

Quasiparticle Kinetic Equation in a Trapped Bose Gas at Low Temperatures

M. Imamovic-Tomasovic and A. Griffin*

Department of Physics, University of Toronto, Toronto, Ontario, Canada M5S 1A7

*E-mail: griffin@physics.utoronto.ca

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Recently the authors used the Kadanoff–Baym non-equilibrium Green’s function formalism to derive kinetic equation for the non-condensate atoms, in conjunction with a consistent generalization of the Gross–Pitaevskii equation for the Bose condensate wavefunction. This work was limited to high temperatures, where the excited atoms could be described by a Hartree–Fock particle-like spectrum. Following the approach of Kane and Kadanoff in 1965, we present the generalization of our recent work which is valid at low temperatures, where the input single-particle spectrum is now described by the Bogoliubov–Popov approximation. We derive a kinetic equation for the quasiparticle distribution function with collision integrals describing scattering between quasiparticles and the condensate atoms. From the general expression for the collision integral for the scattering between quasiparticle excitations, we find the quasiparticle distribution function corresponding to local equilibrium. This expression includes a quasiparticle chemical potential that controls the non-diffusive equilibrium between condensate atoms and the quasiparticle excitations. We derive a generalized Gross–Pitaevskii equation for the condensate wavefunction that also includes the damping effects due to collisions between atoms in the condensate and the thermally excited quasiparticles. For a uniform Bose gas, our kinetic equation for the thermally excited quasiparticles reduces to that found by Eckern, as well as by Kirkpatrick and Dorfman.

1. INTRODUCTION

In a trapped, weakly-interacting Bose gas at $T=0$, the fraction of atoms that are excited out of the condensate is only a few percent.¹ As a result, the dynamics of the trapped Bose gas at low temperatures (compared to T_{BEC}) is well described by the equation of motion for the

macroscopic wavefunction $\Phi(\mathbf{r}, t)$. This is the time-dependent Gross–Pitaevskii equation (GP)¹

$$i \frac{\partial \Phi(\mathbf{r}, t)}{\partial t} = \left[-\frac{1}{2m} \nabla_{\mathbf{r}}^2 + U_{\text{ext}}(\mathbf{r}) + gn_c(\mathbf{r}, t) \right] \Phi(\mathbf{r}, t), \quad (1)$$

where $n_c(\mathbf{r}, t) = |\Phi(\mathbf{r}, t)|^2$ is the non-equilibrium density of the atoms in the condensate and $U_{\text{ext}}(\mathbf{r})$ is a harmonic trap potential (In this paper, we set $\hbar = 1$). For a discussion of the properties of a dilute Bose gas at very low temperatures, only the s -wave component of the two-body interaction $v(\mathbf{r} - \mathbf{r}')$ is important. Thus one can use the pseudopotential

$$v(\mathbf{r}) = g\delta(\mathbf{r}), \quad g = 4\pi a/m, \quad (2)$$

where a is the s -wave scattering length of the true potential. The GP equation describes the motion of the condensate moving in the dynamic Hartree mean-field produced by the other atoms in the condensate and gives a closed equation for the order parameter $\Phi(\mathbf{r}, t)$. The GP equation (1) appears to provide quite a good description of the static and mean-field dynamic properties of the condensate in trapped Bose gases of ⁸⁷Rb and ²³Na atoms for temperatures in the region $T \leq 0.4T_{\text{BEC}}$. This is confirmed by many different kinds of experiments carried out in the last few years.¹ In superfluid ⁴He, in contrast, the non-condensate fraction at $T = 0$ is close to 90%.² Thus in superfluid ⁴He, one always has to deal with *both* the condensate and non-condensate atoms. Clearly a closed GP equation for $\Phi(\mathbf{r}, t)$ like (1) is never valid in superfluid ⁴He.

At finite temperatures, however, the number of atoms thermally excited out of the condensate becomes finite and one must ask how they modify the simple GP equation (1). The simplest way to include the effect of the excited atoms on the condensate is to add the additional Hartree–Fock mean field $V_{\text{HF}} = 2g\tilde{n}(\mathbf{r}, t)$ produced by the non-condensate atoms (here \tilde{n} is the local non-condensate density). One immediately sees this new GP equation is no longer closed since it depends on the dynamics of the non-condensate atoms.

To find the time-dependent non-condensate density, Zaremba, Nikuni, and Griffin (ZNG)³ have used a *quantum Boltzmann equation* for the single-particle distribution function of the non-condensate atoms $f(\mathbf{p}, \mathbf{r}, t)$

$$\left[\frac{\partial}{\partial t} + \frac{\mathbf{p}}{m} \cdot \nabla_{\mathbf{r}} - \nabla_{\mathbf{r}} U(\mathbf{r}, t) \cdot \nabla_{\mathbf{p}} \right] f(\mathbf{p}, \mathbf{r}, t) = \left[\frac{\partial f(\mathbf{p}, \mathbf{r}, t)}{\partial t} \right]_{\text{coll}}. \quad (3)$$

Here, the thermally excited atoms are assumed to be well described by the single-particle spectrum $(p^2/2m) + U(\mathbf{r}, t)$, where

$$U(\mathbf{r}, t) \equiv U_{\text{ext}}(\mathbf{r}) + 2g[n_c(\mathbf{r}, t) + \tilde{n}(\mathbf{r}, t)] \quad (4)$$

includes the self-consistent Hartree–Fock dynamic mean field involving the *total* time-dependent local density $n(\mathbf{r}, t)$. The right-hand side of (3) describes the effect of collisions between atoms on the time evolution of the distribution function $f(\mathbf{p}, \mathbf{r}, t)$. In Bose-condensed gases, this collision integral has two distinct contributions

$$\left[\frac{\partial f}{\partial t} \right]_{\text{coll}} = C_{12}[f] + C_{22}[f]. \quad (5)$$

Here, C_{22} denotes the part of the collision integral that describes two-body collisions between non-condensate atoms. Above T_{BEC} , this is the only term present. In contrast, C_{12} describes collisions involving non-condensate atoms and *one* condensate atom. The role of C_{12} is crucial since it couples the condensate and non-condensate components.

ZNG also derived a generalized Gross–Pitaevskii equation that includes the effect of the collisions between the atoms in the condensate and the thermal cloud. Recently, we obtained the equations of motion derived by ZNG at finite temperatures in a more elegant way using the well-known Kadanoff–Baym Green’s functions formalism.⁴ We also note that several other groups have also recently discussed the finite temperature dynamics of a trapped Bose-condensed gas.^{5–8} These include the quantum kinetic master equation approach⁷ and the Fokker–Planck equation approach based on the Schwinger–Keldysh formalism.⁵ It would be useful to have a detailed comparison between these various formalisms, but this is not done in the present paper.

The kinetic equation (3) is valid in the semiclassical limit only: it assumes that the thermal energy is much greater than the spacing between the trap SHO energy levels ($k_B T \gg \hbar\omega_0$, where ω_0 is the harmonic trap frequency) as well as the average interaction energy ($k_B T \gg gn$). ZNG have given a detailed derivation³ of (3) at finite temperatures for a trapped Bose gas using the approach of Kirkpatrick and Dorfman,⁹ who considered a uniform Bose gas. From various studies,¹ one expects that ZNG type of model based on particle-like excitations will be adequate for temperatures higher than $T \geq 0.5T_{BEC}$ and perhaps much lower. However, the ZNG model ultimately breaks down at low temperatures because the thermal excitations on which it is based do not include the collective (or phonon) part of the Bogoliubov spectrum $E_p = \sqrt{(p^2/2m)^2 + gn_c p^2/m}$. To make this

generalization, one has to formulate a kinetic theory in terms of such Bogoliubov quasiparticle excitations. In this paper, we do this by using the Kadanoff–Baym (KB) non-equilibrium Green’s functions method¹⁰ to derive a generalized kinetic equation for the thermally excited Bogoliubov quasiparticles. To do this, we work within the second-order Beliaev–Popov approximation.¹¹ One expects that our equations of motion will be needed to describe the very low temperature dynamics of a trapped Bose gases which are not included in the simple GP equation (1).

Kadanoff and Baym first formulated the general method of deriving a kinetic equation for a normal interacting system using non-equilibrium Green’s functions.¹⁰ Kane and Kadanoff (KK)^{12, 13} generalized this method to deal with a Bose-condensed gas, with the specific goal of deriving the phenomenological two-fluid hydrodynamics equations of Landau.¹⁴ Their calculation made use of a “conserving” approximation for the generating Beliaev self-energies, rather than the Beliaev second-order approximation we use. The Kane and Kadanoff paper¹² is the only previous paper that applies the Kadanoff–Baym method to a Bose-condensed fluid. We remark that the style of Ref. 12 is very compact, making it a very difficult paper to read. We have tried to show how the KB formalism can be used to derive such equations in a trapped Bose gas in a way that sets the stage for further extensions. An excellent review of the nonequilibrium real-time Green’s functions and the generalized kinetic equation for the normal systems (non Bose-condensed) can be found in the recent book by Zubarev, Morozov, and Ropke.¹⁵

It is important to emphasize that although our analysis involves the non-equilibrium generalization of the Beliaev second-order self-energy used for systems in thermal equilibrium, our work is quite different from the recent papers discussing the poles of equilibrium Green’s functions within the second-order Beliaev approximation.^{11, 16, 17} In a very elegant formulation, Giorgini¹⁶ has calculated the quasiparticle energy and damping at finite temperatures in a dilute Bose gas in the collisionless regime by linearizing the equations of motion for fluctuations using the first-order dynamic Hartree–Fock–Bogoliubov approximation. This leads, as expected, to the same excitations spectrum found by Shi and Griffin¹¹ who calculated directly the poles of the single-particle equilibrium Green’s functions using the second-order Beliaev self-energy diagram contributions.

