - 1}. \quad (17)$$

This function uniquely satisfies  $C_{22}[\tilde{f}, \Phi] = 0$ . With this local equilibrium distribution, the thermal atoms are now completely described in terms of the  $\beta$ ,  $\mathbf{v}_n$ ,  $\tilde{\mu}$  and  $\tilde{U}$  variables, all dependent on  $(\mathbf{r}, t)$ . Inserting  $\tilde{f}$  given by (17) into the Boltzmann equation (3) and taking moments with respect to

the momentum  $\mathbf{p}$ , one easily derives a set of hydrodynamic equations for the non-condensate variables  $\tilde{n}$ ,  $\mathbf{v}_n$  and  $\tilde{\mu}$ . This procedure was first developed by Boltzmann for deriving the hydrodynamic equations for a classical gas.

To summarize, at this stage we have arrived at a closed set of coupled equations<sup>9,10,14</sup> for the condensate variables ( $n_c$ ,  $\mathbf{v}_s$ ,  $\mu_c$ ) and the non-condensate variables ( $\tilde{n}$ ,  $\mathbf{v}_n$ ,  $\tilde{\mu}$ ). Linearizing these equations around static equilibrium, one can derive the hydrodynamic collective modes which correspond to coherent motions of both the condensate and non-condensate. Our new two-fluid equations exhibit a new relaxation time  $\tau_\mu$ , which describes how fast the two components come into diffusive equilibrium with each other (i.e., how fast  $\mu_{diff}(\mathbf{r}, t) \equiv \tilde{\mu}(\mathbf{r}, t) - \mu_c(\mathbf{r}, t)$  relaxes to zero). Of course, in static equilibrium, we have  $\tilde{\mu}_0 = \mu_{c0}$  but when we perturb the trapped Bose gas from equilibrium, one can have  $\delta\mu_{diff}(\mathbf{r}, t) \neq 0$  and it relaxes to zero in a time  $\tau_\mu$ . As one would expect,  $\tau_\mu$  is proportional to the collision time  $\tau_{12}$  associated with  $C_{12}$  collisions.

How is all this related to Landau's famous two-fluid hydrodynamic equations? These equations are expressed using different local variables. The linearized form of these two-fluid equations are<sup>11</sup>

$$\begin{aligned}
 \frac{\partial \delta n}{\partial t} &= -\nabla \cdot \delta \mathbf{j} \\
 m \frac{\partial \delta j_\mu}{\partial t} &= -\frac{\partial \delta P}{\partial x_\mu} - \partial n \frac{\partial V_{ex}}{\partial x_\mu} + \frac{\partial}{\partial x_\nu} \left[ 2\eta \left( D_{\mu\nu} - \frac{1}{3} T r D \delta_{\mu\nu} \right) \right] \\
 &\quad + \frac{\partial}{\partial x_\mu} [\zeta_1 \nabla \cdot (m n_{c0} (\delta \mathbf{v}_s - \delta \mathbf{v}_n)) + \zeta_2 \nabla \cdot \delta \mathbf{v}_n] \\
 m \frac{\partial \delta \mathbf{v}_s}{\partial t} &= -\nabla [\delta \mu + m \zeta_3 \nabla \cdot m n_{c0} (\delta \mathbf{v}_s - \delta \mathbf{v}_n) + m \zeta_4 \nabla \cdot \delta \mathbf{v}_n] \\
 \frac{\partial \delta s}{\partial t} &= -\nabla \cdot (s_0 \delta \mathbf{v}_n) + \frac{1}{T} \nabla \cdot (\kappa \nabla \delta T). \tag{18}
 \end{aligned}$$

Here  $P(\mathbf{r}, t)$  is the local pressure,  $s(\mathbf{r}, t)$  is the local entropy density (entirely associated with the normal fluid), and  $D_{\mu\nu} \equiv \frac{1}{2} \left( \frac{\partial v_{n\mu}}{\partial x_\nu} + \frac{\partial v_{n\nu}}{\partial x_\mu} \right)$ . The expressions

$$\begin{aligned}
 \delta \rho &\equiv m \delta n = \delta \rho_s + \delta \rho_n \\
 m \delta \mathbf{j} &\equiv \rho_{s0} \delta \mathbf{v}_s + \rho_{n0} \delta \mathbf{v}_n. \tag{19}
 \end{aligned}$$

make the two-fluid nature of the theory clear. We have included hydrodynamic damping from various transport processes (the thermal conductivity  $\kappa$ , the shear viscosity  $\eta$  and the four second viscosity coefficients  $\zeta_i$ , all dependent on position through the local condensate density  $n_{c0}(\mathbf{r})$ ). In making the detailed comparison between the Landau two-fluid equations and our

two-fluid equations, a key role is played by the condensate relaxation time  $\tau_\mu$  introduced above.

In the limit  $\omega\tau_\mu \rightarrow 0$ , one can show<sup>9</sup> that our two-fluid equations are precisely equivalent to the Landau equations summarized in (18), but without the dissipative terms. This makes sense, since these equations are derived<sup>12</sup> under the assumption that the superfluid and normal fluid are always in local equilibrium with each other. This corresponds to  $\tilde{\mu}(\mathbf{r}, t) = \mu_c(\mathbf{r}, t)$ , ie,  $\tau_\mu \rightarrow 0$ . However, even in this limit, it turns out that  $\Gamma_{12}$  in (15) is finite and is crucial to ensure the equivalence with the Landau equations. One finds one can make the identification (valid within our model)

$$\begin{aligned}\rho_s(\mathbf{r}, t) &= mn_c(\mathbf{r}, t) \\ \rho_n(\mathbf{r}, t) &= m\tilde{n}(\mathbf{r}, t).\end{aligned}\quad (20)$$

In the limit of  $\omega\tau_\mu \ll 1, \omega\tau_{22} \ll 1$ , our equations can be shown<sup>16</sup> to reduce precisely to the Landau-Khalatnikov two-fluid equations<sup>11</sup> in (18). What is very satisfactory is that, in our model, the second viscosity damping terms turn out to be proportional to  $\delta\mu_{diff}$ , i.e., to the fact that  $\mu_c \neq \tilde{\mu}$ . This is physically quite reasonable and expected. Thus the four second viscosities are all proportional to the collision time  $\tau_{12}$  associated with the  $C_{12}$  collision term. In contrast, the other transport coefficients ( $\kappa$  and  $\eta$ ) are associated with deviations for  $f(\mathbf{p}, \mathbf{r}, t)$  from the local equilibrium distribution (17) and have contributions from *both* the  $C_{22}$  and  $C_{12}$  collision integrals.

Finally, one can consider the new limit  $\omega\tau_\mu \gg 1, \omega\tau_{22} \ll 1$ , which can arise<sup>10</sup> at temperatures close to  $T_{BEC}$  (where  $\tau_\mu$  becomes very large due to a sort of “critical slowing down”). This is outside the region of validity of the usual Landau two-fluid equations. Surprisingly, however, our new linearized two-fluid equations lead again<sup>16</sup> to the Landau-Khalatnikov two-fluid equations<sup>11</sup> but now with frequency-dependent second viscosities

$$\zeta_i(\omega) = \frac{\zeta_i}{1 - i\omega\tau_\mu}.\quad (21)$$

This form could have been predicted, since it is typical of a situation where a fluid is coupled into an “internal” degree of freedom with a long relaxation time - in the present case, this is the Bose condensate.

As one might expect, our new form of the two-fluid equation leads naturally to a central zero-frequency relaxational mode which is missed by the usual LK two-fluid hydrodynamic equations. One can show<sup>17,14</sup> that as  $T \rightarrow T_{BEC}$  from below, this mode (which is strongly coupled to the thermal conductivity) becomes the usual zero frequency thermal diffusion mode. Observation of this mode below  $T_{BEC}$  is a goal for future BEC studies since this mode is a unique feature of the hydrodynamics of a trapped Bose gas.

In conclusion, we have sketched recent progress in deriving the equivalent of the dissipative Landau-Khalatnikov two-fluid hydrodynamic equations for a trapped Bose-condensed gas. The nice feature about our derivation is that our microscopic model allows us to compute all the thermodynamic functions, transport coefficients and relaxation times which enter into these equations. What is needed now is a sustained experimental effort to check the predictions of this two-fluid theory.

## ACKNOWLEDGMENTS

Much of the work presented in Section 3 has been done in close collaboration with Eugene Zaremba. A.G. is supported by NSERC of Canada and T.N. is supported by the JSPS.

## REFERENCES

1. P.C. Hohenberg and P.C. Martin, *Ann. Phys. (N. Y.)* **34**, 291 (1965).
2. For a review, see A. Griffin, *Excitations in a Bose-condensed Liquid (Cambridge, N. Y., 1993)*.
3. K.W. Madison *et al*, *Phys. Rev. Lett.* **84**, 806 (2000).
4. M.R. Matthews *et al*, *Phys. Rev. Lett.* **83**, 2498 (1999).
5. S. Cornish *et al*, *Phys. Rev. Lett.* **85**, 1785 (2000).
6. *Bose-Einstein Condensation in Atomic Gases*, Proceedings of the International School of Physics "Enrico Fermi", ed. M. Inguscio, S. Stringari and C. E. Wieman (*IOS Press, Amsterdam, 1999*).
7. M.R. Matthews *et al*, *Phys. Rev. Lett.* **83**, 3358 (1999).
8. D.M. Stamper-Kurn *et al*, *Phys. Rev. Lett.* **83**, 2876 (1999).
9. E. Zaremba, T. Nikuni and A. Griffin, *J. Low Temp. Phys.* **116**, 227 (1999).
10. T. Nikuni, E. Zaremba and A. Griffin, *Phys. Rev. Lett.* **83**, 10 (1999).
11. I.M. Khalatnikov, *An Introduction to the Theory of Superfluidity (W. A. Benjamin, N. Y., 1965)*.
12. N.N. Bogoliubov, *Lectures on Quantum Statistics, Vol. 2 Gordon and Breach, N. Y., (1970)*, p.148.
13. F. Dalfovo, S. Giorgini, L.P. Pitaevskii and S. Stringari, *Rev. Mod. Phys.* **71**, 463 (1999).
14. A. Griffin, in 13<sup>th</sup> Physics Summer School: Bose-Einstein Condensation, ed by C. Savage and M. Das (*World Scientific, Singapore, 2000*), in press; cond-mat/0006382.
15. S. Stringari, *Phys. Rev. Lett.* **77**, 2360 (1996).
16. T. Nikuni, private communication; T. Nikuni and A. Griffin, to be published.
17. T. Nikuni, A. Griffin and E. Zaremba, cond-mat/9911336.