In the present paper, we use the second-order Beliaev approximation to discuss the *non-equilibrium dynamics* of a trapped Bose-condensed gas at finite temperatures. We use the second-order Beliaev self-energies with the lower order Bogoliubov excitation spectrum, including off-diagonal single-particle propagators, but we ignore the anomalous correlation function \tilde{m} . This last assumption defines what we call the Bogoliubov–Popov

approximation.¹⁸ It is important to note that, in this paper, we are primarily interested in the damping effect arising from the collisions between atoms. We do not explicitly calculate the second-order corrections in g to the quasiparticle spectrum or to the condensate chemical potential, both of which are associated with the real parts of the second-order Beliaev self-energies. One effect of including the real part of the second-order self-energies in an improved kinetic equation is the appearance of a renormalized quasiparticle energy (see Eq. (6.3.77) in Ref. 15). In this context, the original work of Kane and Kadanoff¹² is superior to our present study since they use the KB method to derive equations of motion for the order parameter and non-condensate Green's functions keeping the real parts of their (simplified) second-order self energies.

We derive the kinetic equation for the distribution function for the thermally excited quasiparticles, as well as a generalized equation for the Bose condensate order parameter. The kinetic equation we obtain is the same as the one derived for a uniform Bose gas by Eckern¹⁹ in 1984, and by Kirkpatrick and Dorfman (KD) in 1985.⁹ The KD derivation was based on a direct extension of the traditional method used to derive kinetic equations for classical gases, which we do not believe is as transparent as the KB approach.¹⁰ Moreover, KD did not explicitly derive a GP-like equation of motion for the condensate degree of freedom. In a non-Bose condensed uniform gas, similar kinetic equations are derived in Ref. 15 using the related Keldysh formalism.

The kinetic equation for the quasiparticle excitations which we derive in this paper, coupled to a generalized GP equation, provides a platform for studying different non-equilibrium aspects of a dynamics of a trapped, Bose-condensed gas at all temperatures, both in the collisionless and hydrodynamic domains. Linearizing our equations of motion around static equilibrium, one could calculate the density response functions which would exhibit collective mode resonances with a spectrum which goes past the generating Beliaev approximation. However, we have no guarantee that this spectrum would be consistent with the conservation laws since our calculation is not based on a "conserving approximation" (for further discussion, see Refs. 20, 15, 12, 13, and 4). The high temperature limit of these coupled equations^{3,4} has been recently used to study the dynamics of the condensate formation and growth in an inhomogeneous, trapped Bose gas.²¹ By taking the moments of the kinetic equation, the two-fluid hydrodynamic equations have also been derived recently for superfluid Bose gases.^{3,22}

The present paper is a natural generalization of our two earlier papers based on the KB formalism. In Ref. 23, we have derived kinetic equations within the full Hartree-Fock-Bogoliubov approximation but ignored

collisions. In Ref. 4, we derived a kinetic equation including collisions, but which was based on a simple Hartree–Fock particle-like spectrum and hence was not valid at very low temperatures.

In Sec. 2, we review the general equations of motion for the non-equilibrium Green's functions describing the non-condensate atoms as well as the equation of motion for the macroscopic order parameter. In Sec. 3, we transform these equations to a local rest frame of reference where the order parameter $\Phi(\mathbf{r}, t)$ is real, i.e., to a frame where the local superfluid velocity is zero. This naturally introduces the superfluid velocity and the local chemical potential as the spatial and time derivatives, respectively, of the phase of the order parameter.¹² We then specialize our equations of motion for the non-equilibrium Green's functions for the case of slowly varying external perturbations. We use the key assumption that all correlation functions vary slowly as a function of center-of-mass space-time coordinates but are dominated by small values of the relative coordinates. Following the Kadanoff-Baym approach,^{10, 12, 13} we derive a generalized quantum Boltzmann equation for the frequency-dependent quasiparticle distribution function $f(\mathbf{p}, \omega; \mathbf{R}, T)$.

In Sec. 4, we use this generalized KB quantum Boltzmann equation to derive the kinetic equation for quasiparticles at low temperatures, with the collision integral describing collisions between quasiparticles in the Bogoliubov–Popov approximation. In Sec. 5, we discuss a general form of the local quasiparticle distribution function and introduce a new quasiparticle chemical potential for situations in which the condensate and thermal excitations are not in diffusive equilibrium. In Sec. 6, we derive, in a self-consistent manner, a generalized Gross–Pitaevskii equation. In Sec. 7, we verify that our coupled equations exhibit the Kohn mode corresponding to the harmonic oscillations of the center-of-mass of the equilibrium condensate and non-condensate density profiles.

2. EQUATIONS OF MOTION FOR NON-EQUILIBRIUM GREEN'S FUNCTIONS

For convenience, we first review the KB formalism¹⁰ already used in our earlier work.^{4, 23} In terms of quantum field operators, the many-body Hamiltonian ($\hat{K} = \hat{H} - \mu_0 \hat{N}$) describing interacting Bosons confined by an external harmonic potential $U_{\text{ext}}(\mathbf{r})$ is given by:

$$\begin{aligned} \hat{K} = \int d\mathbf{r} \psi^\dagger(\mathbf{r}) \left[-\frac{1}{2m} \nabla_{\mathbf{r}}^2 + U_{\text{ext}}(\mathbf{r}) - \mu_0 \right] \psi(\mathbf{r}) \\ + \frac{1}{2} \int d\mathbf{r} d\mathbf{r}' \psi^\dagger(\mathbf{r}) \psi^\dagger(\mathbf{r}') v(\mathbf{r} - \mathbf{r}') \psi(\mathbf{r}) \psi(\mathbf{r}'). \end{aligned} \quad (6)$$

We separate out the condensate part of the field operator in the usual fashion.^{24, 25} Statically, this decomposition is given by

$$\psi(\mathbf{r}) = \langle \psi(\mathbf{r}) \rangle + \tilde{\psi}(\mathbf{r}), \quad (7)$$

where $\langle \tilde{\psi}(\mathbf{r}) \rangle = 0$ and $\langle \psi(\mathbf{r}) \rangle = \Phi(\mathbf{r})$ is the Bose macroscopic wavefunction. The non-condensate (or excited-atom component) field operators $\tilde{\psi}(\mathbf{r})$ and $\tilde{\psi}^\dagger(\mathbf{r})$ satisfy the usual Bose commutation relations. Here, $\langle \psi(\mathbf{r}) \rangle$ involves a broken-symmetry thermal average in the presence of external driving fields, as described by (12).

In a Bose-condensed system, the finite value of $\Phi(\mathbf{r}, t)$ leads to finite values of the off-diagonal (or anomalous) propagators $\langle \tilde{\psi}(1) \tilde{\psi}(1') \rangle$ and $\langle \tilde{\psi}^\dagger(1) \tilde{\psi}^\dagger(1') \rangle$. These must be dealt with on an equal basis with the diagonal (or normal) propagators, and thus it is convenient to work with a single-particle 2×2 matrix Green's function defined by^{12, 20}

$$\hat{g}(1, 1'; U) = -i \begin{pmatrix} \langle T\psi(1) \psi^\dagger(1') \rangle & \langle T\psi(1) \psi(1') \rangle \\ \langle T\psi^\dagger(1) \psi^\dagger(1') \rangle & \langle T\psi^\dagger(1) \psi(1') \rangle \end{pmatrix}. \quad (8)$$

Here, T represents the time-ordering operator and we use the usual KB abbreviated notation, $1 \equiv (\mathbf{r}, t)$ and $1' \equiv (\mathbf{r}', t')$. We define $\hat{g}^<$ and $\hat{g}^>$ by

$$\begin{aligned} \hat{g}(1, 1'; U) &= \hat{g}^>(1, 1'; U) & t_1 > t_{1'} \\ &= \hat{g}^<(1, 1'; U) & t_1 < t_{1'}. \end{aligned} \quad (9)$$

Using (7), the matrix propagator in (8) splits into two parts

$$\hat{g}(1, 1'; U) = \hat{\hat{g}}(1, 1'; U) + \hat{h}(1, 1'; U). \quad (10)$$

Here $\hat{\hat{g}}$ is identical to (8), except that it involves the non-condensate part of the field operators. The condensate propagator is given by

$$\hat{h}(1, 1'; U) \equiv -i \begin{pmatrix} \Phi(1) \Phi^*(1') & \Phi(1) \Phi(1') \\ \Phi^*(1) \Phi^*(1') & \Phi^*(1) \Phi(1') \end{pmatrix}, \quad (11)$$

with $\langle \psi(\mathbf{r}, t) \rangle \equiv \Phi(\mathbf{r}, t)$ and $\langle \psi^\dagger(\mathbf{r}, t) \rangle \equiv \Phi^*(\mathbf{r}, t)$. The latter are time-dependent generalization of (7) describing a dynamical local condensate.

A very useful and elegant way of generating the equations of motion for both $\hat{\hat{g}}$ and Φ is to use functional derivatives with respect to weak external fields,^{10, 20, 26}

$$H'(t_1) = \frac{1}{2} \int d\mathbf{r}_1 d2 \psi^\dagger(1) U(1, 2) \psi(2) + \int d\mathbf{r}_1 [\psi^\dagger(1) \eta_{\text{ext}}(1) + \psi(1) \eta_{\text{ext}}^*(1)]. \quad (12)$$

Here $U(1, 2)$ is an external generating scalar field non-local in space and time. It represents a perturbation in which a particle is removed from the system at point 1 and added at 2. The symmetry-breaking fields η_{ext} and η_{ext}^* describe particle creation and destruction.^{20, 27} All higher-order Green's functions can be neatly expressed as functional derivatives of single-particle Green's functions with respect to such generating fields.

Following the Kane–Kadanoff (KK) analysis,^{12, 10} the Dyson–Beliaev equations of motion for the real-time non-condensate propagators $\hat{g}(1, 1')$ can be written in the following 2×2 matrix form

$$\begin{aligned} & \int d\bar{1} [\hat{g}_0^{-1}(1, \bar{1}) - \hat{\Sigma}^{HF}(1, \bar{1})] \hat{g}^{\lessgtr}(\bar{1}, 1') \\ &= \int_{-\infty}^{t_1} d\bar{1} \hat{\Gamma}(1, \bar{1}) \hat{g}^{\lessgtr}(\bar{1}, 1') - \int_{-\infty}^{t_1'} d\bar{1} \hat{\Sigma}_c^{\lessgtr}(1, \bar{1}) \hat{a}(\bar{1}, 1'), \end{aligned} \quad (13)$$

and

$$\begin{aligned} & \int d\bar{1} \hat{g}^{\lessgtr}(1, \bar{1}) [\hat{g}_0^{-1}(\bar{1}, 1') - \hat{\Sigma}^{HF}(\bar{1}, 1')] \\ &= \int_{-\infty}^{t_1} d\bar{1} \hat{a}(1, \bar{1}) \hat{\Sigma}_c^{\lessgtr}(\bar{1}, 1') - \int_{-\infty}^{t_1'} d\bar{1} \hat{g}^{\lessgtr}(1, \bar{1}) \hat{\Gamma}(\bar{1}, 1'). \end{aligned} \quad (14)$$

Here $\hat{a}(1, 1')$ and $\hat{\Gamma}(1, 1')$ are defined by the matrix elements

$$\begin{aligned} a_{\alpha\beta}(1, 1') &\equiv \tilde{g}_{\alpha\beta}^>(1, 1') - \tilde{g}_{\alpha\beta}^<(1, 1') \\ \Gamma_{\alpha\beta}(1, 1') &\equiv \Sigma_{\alpha\beta}^>(1, 1') - \Sigma_{\alpha\beta}^<(1, 1'). \end{aligned} \quad (15)$$

The *non-equilibrium* single-particle spectral density $a_{\alpha\beta}(1, 1')$ will play a crucial role in this paper. In the above equations and elsewhere, integration over $d\bar{1}$ means integration over the coordinates (\mathbf{r}_1, t_1) and a trace over the matrix index α_1 ; and $\delta(11') \equiv \delta(\mathbf{r} - \mathbf{r}') \delta(t - t')$.

The single-particle self-energy which is involved in (13) and (14) has already been split into two parts^{12, 20}

$$\hat{\Sigma}(1, 1') = \hat{\Sigma}^{HF}(1, 1') + \hat{\Sigma}_c(1, 1'). \quad (16)$$

The first-order Hartree–Fock self-energies are given by

$$\hat{\Sigma}^{HF}(11') = g \begin{pmatrix} 2n(1), m(1) \\ m^*(1), 2n(1) \end{pmatrix} \delta(11'), \quad (17)$$

and Σ_c is the second-order “collisional” Beliaev self-energy. The total density is given by $n(1) \equiv i\tilde{g}_{11}^<(1, 1^+) + |\Phi(1)|^2 = \tilde{n}(1) + n_c(1)$ and the total anomalous density by $m(1) \equiv i\tilde{g}_{12}^<(1, 1) + [\Phi(1)]^2$. In addition, we define (see (9))

$$\begin{aligned} \hat{\Sigma}_c(1, 1') &= \hat{\Sigma}_c^>(1, 1') & t_1 > t_{1'}, \\ &= \hat{\Sigma}_c^<(1, 1') & t_1 < t_{1'}. \end{aligned} \tag{18}$$

In (13) and (14), the inverse of the non-interacting 2×2 matrix Bose gas propagator $\hat{g}_0(1, 1')$ is defined by

$$\hat{g}_0^{-1}(1, 1') = \left[i\hat{\tau}_3 \frac{\partial}{\partial t_1} + \frac{\nabla_1^2}{2m} - U_{\text{ext}}(\mathbf{r}_1) + \mu_0 \right] \hat{\mathbb{I}}\delta(1, 1'). \tag{19}$$

where $\hat{\tau}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $\hat{\mathbb{I}}$ is the 2×2 identity matrix. We note that the equations in (13) and (14) already have the “structure” of a kinetic equation such as (3). The Hartree–Fock part of the self-energy has been included into the left-hand side of (13) and (14), giving the mean-field contribution to the “streaming” term. The second-order self-energy describing binary collisions is included on the right-hand side of (13) and (14), and it will be shown to give rise to collision integrals in the quasiparticle kinetic equation we derive.

In this paper, we work with the second-order self-energy Σ_c as given by the Beliaev (gapless) approximation.^{11, 18, 20} The advantage of the Beliaev approximation is that the non-condensate single-particle Green’s function exhibits the correct quasiparticle spectrum (phonon-like in the long-wavelength, uniform gas limit). In the second-order Beliaev approximation, the second-order self-energy Σ^{\cong} is given by⁴

$$\begin{aligned} \hat{\Sigma}_c^{\cong}(1, 1') &= -\frac{1}{2}v(13)v(21')\hat{g}^{\cong}(11')[\hat{g}^{\leq}(23)\hat{g}^{\cong}(32) + \hat{g}^{\leq}(23)\hat{h}(32) \\ &\quad + \hat{h}(23)\hat{g}^{\cong}(32)] - v(13)v(21')\hat{g}^{\cong}(12)[\hat{g}^{\leq}(23)\hat{h}(31') \\ &\quad + \hat{h}(23)\hat{g}^{\cong}(31') + \hat{g}^{\leq}(23)\hat{g}^{\cong}(31')] \\ &\quad - \frac{1}{2}v(13)v(21')[\hat{h}(11')\hat{g}^{\leq}(23)\hat{g}^{\cong}(32) \\ &\quad + 2\hat{h}(12)\hat{g}^{\leq}(23)\hat{g}^{\cong}(31')]. \end{aligned} \tag{20}$$

The equation for the condensate can be written (see Ref. 4 for more details) in terms of the 2-component order parameter $\hat{G}_{1/2}(1) \equiv \sqrt{-i} \langle \hat{\Psi}(1) \rangle$, where $\hat{\Psi}$ is defined as

$$\hat{\Psi}(1) \equiv \begin{pmatrix} \psi(1) \\ \psi^\dagger(1) \end{pmatrix}. \tag{21}$$

One finds that equation of motion for the $\hat{G}_{1/2}$ is given by^{4, 20, 26}

$$\int d\bar{1} \hat{g}_0^{-1}(1, \bar{1}) \hat{G}_{1/2}(\bar{1}) = \sqrt{-i} \hat{\eta}(1) + \sqrt{-i} \hat{\eta}_{\text{ext}}(1), \quad (22)$$

where the so-called condensate source function η is defined by the three-field correlation function

$$\hat{\eta}(1) \equiv \frac{1}{2} \int d\bar{2} v(1\bar{2}) \langle T \hat{\Psi}(1) \hat{\Psi}^\dagger(\bar{2}) \hat{\Psi}(\bar{2}) \rangle. \quad (23)$$

The external particle-source fields are defined in (12), with

$$\hat{\eta}_{\text{ext}}(1) \equiv \begin{pmatrix} \eta_{\text{ext}}(1) \\ \eta_{\text{ext}}^*(1) \end{pmatrix}. \quad (24)$$

The exact coupled equations of motion (13), (14) and (22) are the starting point of our analysis. The external generating fields U and η_{ext} will be left implicit in the rest of this paper.

Using (7), one can also decompose the three-field correlation function involved in η defined in (23). For example, one has

$$\begin{aligned} \langle T \psi(1) \psi^\dagger(2) \psi(2) \rangle &= \Phi(1) n_c(2) + \Phi(2) \langle T \tilde{\psi}(1) \tilde{\psi}^\dagger(2) \rangle \\ &+ \Phi^*(2) \langle T \tilde{\psi}(1) \tilde{\psi}(2) \rangle + \langle T \tilde{\psi}(1) \tilde{\psi}^\dagger(2) \tilde{\psi}(2) \rangle. \end{aligned} \quad (25)$$

In the first order Hartree–Fock–Bogoliubov (HFB) approximation, one neglects the three-field correlation function $\langle \tilde{\psi} \tilde{\psi}^\dagger \tilde{\psi} \rangle$ for the non-condensate atoms. In this approximation, (25) only involves the condensate density n_c and the two-field correlation functions \tilde{n} and \tilde{m} , as defined earlier. In this paper, in contrast, we keep the non-condensate three-field correlations but will eventually set $\tilde{m} \equiv \langle \tilde{\psi}(1) \tilde{\psi}(1) \rangle = 0$ (the Popov approximation¹⁸).

Equation (22) can be rewritten in terms of a condensate self-energy function S , defined by

$$\int d\bar{1} \hat{S}(1, \bar{1}) \hat{h}(\bar{1}, 1') \equiv \sqrt{-i} \eta(1) \hat{G}_{1/2}^\dagger(1'), \quad (26)$$

with the condensate propagator h given in (11). In place of (22), we have

$$\int d\bar{1} [\hat{g}_0^{-1}(1, \bar{1}) - \hat{S}^{HF}(1, \bar{1})] h(\bar{1}, 1') = \int_{-\infty}^{t_1} d\bar{1} (\hat{S}^>(1, \bar{1}) - \hat{S}^<(1, \bar{1})) \hat{h}(\bar{1}, 1'), \quad (27)$$

where, as before, the mean-field contributions (see (25)) are included in the Hartree–Fock part of the condensate self-energy S^{HF}

$$\hat{S}^{HF}(11') = g \left(\begin{array}{c} n(1) + \tilde{n}(1), \tilde{m}(1) \\ \tilde{m}^*(1), n(1) + \tilde{n}(1) \end{array} \right) \delta(1, 1'). \quad (28)$$

The functions \hat{S}^{\cong} on the right-hand side of (27) contain the second-order contributions to the condensate self-energy⁴

$$\begin{aligned} \hat{S}^{\cong}(1, 1') &= -\frac{1}{2}v(13)v(21')\hat{g}^{\cong}(11')[\hat{g}^{\leq}(23)\hat{g}^{\cong}(32)] \\ &\quad -v(13)v(21')\hat{g}^{\cong}(12)[\hat{g}^{\leq}(23)\hat{g}^{\cong}(31')]. \end{aligned} \quad (29)$$

As we will show in Sec. 6, these contributions give rise to a dissipative term in a generalized GP equation. If we recall the definition for the condensate propagator \hat{h} in (11) and use the explicit form for the condensate Hartree–Fock self energy \hat{S}^{HF} in (28), we obtain a generalized GP equation for the macroscopic wave function $\Phi(\mathbf{r}, t)$, namely

$$\begin{aligned} &\left[-i\frac{\partial}{\partial t_1} + \frac{\nabla_1^2}{2m} - U_{\text{ext}}(\mathbf{r}_1) + \mu_0 - g(n_c(1) + 2\tilde{n}(1)) \right] \Phi(1) \\ &= g\tilde{m}(1)\Phi^*(1) + \int_{-\infty}^{t_1} d\bar{1}(S_{1\bar{1}}^> - S_{1\bar{1}}^<)(1, \bar{1})\Phi(\bar{1}) \\ &\quad + \int_{-\infty}^{t_1} d\bar{1}(S_{1\bar{2}}^> - S_{1\bar{2}}^<)(1, \bar{1})\Phi^*(\bar{1}). \end{aligned} \quad (30)$$

Initially, the equation of motion for the macroscopic order parameter (22) was given in terms of the condensate three-field source function η in (23). We have rewritten the equation for Φ in terms of the condensate self-energies S defined by (26) because, as we will see in Sec. 6, S is more convenient to work with than the condensate source function η .

3. GENERALIZED KINETIC EQUATION

In general, the order parameter $\Phi(\mathbf{r}, t)$ in a Bose fluid is complex. It is often written in terms of the condensate amplitude and phase

$$\Phi(\mathbf{r}, t) = \sqrt{n_c(\mathbf{r}, t)} e^{i\theta(\mathbf{r}, t)}. \quad (31)$$

To derive a generalized kinetic equation, the rapid oscillations of the phase in (31) cause difficulties. Following KK, we first gauge transform (13) and (14) to the local rest frame in which the superfluid velocity is zero. This

corresponds to removing the phase of the macroscopic wavefunction. The required gauge transformations on $\hat{h}(1, 1')$ and $\hat{g}^{\leq}(1, 1')$ are¹²

$$\begin{aligned}\hat{h}'(1, 1') &= e^{-i\theta(1)\hat{\tau}_3}\hat{h}(1, 1')e^{i\theta(1')\hat{\tau}_3}, \\ \hat{g}'^{\leq}(1, 1') &= e^{-i\theta(1)\hat{\tau}_3}\hat{g}^{\leq}(1, 1')e^{i\theta(1')\hat{\tau}_3}.\end{aligned}\quad (32)$$

The physical interpretation of (32) is that it involves a transformation to a coordinate system in which non-condensate atoms are moving with average velocity \mathbf{v}_s relative to a stationary condensate. For example, the transformation (32) gives

$$\begin{aligned}\tilde{g}'_{11}^{\geq}(1, 1') &= e^{-i(\theta(1)-\theta(1'))}\tilde{g}_{11}^{\geq}(1, 1'), \\ h'_{11}(1, 1') &= -i\sqrt{n_c(1)n_c(1')}.\end{aligned}\quad (33)$$

One sees that, in the local rest frame (denoted by a prime), the order parameter $\Phi'(\mathbf{r}, t)$ is real (see (11)).

Equations (13) and (14) remain unchanged in form after this transformation to the local rest frame as long as \hat{g}_0^{-1} is replaced by (compare with (19) in the lab frame):

$$\begin{aligned}\hat{g}_0^{-1}(1, 1') &= \left[i\hat{\tau}_3 \frac{\partial}{\partial t_1} - \frac{\partial\theta(1)}{\partial t_1} + \frac{1}{2} [\nabla_1 + i\hat{\tau}_3\nabla_1\theta(1)]^2 \right. \\ &\quad \left. + \mu_0 - U_{\text{ext}}(\mathbf{r}_1) \right] \hat{\delta}(1, 1').\end{aligned}\quad (34)$$

We recall that the superfluid velocity $\mathbf{v}_s(\mathbf{R}, T)$ and the local chemical potential $\mu_c(\mathbf{R}, T)$ are defined as the spatial and time derivatives of the phase,¹² namely

$$\begin{aligned}m\mathbf{v}_s(\mathbf{R}, T) &\equiv \nabla_{\mathbf{R}}\theta(\mathbf{R}, T) \\ \frac{\partial\theta(\mathbf{R}, T)}{\partial T} &\equiv -\left[\mu_c(\mathbf{R}, T) - \mu_0 + \frac{1}{2}mv_s^2(\mathbf{R}, T) \right],\end{aligned}\quad (35)$$

where, in the lab frame, the condensate wavefunction is given by $\Phi(\mathbf{R}, T) \equiv \sqrt{n_c(\mathbf{R}, T)}e^{i\theta(\mathbf{R}, T)}$. Therefore, the gauge transformation changes the momentum $\mathbf{p} \rightarrow \mathbf{p} - m\mathbf{v}_s$ as expected for the momentum in the local rest frame (when we Fourier transform, the gradient operator in (34) becomes the momentum).

In Sec. 2, we have written down the equations of motion (13) and (14) for the non-equilibrium real-time Green's functions. We now want to use these to derive a kinetic equation for the quasiparticle distribution function.

We recall that (13) and (14) are matrix equations. In addition, (13) involves differential operators with the respect to the coordinates (\mathbf{r}, t) and (14) involves derivatives with respect to the coordinates (\mathbf{r}', t') . Since our single-particle Green's functions are functions of both coordinates $(1, 1')$, one has to find a way to combine both equations to derive a single kinetic equation for a quasiparticle distribution function. We will discuss later how these Green's functions are related to the quasiparticle distribution function. However, we emphasize that we will need to use both (13) and (14) to derive a kinetic equation for quasiparticles.

In the Kadanoff–Baym procedure,^{4, 10} one rewrites the equations of motion in terms of relative and center-of-mass space-time coordinates, defined by

$$\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_{1'}, \quad t = t_1 - t_{1'}; \quad \mathbf{R} = \frac{\mathbf{r}_1 + \mathbf{r}_{1'}}{2}, \quad T = \frac{t_1 + t_{1'}}{2}. \quad (36)$$

In the literature, this is sometimes known as the Wigner representation.^{15, 28} It allows one to separate variables describing “slow” and “fast” processes in the system. In thermal equilibrium, the Green's functions are only functions of the relative space-time coordinates \mathbf{r} and t , and moreover are sharply peaked about $\mathbf{r} = 0$ and $t = 0$.¹⁰ Since we assume that the external disturbances are slowly varying in space and time (with a wavelength much larger than the thermal deBroglie wavelength), we expect that these slowly varying external disturbances will not change this dependence of $\hat{g}^{\cong}(\mathbf{r}, t; \mathbf{R}, T)$ on small values of \mathbf{r} and t . Therefore, our non-equilibrium correlation functions (like $\hat{g}, \hat{\Sigma}$, etc.) are assumed to be dominated by the small values of relative coordinates (\mathbf{r}, t) (equivalently, by large momenta and frequencies in the Fourier transforms), but vary slowly as functions of the center-of-mass coordinates (\mathbf{R}, T) .

Using these key properties of the non-equilibrium correlation functions to simplify the equations, we now write (13) and (14) in terms of the center-of-mass and relative coordinates,^{4, 12, 13, 23} and take the trace of the resulting matrix equation to obtain:

$$\begin{aligned} & \hat{\mathcal{L}}_{11} \tilde{g}_{11}^<(\mathbf{r}, t; \mathbf{R}, T) + \hat{\mathcal{L}}_{22} \tilde{g}_{22}^<(\mathbf{r}, t; \mathbf{R}, T) \\ &= g \left[\left(\left(\mathbf{r} \cdot \nabla_{\mathbf{R}} + t \frac{\partial}{\partial T} \right) m \right) \tilde{g}_{21}^< + \left(\left(\mathbf{r} \cdot \nabla_{\mathbf{R}} + t \frac{\partial}{\partial T} \right) m^* \right) \tilde{g}_{12}^< \right] \\ &+ \int_{-\infty}^{\infty} d\bar{\mathbf{r}} d\bar{t} \text{Tr} \left(\Sigma^>(\mathbf{r} - \bar{\mathbf{r}}, t - \bar{t}) \hat{g}^<(\bar{\mathbf{r}}, \bar{t}) - \Sigma^<(\mathbf{r} - \bar{\mathbf{r}}, t - \bar{t}) \hat{g}^>(\bar{\mathbf{r}}, \bar{t}) \right). \end{aligned} \quad (37)$$

As usual, for simplicity, the (\mathbf{R}, T) dependence of the \tilde{g} , Σ and m is left implicit. The operators \mathcal{L}_{11} and \mathcal{L}_{22} on the left-hand side of (37) are defined as

$$\begin{aligned} \mathcal{L}_{11} &\equiv i \frac{\partial}{\partial T} + \left(\mathbf{r} \cdot \nabla_{\mathbf{R}} + t \frac{\partial}{\partial T} \right) \left(\mu(\mathbf{R}, T) - U_{\text{eff}}(\mathbf{R}, T) \right) + \frac{1}{m} \nabla_{\mathbf{R}} \cdot \nabla_{\mathbf{r}} \\ &\quad + i \left(\left(\mathbf{r} \cdot \nabla_{\mathbf{R}} + t \frac{\partial}{\partial T} \right) \mathbf{v}_s(\mathbf{R}, T) \right) \cdot \nabla_{\mathbf{r}} + i \mathbf{v}_s(\mathbf{R}, T) \cdot \nabla_{\mathbf{R}} + i \nabla_{\mathbf{R}} \cdot \mathbf{v}_s(\mathbf{R}, T) \\ \mathcal{L}_{22} &\equiv -i \frac{\partial}{\partial T} + \left(\mathbf{r} \cdot \nabla_{\mathbf{R}} + t \frac{\partial}{\partial T} \right) \left(\mu(\mathbf{R}, T) - U_{\text{eff}}(\mathbf{R}, T) \right) + \frac{1}{m} \nabla_{\mathbf{R}} \cdot \nabla_{\mathbf{r}} \\ &\quad - i \left(\left(\mathbf{r} \cdot \nabla_{\mathbf{R}} + t \frac{\partial}{\partial T} \right) \mathbf{v}_s(\mathbf{R}, T) \right) \cdot \nabla_{\mathbf{r}} - i \mathbf{v}_s(\mathbf{R}, T) \cdot \nabla_{\mathbf{R}} - i \nabla_{\mathbf{R}} \cdot \mathbf{v}_s(\mathbf{R}, T), \end{aligned} \quad (38)$$

where the effective dynamic HF field $U_{\text{eff}}(\mathbf{R}, T)$ is given by

$$U_{\text{eff}}(\mathbf{R}, T) = U_{\text{ext}}(\mathbf{R}) + 2g(n_c(\mathbf{R}, T) + \tilde{n}(\mathbf{R}, T)). \quad (39)$$

We emphasize that in the expansion for the small values of relative coordinates (\mathbf{r}, t) , we did not keep all terms of order $\partial/\partial T$ and $\nabla_{\mathbf{R}}$ in (37). These additional terms that we have neglected contribute to the many-body renormalization effects, i.e., how the two-particle interaction changes the dispersion relation of the quasiparticles due to terms of second order in g . Such corrections involve the real part of the second-order Beliaev self-energies. The Bogoliubov–Popov quasiparticle approximation we use for the spectral densities $a_{\alpha\beta}$ in (15) do not include such second-order effects. In the present paper, we concentrate on the damping effects associated with the collisional self-energies Σ^{\cong} on the right-hand side of (37). For further discussion of the KB formalism related to going past the simple Bogoliubov quasiparticle approximation, see Chap. 9 of Ref. 10 and Chap. 6 in Ref. 15. We also note that Kane and Kadanoff^{12, 13} have, working within a different second-order approximation, derived equation of motions which keep the extra terms left out of (37) and (38) which include the many-body renormalization effects. In that sense, their work is more sophisticated implementation of the Kadanoff–Baym formalism.

The double Fourier transform of (37) gives

$$\begin{aligned} \mathcal{L}_{11} \tilde{g}_{11}^< + \mathcal{L}_{22} \tilde{g}_{22}^< - g \nabla_{\mathbf{R}} m^* \cdot \nabla_{\mathbf{p}} \tilde{g}_{12}^< - g \nabla_{\mathbf{R}} m \cdot \nabla_{\mathbf{p}} \tilde{g}_{21}^< + g \frac{\partial m^*}{\partial T} \frac{\partial \tilde{g}_{12}^<}{\partial \omega} + g \frac{\partial m}{\partial T} \frac{\partial \tilde{g}_{21}^<}{\partial \omega} \\ = \text{Tr} \left(\hat{\Sigma}^>(\mathbf{p}, \omega; \mathbf{R}, T) \hat{g}^<(\mathbf{p}, \omega; \mathbf{R}, T) - \hat{\Sigma}^<(\mathbf{p}, \omega; \mathbf{R}, T) \hat{g}^>(\mathbf{p}, \omega; \mathbf{R}, T) \right), \end{aligned} \quad (40)$$

with $\tilde{g}_{\alpha\beta}^{\leq} \equiv \tilde{g}_{\alpha\beta}^{\leq}(\mathbf{p}, \omega; \mathbf{R}, T)$ and

$$\begin{aligned} \mathcal{L}_{11} &= \frac{\partial}{\partial T} + \nabla_{\mathbf{p}}[\tilde{\varepsilon}_p + \mathbf{p} \cdot \mathbf{v}_s] \cdot \nabla_{\mathbf{R}} - \nabla_{\mathbf{R}}[\tilde{\varepsilon}_p + \mathbf{p} \cdot \mathbf{v}_s] \cdot \nabla_{\mathbf{p}} \\ &\quad + \frac{\partial}{\partial T}[\tilde{\varepsilon}_p + \mathbf{p} \cdot \mathbf{v}_s] \frac{\partial}{\partial \omega} \\ \mathcal{L}_{22} &= -\frac{\partial}{\partial T} + \nabla_{\mathbf{p}}[\tilde{\varepsilon}_p - \mathbf{p} \cdot \mathbf{v}_s] \cdot \nabla_{\mathbf{R}} - \nabla_{\mathbf{R}}[\tilde{\varepsilon}_p - \mathbf{p} \cdot \mathbf{v}_s] \cdot \nabla_{\mathbf{p}} \\ &\quad + \frac{\partial}{\partial T}[\tilde{\varepsilon}_p - \mathbf{p} \cdot \mathbf{v}_s] \frac{\partial}{\partial \omega}. \end{aligned} \quad (41)$$

The result in (40) gives an equation closely related to the quasiparticle kinetic equation we are trying to derive. Here, $\tilde{\varepsilon}_p$ is defined by

$$\tilde{\varepsilon}_p(\mathbf{R}, T) \equiv \frac{p^2}{2m} + U_{\text{ext}}(\mathbf{R}) + 2gn(\mathbf{R}, T) - \mu_c(\mathbf{R}, T). \quad (42)$$

In Sec. 6, we shall see that the condensate chemical potential $\mu_c(\mathbf{R}, T)$ is given by

$$\mu_c = -\frac{\nabla_{\mathbf{R}}^2 \sqrt{n_c(\mathbf{R}, T)}}{2m \sqrt{n_c(\mathbf{R}, T)}} + U_{\text{ext}}(\mathbf{R}) + g[2\tilde{n}(\mathbf{R}, T) + n_c(\mathbf{R}, T)]. \quad (43)$$

A kinetic equation for thermally excited atoms in a trapped Bose gas can be written in terms of distribution functions for either atoms or for quasiparticle excitations. In our earlier papers,^{4, 23} we transformed the equations of motion for a non-equilibrium Green's functions at high temperatures into a kinetic equation for a single-particle distribution function $f(\mathbf{p}, \mathbf{R}, T)$ describing the non-condensate atoms. The latter are assumed to have a Hartree–Fock spectrum. If one wants to use a more realistic spectrum valid at low temperatures, it is much more convenient to work within a quasiparticle picture. In the theory of Bose-condensed trapped gases, one introduces quasiparticles by expressing the quantum field operators for the non-condensate atoms as a coherent superposition of creation and annihilation operators for Bose quasiparticles, with the weights given by the usual Bose-coherence factors u and v

$$\tilde{\psi}(\mathbf{R}, T) \equiv \sum_i [u_i(\mathbf{R}) \hat{\alpha}_i e^{-iE_i T/\hbar} + v_i^*(\mathbf{R}) \hat{\alpha}_i^\dagger e^{iE_i T/\hbar}]. \quad (44)$$

In the semiclassical approximation, (44) becomes

$$\tilde{\psi}(\mathbf{R}, T) \equiv \int \frac{d\mathbf{p}}{(2\pi)^3} [u_p(\mathbf{R}) \hat{\alpha}_p e^{-iE_p T/\hbar} + v_p^*(\mathbf{R}) \hat{\alpha}_p^\dagger e^{iE_p T/\hbar}]. \quad (45)$$

Here $\hat{\alpha}_p^\dagger$ and $\hat{\alpha}_p$ are the Bogoliubov quasiparticle creation and annihilation operators, respectively, which obey the usual Bose commutation relations. One can see that creating an atom with momentum \mathbf{p} is equivalent to creating a quasiparticle with momentum \mathbf{p} with amplitude u_p and at the same time, destroying a quasiparticle with momentum $-\mathbf{p}$ and amplitude v_p . The quasiparticle distribution function is given by the statistical average of the quasiparticle operators, i.e., $f(\mathbf{p}) \equiv \langle \hat{\alpha}_p^\dagger \hat{\alpha}_p \rangle$. We recall^{4, 23} that distribution function for atoms $f_{\text{at}}(\mathbf{p}, \omega; \mathbf{R}, T)$ is directly related to the diagonal Green's function, namely

$$f_{\text{at}}(\mathbf{p}, \omega; \mathbf{R}, T) = -i\tilde{g}_{11}^<(\mathbf{p}, \omega; \mathbf{R}, T). \quad (46)$$

In terms of quantum field operators, $\tilde{g}_{11}^<$ is given by (see Chap. 9 of Ref. 10)

$$\tilde{g}_{11}^<(\mathbf{p}, \omega; \mathbf{R}, T) = i \int d\mathbf{r} dt e^{-i\mathbf{p} \cdot \mathbf{r} + i\omega t} \left\langle \tilde{\psi}^\dagger \left(\mathbf{R} - \frac{\mathbf{r}}{2}, T - \frac{t}{2} \right) \tilde{\psi} \left(\mathbf{R} + \frac{\mathbf{r}}{2}, T + \frac{t}{2} \right) \right\rangle. \quad (47)$$

The usual Boltzmann equation is expressed in terms of the Wigner distribution function $f_W(\mathbf{p}, \mathbf{R}, T)$.^{10, 4} This limits the description to the semiclassical approximation because it is assumed that the position and momentum of the particles can be defined simultaneously. In order to use this kind of distribution function for quantum systems, it is necessary to perform some type of averaging in order to remove effects due to the uncertainty principle. In our work, we want to derive a kinetic equation for the quasiparticles which is valid at all temperatures and therefore the semiclassical approximation will no longer be valid. To include the quantum effects, we introduce the *quasiparticle* distribution function $f(\mathbf{p}, \omega; \mathbf{R}, T)$ with an additional variable ω in the following way^{12, 15, 28, 29}

$$\begin{aligned} \tilde{g}_{\alpha\beta}^<(\mathbf{p}, \omega; \mathbf{R}, T) &\equiv ia_{\alpha\beta}(\mathbf{p}, \omega; \mathbf{R}, T) f(\mathbf{p}, \omega; \mathbf{R}, T) \\ \tilde{g}_{\alpha\beta}^>(\mathbf{p}, \omega; \mathbf{R}, T) &\equiv ia_{\alpha\beta}(\mathbf{p}, \omega; \mathbf{R}, T)[1 + f(\mathbf{p}, \omega; \mathbf{R}, T)], \end{aligned} \quad (48)$$

where the spectral density $a_{\alpha\beta}$ is defined in (15). Using the Bogoliubov–Popov approximation for the spectral density in (54), one can use (48) to

obtain the well-known relation between the quasiparticle distribution function $f(\mathbf{p}, \omega; \mathbf{R}, T)$ and the atom distribution function f_{at} in (46), namely

$$f_{\text{at}}(\mathbf{p}, \omega; \mathbf{R}, T) = \left(u_p^2(\mathbf{R}, T) + v_p^2(\mathbf{R}, T) \right) f(\mathbf{p}, \omega; \mathbf{R}, T) + v_p^2(\mathbf{R}, T). \quad (49)$$

The semiclassical Wigner distribution function is obtained by taking the frequency integral of the atom distribution function defined in (49).

One can show, using (13) and (14), that the spectral function $a_{\alpha\beta}$ in (15) satisfies

$$\begin{aligned} \mathcal{L}_{11} a_{11} + \mathcal{L}_{22} a_{22} - g \nabla_{\mathbf{R}} m^* \cdot \nabla_{\mathbf{p}} a_{12} - g \nabla_{\mathbf{R}} m \cdot \nabla_{\mathbf{p}} a_{21} \\ - g \frac{\partial m^*}{\partial T} \frac{\partial a_{12}}{\partial \omega} - g \frac{\partial m}{\partial T} \frac{\partial a_{21}}{\partial \omega} = 0. \end{aligned} \quad (50)$$

Using (48) and (50), one can rewrite the kinetic equation (40) for $\tilde{g}_{\alpha\beta}^{\lessgtr}$ to obtain a new kinetic equation specifically for the quasiparticle distribution function $f(\mathbf{p}, \omega; \mathbf{R}, T)$ in the following form:

$$\begin{aligned} a_{11} \mathcal{L}_{11} f + a_{22} \mathcal{L}_{22} f - a_{12} g \nabla_{\mathbf{R}} m^* \cdot \nabla_{\mathbf{p}} f - a_{21} g \nabla_{\mathbf{R}} m \cdot \nabla_{\mathbf{p}} f \\ + a_{12} \frac{\partial m^*}{\partial T} \frac{\partial f}{\partial \omega} + a_{21} \frac{\partial m}{\partial T} \frac{\partial f}{\partial \omega} = f \text{Tr}(\Sigma^> \hat{a}) - (1 + f) \text{Tr}(\Sigma^< \hat{a}). \end{aligned} \quad (51)$$

We notice that (51) includes terms involving $\partial/\partial\omega$. The additional variable ω in $f(\mathbf{p}, \omega; \mathbf{R}, T)$ results in new streaming terms on the left side of (51). These terms are not present in the semiclassical kinetic equation (which is obtained from (51) integrating over ω), which shows that the terms involving $\partial/\partial\omega$ are of quantum origin.^{15, 28, 29}

Equation (51) is the most general form for a kinetic equation for the quasiparticle distribution function f within our model. To derive (51), we have only assumed that the external disturbances vary slowly in space and time, and therefore all relevant physical quantities vary slowly as function of center-of-mass coordinates (\mathbf{R}, T) defined in (36). The other assumption that we made is that one can introduce a quasiparticle distribution function f through the definition in (48). Of course, at this stage, one could say that we are only replacing one unknown function with another. The generalized kinetic equation for a Bose-condensed system (51) was first derived by Kane.¹³ From (51), we see that the general structure of the collision integral I has the following form^{12, 13, 5}

$$I[f(\mathbf{p}, \mathbf{R}, T)] \equiv \int \frac{d\omega}{2\pi} [f \text{Tr}(\hat{\Sigma}^> \hat{a}) - (1 + f) \text{Tr}(\hat{\Sigma}^< \hat{a})]. \quad (52)$$

Using (48) in the the general expression for the non-equilibrium Beliaev self-energies Σ^{\geq} given in (20), one can prove that the collision integral given by (52) conserves momentum (see Appendix for details),

$$\int d\mathbf{p} \mathbf{p} I[f(\mathbf{p}, \mathbf{R}, T)] = 0. \quad (53)$$

One can also prove that (52) conserves energy as well. To prove this, however, we need to work within a specific approximation for the single-particle spectral density $a_{\alpha\beta}(\mathbf{p}, \omega; \mathbf{R}, T)$.

4. KINETIC EQUATION IN THE BOGOLIUBOV-POPOV APPROXIMATION

In this section, we use the general equation in (51) derived in previous section to derive a quasiparticle kinetic equation within the Bogoliubov-Popov approximation. More precisely, this means that we will use the spectral densities with the Bogoliubov-Popov quasiparticle excitation energies^{10, 13, 24}

$$\begin{aligned} a_{11}(\mathbf{p}, \omega; \mathbf{R}, T) &= 2\pi[u_p^2\delta(\omega - \mathbf{v}_s \cdot \mathbf{p} - E_p) - v_p^2\delta(\omega - \mathbf{v}_s \cdot \mathbf{p} + E_p)] \\ a_{12}(\mathbf{p}, \omega; \mathbf{R}, T) &= -2\pi u_p v_p [\delta(\omega - \mathbf{v}_s \cdot \mathbf{p} - E_p) - \delta(\omega - \mathbf{v}_s \cdot \mathbf{p} + E_p)] \\ a_{21}(\mathbf{p}, \omega; \mathbf{R}, T) &= a_{12}(\mathbf{p}, \omega; \mathbf{R}, T) \\ a_{22}(\mathbf{p}, \omega; \mathbf{R}, T) &= -a_{11}(-\mathbf{p}, -\omega; \mathbf{R}, T). \end{aligned} \quad (54)$$

Here, the Bose-coherence factors $u(\mathbf{R}, T)$ and $v(\mathbf{R}, T)$ are given by²⁴

$$u_p^2(\mathbf{R}, T) = \frac{\tilde{\varepsilon}_p(\mathbf{R}, T) + E_p(\mathbf{R}, T)}{2E_p(\mathbf{R}, T)}, \quad u_p^2 - v_p^2 = 1, \quad u_p v_p = \frac{gn_c(\mathbf{R}, T)}{2E_p(\mathbf{R}, T)} \quad (55)$$

and the quasiparticle energy E_p is given by

$$E_p(\mathbf{R}, T) = \sqrt{\tilde{\varepsilon}_p^2(\mathbf{R}, T) - (gn_c(\mathbf{R}, T))^2}. \quad (56)$$

We emphasize that the spectral densities in (54) could be derived in the quasiparticle approximation from the general equations of motion for the Green's functions as it has been shown in Ref. 13. We simply start with them as input into our general formalism.

In the Thomas–Fermi approximation,¹ one neglects the quantum pressure term in (43), in which case the quasiparticle energy E_p reduces to the usual Bogoliubov excitation energy

$$E_p(\mathbf{R}, T) = \sqrt{\varepsilon_p^2 + 2gn_c(\mathbf{R}, T) \varepsilon_p}, \quad (57)$$

where $\varepsilon_p = p^2/2m$. We note that spectral densities $a_{\alpha\beta}(\mathbf{p}, \omega; \mathbf{R}, T)$ in (54) exhibit both positive and negative energy poles. In the Hartree–Fock approximation used in our earlier work,⁴ $u_p^2 = 1$ and $v_p^2 = 0$. Physically, (54) corresponds to the assumption that the thermal cloud can be considered as a gas of weakly-interacting single-particle excitations with the excitation energy given by (57). One can check explicitly that (54) do satisfy the general equation of motion given in (50). Note that in the literature, v_p is sometimes defined with the opposite sign, such that $u_p v_p$ in (55) is negative.

Substituting the spectral densities (54) into (51), we can now derive a kinetic equation for the quasiparticles. After lengthy algebra, we obtain

$$\begin{aligned} & \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \delta(\omega - \mathbf{v}_s \cdot \mathbf{p} - E_p) \left[\frac{\partial f}{\partial T} + \nabla_{\mathbf{p}}(E_p + \mathbf{v}_s \cdot \mathbf{p}) \cdot \nabla_{\mathbf{R}} f - \nabla_{\mathbf{R}}(E_p + \mathbf{v}_s \cdot \mathbf{p}) \cdot \nabla_{\mathbf{p}} f \right. \\ & + \frac{\partial}{\partial T}(E_p + \mathbf{v}_s \cdot \mathbf{p}) \frac{\partial f}{\partial \omega} + u_p^2 \left((1+f) \Sigma_{11}^{\leq} - f \Sigma_{11}^{\geq} \right) + v_p^2 \left((1+f) \Sigma_{22}^{\leq} - f \Sigma_{22}^{\geq} \right) \\ & \left. - u_p v_p \left((1+f)(\Sigma_{12}^{\leq} + \Sigma_{21}^{\leq}) - f(\Sigma_{12}^{\geq} + \Sigma_{21}^{\geq}) \right) \right] \\ & + \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \delta(\omega - \mathbf{v}_s \cdot \mathbf{p} + E_p) \left[\frac{\partial f}{\partial T} + \nabla_{\mathbf{p}}(-E_p + \mathbf{v}_s \cdot \mathbf{p}) \cdot \nabla_{\mathbf{R}} f \right. \\ & - \nabla_{\mathbf{R}}(-E_p + \mathbf{v}_s \cdot \mathbf{p}) \cdot \nabla_{\mathbf{p}} f \\ & + \frac{\partial}{\partial T}(-E_p + \mathbf{v}_s \cdot \mathbf{p}) \frac{\partial f}{\partial \omega} + v_p^2 \left((1+f) \Sigma_{11}^{\leq} - f \Sigma_{11}^{\geq} \right) \\ & + u_p^2 \left((1+f) \Sigma_{22}^{\leq} - f \Sigma_{22}^{\geq} \right) \\ & \left. - u_p v_p \left((1+f)(\Sigma_{12}^{\leq} + \Sigma_{21}^{\leq}) - f(\Sigma_{12}^{\geq} + \Sigma_{21}^{\geq}) \right) \right] = 0, \quad (58) \end{aligned}$$

where $\Sigma_{\alpha\beta} = \Sigma_{\alpha\beta}(\mathbf{p}, \omega; \mathbf{R}, T)$. This is a kinetic equation for the frequency dependent quasiparticle distribution function f expressed in terms of an integral over both positive and negative energy poles. If we recall the expression for Bose coherence factors u and v given by (55), we note that the second term in (58) is the same as the first term in (58) if we replace

$-E_p$ with E_p . Therefore, it is sufficient only to consider the first term to obtain the kinetic equation for the quasiparticle distribution function defined by

$$f_{qp}(\mathbf{p}, \mathbf{R}, T) \equiv f(\mathbf{p}, \omega = E_p + \mathbf{v}_s \cdot \mathbf{p}; \mathbf{R}, T). \quad (59)$$

We obtain finally

$$\left[\frac{\partial f_{qp}}{\partial T} + \nabla_{\mathbf{p}}(E_p + \mathbf{v}_s \cdot \mathbf{p}) \cdot \nabla_{\mathbf{R}} f_{qp} - \nabla_{\mathbf{R}}(E_p + \mathbf{v}_s \cdot \mathbf{p}) \cdot \nabla_{\mathbf{p}} f_{qp} \right] = I[f_{qp}]. \quad (60)$$

Here the collision integral $I[f_{qp}(p, \mathbf{R}, T)]$, defined in (52), becomes

$$\begin{aligned} I[f_{qp}(p, \mathbf{R}, T)] &= \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \left[u_p^2 \left((1 + f_{qp}) \Sigma_{11}^{\leq} - f_{qp} \Sigma_{11}^{\geq} \right) + v_p^2 \left((1 + f_{qp}) \Sigma_{22}^{\leq} - f_{qp} \Sigma_{22}^{\geq} \right) \right. \\ &\quad \left. + u_p v_p \left((1 + f_{qp})(\Sigma_{12}^{\leq} + \Sigma_{21}^{\leq}) - f_{qp}(\Sigma_{12}^{\geq} + \Sigma_{21}^{\geq}) \right) \right]. \quad (61) \end{aligned}$$

Such a kinetic equation is also derived by Kane in his thesis,¹³ but this is not discussed in Ref. 12. A similar quasiparticle kinetic equation has also been derived by Stoof⁵ using the related Schwinger–Keldysh formalism for non-equilibrium processes. However, Stoof mainly studied the high temperature limit of the collision integral.

What remains to be done is to choose a specific approximation for the second-order self-energy $\Sigma_{\alpha\beta}$, so that we can evaluate the collision integral in (61). Here, we use the second order Beliaev approximation given by (20), the Fourier transform of which is

$$\begin{aligned} \hat{\Sigma}^{\cong}(\mathbf{p}, \omega; \mathbf{R}, T) &= -\frac{1}{2} g^2 \int \frac{d\mathbf{p}_i d\omega_i}{(2\pi)^8} \delta(\omega + \omega_1 - \omega_2 - \omega_3) \delta(\mathbf{p} + \mathbf{p}_1 - \mathbf{p}_2 - \mathbf{p}_3) \\ &\quad \times [\hat{g}^{\cong}(\mathbf{p}_2, \omega_2) \text{Tr}[\hat{g}^{\leq}(\mathbf{p}_1, \omega_1) \hat{g}^{\cong}(\mathbf{p}_3, \omega_3)] \\ &\quad + 2\hat{g}^{\cong}(\mathbf{p}_2, \omega_2) \hat{g}^{\leq}(\mathbf{p}_1, \omega_1) \hat{g}^{\cong}(\mathbf{p}_3, \omega_3) \\ &\quad + \hat{g}^{\cong}(\mathbf{p}_2, \omega_2) \text{Tr}[\hat{h}(\mathbf{p}_1, \omega_1) \hat{g}^{\cong}(\mathbf{p}_3, \omega_3) + \hat{g}^{\leq}(\mathbf{p}_1, \omega_1) \hat{h}(\mathbf{p}_3, \omega_3)] \\ &\quad + \hat{h}(\mathbf{p}_2, \omega_2) \text{Tr}[\hat{g}^{\leq}(\mathbf{p}_1, \omega_1) \hat{g}^{\cong}(\mathbf{p}_3, \omega_3)] \\ &\quad + 2\hat{h}(\mathbf{p}_2, \omega_2) \hat{g}^{\leq}(\mathbf{p}_1, \omega_1) \hat{g}^{\cong}(\mathbf{p}_3, \omega_3) \\ &\quad + 2\hat{g}^{\cong}(\mathbf{p}_2, \omega_2) [\hat{h}(\mathbf{p}_1, \omega_1) \hat{g}^{\cong}(\mathbf{p}_3, \omega_3) + \hat{g}^{\leq}(\mathbf{p}_1, \omega_1) \hat{h}(\mathbf{p}_3, \omega_3)]]. \quad (62) \end{aligned}$$

As usual, the (\mathbf{R}, T) dependence of the functions Σ , \tilde{g} and h on the right-hand side has been suppressed for simplicity of notation. The quasiparticle energy $E_p(\mathbf{R}, T)$ in (60) is the energy of the quasiparticles in the local rest frame ($\mathbf{v}_s = 0$). The Beliaev second-order expression (62) consists of two kinds of contributions: (1) Terms that include both the condensate propagator h and the non-condensate propagators \tilde{g} ; (2) Terms that include the non-condensate propagators \tilde{g} only. The first type of contribution will give rise to the collision integral that describe collisions that include one condensate atom interacting with the thermally excited quasiparticles. As in earlier work,^{3,4,9} we denote this part of the collision integral as C_{12} , indicating that we go from 1 thermally excited quasiparticle (and one condensate atom) to 2 thermally excited quasiparticles. The second type of contribution in (62) only includes non-condensate propagators. We denote this part of the collision integral as C_{22} , indicating that it describes collisions where 2 thermally excited quasiparticles are scattered into 2 excited quasiparticles. At low temperatures, when the number of thermally excited quasiparticles is small, we can neglect the C_{22} collision integral relative to C_{12} .

From (53), it follows that both C_{12} and C_{22} conserve momentum (see Appendix for details), i.e.,

$$\int d\mathbf{p} \mathbf{p} C_{12} = 0$$

$$\int d\mathbf{p} \mathbf{p} C_{22} = 0.$$
(63)

In addition, using (54) in (62) one can show that the collision integral in (52) conserves quasiparticle energy E_p and therefore both C_{12} and C_{22} will satisfy the conditions

$$\int d\mathbf{p} E_p C_{12} = 0$$

$$\int d\mathbf{p} E_p C_{22} = 0.$$
(64)

One can show that for slowly varying external disturbances, the condensate propagator in (11) can be approximated by

$$\hat{h}(\mathbf{p}, \omega; \mathbf{R}, T) = n_c(\mathbf{R}, T)(2\pi)^4 \delta(\mathbf{p}) \delta(\omega) \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}. \quad (65)$$

To evaluate the collision integral C_{12} in terms of the Bose coherence factors u and v , we need to substitute (54) and (65) into (62). One can simplify (62) greatly using the following exact symmetry relations;

$$\begin{aligned}\tilde{g}_{22}^{\leq}(\mathbf{p}, \omega; \mathbf{R}, T) &= \tilde{g}_{11}^{\geq}(-\mathbf{p}, -\omega; \mathbf{R}, T), \\ \tilde{g}_{12}^{\leq}(\mathbf{p}, \omega; \mathbf{R}, T) &= \tilde{g}_{12}^{\geq}(-\mathbf{p}, -\omega; \mathbf{R}, T), \\ \tilde{g}_{21}^{\leq}(\mathbf{p}, \omega; \mathbf{R}, T) &= \tilde{g}_{21}^{\geq}(-\mathbf{p}, -\omega; \mathbf{R}, T).\end{aligned}\quad (66)$$

After considerable algebra, one finds the following expressions for the non-equilibrium self-energy (considering now only the C_{12} terms, which include one condensate propagator):

$$\begin{aligned}\Sigma_{11}^{\geq}(\mathbf{p}, \omega; \mathbf{R}, T) &= -g^2 \int \frac{d\mathbf{p}_2 d\omega_2}{(2\pi)^4} n_c(\mathbf{R}, T) [2\tilde{g}_{11}^{\geq}(\mathbf{p}_2, \omega_2) \tilde{g}_{11}^{\geq}(\mathbf{p} - \mathbf{p}_2, \omega - \omega_2) \\ &\quad + 4\tilde{g}_{11}^{\geq}(\mathbf{p}_2, \omega_2) \tilde{g}_{11}^{\leq}(\mathbf{p}_2 - \mathbf{p}, \omega_2 - \omega) \\ &\quad + 8\tilde{g}_{12}^{\geq}(\mathbf{p}_2, \omega_2) \tilde{g}_{11}^{\geq}(\mathbf{p} - \mathbf{p}_2, \omega - \omega_2) \\ &\quad + 4\tilde{g}_{12}^{\geq}(\mathbf{p}_2, \omega_2) \tilde{g}_{12}^{\geq}(\mathbf{p} - \mathbf{p}_2, \omega - \omega_2)],\end{aligned}\quad (67)$$

$$\begin{aligned}\Sigma_{22}^{\geq}(\mathbf{p}, \omega; \mathbf{R}, T) &= -g^2 \int \frac{d\mathbf{p}_2 d\omega_2}{(2\pi)^4} n_c(\mathbf{R}, T) [4\tilde{g}_{12}^{\geq}(\mathbf{p}_2, \omega_2) \tilde{g}_{12}^{\geq}(\mathbf{p} - \mathbf{p}_2, \omega - \omega_2) \\ &\quad + 4\tilde{g}_{11}^{\geq}(\mathbf{p}_2, \omega_2) \tilde{g}_{11}^{\leq}(\mathbf{p}_2 - \mathbf{p}, \omega_2 - \omega) \\ &\quad + 8\tilde{g}_{12}^{\geq}(\mathbf{p}_2, \omega_2) \tilde{g}_{11}^{\leq}(\mathbf{p}_2 - \mathbf{p}, \omega_2 - \omega) \\ &\quad + 2\tilde{g}_{11}^{\leq}(-\mathbf{p}_2, -\omega_2) \tilde{g}_{11}^{\leq}(\mathbf{p}_2 - \mathbf{p}, \omega_2 - \omega)],\end{aligned}\quad (68)$$

$$\begin{aligned}\Sigma_{12}^{\geq}(\mathbf{p}, \omega; \mathbf{R}, T) &= -g^2 \int \frac{d\mathbf{p}_2 d\omega_2}{(2\pi)^4} n_c(\mathbf{R}, T) [6\tilde{g}_{12}^{\geq}(\mathbf{p}_2, \omega_2) \tilde{g}_{12}^{\geq}(\mathbf{p} - \mathbf{p}_2, \omega - \omega_2) \\ &\quad + 4\tilde{g}_{12}^{\geq}(\mathbf{p}_2, \omega_2) \tilde{g}_{11}^{\geq}(\mathbf{p} - \mathbf{p}_2, \omega - \omega_2) \\ &\quad + 4\tilde{g}_{11}^{\geq}(\mathbf{p}_2, \omega_2) \tilde{g}_{11}^{\leq}(\mathbf{p}_2 - \mathbf{p}, \omega_2 - \omega) \\ &\quad + 4\tilde{g}_{12}^{\geq}(\mathbf{p}_2, \omega_2) \tilde{g}_{11}^{\leq}(\mathbf{p}_2 - \mathbf{p}, \omega_2 - \omega)],\end{aligned}\quad (69)$$

$$\Sigma_{21}^{\geq}(\mathbf{p}, \omega; \mathbf{R}, T) = \Sigma_{12}^{\geq}(\mathbf{p}, \omega; \mathbf{R}, T).\quad (70)$$

Again, the (\mathbf{R}, T) dependence of the \tilde{g} 's is suppressed on the right-hand side. One notices that these expressions have the same structure as the thermal equilibrium ones obtained by Shi and Griffin as well as others.^{11, 16, 17} This is expected since the whole structure of our theory is only valid for a

systems slightly perturbed from thermal equilibrium, with all physical quantities assumed to vary slowly as functions of center-of-mass coordinates (\mathbf{R}, T) . The entire KB formalism reduces to the usual equilibrium self-energies in the appropriate limit, which is one of its strengths.

One can show using the general properties of the non-equilibrium Green's functions in (66) in conjunction with (48) that the quasiparticle distribution function $f(\mathbf{p}, \omega; \mathbf{R}, T)$ satisfies the exact relation

$$f(-\mathbf{p}, -\omega; \mathbf{R}, T) = -(1 + f(\mathbf{p}, \omega; \mathbf{R}, T)). \quad (71)$$

If we introduce following standard abbreviations for the Bose-coherence factors

$$A_p \equiv u_p^2, \quad B_p \equiv v_p^2, \quad C_p \equiv -u_p v_p, \quad (72)$$

the self-energies in (67)–(69) can be finally written as (using (71))

$$\begin{aligned} & \Sigma_{11}^{\cong}(\mathbf{p}, \omega; \mathbf{R}, T) \\ &= g^2 \int \frac{d\mathbf{p}_2 d\omega_2}{(2\pi)^2} n_c(\mathbf{R}, T) \left(\frac{(1+f_1)(1+f_2)}{f_1 f_2} \right) \\ & \times [(2A_1 A_2 + 8A_1 C_2 + 4C_1 C_2 + 4B_1 A_2) \delta(\omega_2 - E_2) \delta(\omega_1 - E_1) \\ & - (2B_1 A_2 + 8B_1 C_2 + 4C_1 C_2 + 4A_1 A_2) \delta(\omega_2 - E_2) \delta(\omega_1 + E_1) \\ & - (2A_1 B_2 + 8A_1 C_2 + 4C_1 C_2 + 4B_1 B_2) \delta(\omega_2 + E_2) \delta(\omega_1 - E_1) \\ & + (2B_1 B_2 + 8B_1 C_2 + 4C_1 C_2 + 4A_1 B_2) \delta(\omega_2 + E_2) \delta(\omega_1 + E_1)], \end{aligned} \quad (73)$$

$$\begin{aligned} & \Sigma_{12}^{\cong}(\mathbf{p}, \omega; \mathbf{R}, T) \\ &= g^2 \int \frac{d\mathbf{p}_2 d\omega_2}{(2\pi)^2} n_c(\mathbf{R}, T) \left(\frac{(1+f_1)(1+f_2)}{f_1 f_2} \right) \\ & \times [(6C_1 C_2 + 4A_1 C_2 + 4B_1 A_2 + 4B_1 C_2) \delta(\omega_2 - E_2) \delta(\omega_1 - E_1) \\ & - (4B_1 C_2 + 4A_1 C_2 + 6C_1 C_2 + 4A_1 A_2) \delta(\omega_2 - E_2) \delta(\omega_1 + E_1) \\ & - (4A_1 C_2 + 4B_1 C_2 + 6C_1 C_2 + 4B_1 B_2) \delta(\omega_2 + E_2) \delta(\omega_1 - E_1) \\ & + (4A_1 B_2 + 4B_1 C_2 + 6C_1 C_2 + 4A_1 C_2) \delta(\omega_2 + E_2) \delta(\omega_1 + E_1)], \end{aligned} \quad (74)$$

and

$$\begin{aligned}
& \Sigma_{22}^{\approx}(\mathbf{p}, \omega; \mathbf{R}, T) \\
&= g^2 \int \frac{d\mathbf{p}_2 d\omega_2}{(2\pi)^2} n_c(\mathbf{R}, T) \begin{pmatrix} (1+f_1)(1+f_2) \\ f_1 f_2 \end{pmatrix} \\
& \times [(2B_1 B_2 + 8B_1 C_2 + 4C_1 C_2 + 4B_1 A_2) \delta(\omega_2 - E_2) \delta(\omega_1 - E_1) \\
& - (2A_1 B_2 + 8A_1 C_2 + 4C_1 C_2 + 4A_1 A_2) \delta(\omega_2 - E_2) \delta(\omega_1 + E_1) \\
& - (2B_1 A_2 + 8B_1 C_2 + 4C_1 C_2 + 4B_1 B_2) \delta(\omega_2 + E_2) \delta(\omega_1 - E_1) \\
& + (2A_1 A_2 + 8A_1 C_2 + 4C_1 C_2 + 4A_1 B_2) \delta(\omega_2 + E_2) \delta(\omega_1 + E_1)], \quad (75)
\end{aligned}$$

where $\mathbf{p} - \mathbf{p}_2 \equiv \mathbf{p}_1$ and $\omega - \omega_2 \equiv \omega_1$.

Using these results, we can finally evaluate the C_{12} collision integral given in (61)

$$\begin{aligned}
C_{12}[f] &= 2g^2 n_c(\mathbf{R}, T) \int \frac{d\mathbf{p}_1 d\mathbf{p}_2}{(2\pi)^2} [(1+f) f_1 f_2 - f(1+f_1)(1+f_2)] \\
& \times \delta(\mathbf{p} - \mathbf{p}_1 - \mathbf{p}_2) \left[\left((u_1 - v_1)(u_p u_2 + v_p v_2) + (u_2 - v_2)(u_p u_1 + v_p v_1) \right. \right. \\
& \left. \left. - (u_p - v_p)(u_1 v_2 + v_1 u_2) \right)^2 \delta(E_p - E_1 - E_2) \right. \\
& + 2 \left((u_1 - v_1)(u_p u_2 + v_p v_2) + (u_p - v_p)(u_1 u_2 + v_1 v_2) \right. \\
& \left. \left. - (u_2 - v_2)(u_p v_1 + u_1 v_p) \right)^2 \delta(E_p + E_1 - E_2) \right. \\
& + \left(v_p u_1 u_2 + u_p u_2 v_1 + u_1 v_2 u_p - v_p v_2 u_1 \right. \\
& \left. \left. - v_1 u_2 v_p - u_p v_2 v_1 \right)^2 \delta(E_p + E_1 + E_2) \right]. \quad (76)
\end{aligned}$$

The last term in (76) clearly vanishes because of the energy delta function. We recall that all u 's, v 's and the quasiparticle energy E_p in (76) have an implicit (\mathbf{R}, T) dependence. If we change $\mathbf{p}_1 \rightarrow -\mathbf{p}_1$ in the second term in (76), and use (71), we can simplify (76) slightly to obtain

$$\begin{aligned}
C_{12}[f] &= 2g^2 n_c(\mathbf{R}, T) \int \frac{d\mathbf{p}_1 d\mathbf{p}_2}{(2\pi)^2} [(1+f) f_1 f_2 - f(1+f_1)(1+f_2)] \\
& \times \left[\left((u_1 - v_1)(u_p u_2 + v_p v_2) + (u_2 - v_2)(u_p u_1 + v_p v_1) \right. \right. \\
& \left. \left. - (u_p - v_p)(u_1 v_2 + v_1 u_2) \right)^2 \delta(\mathbf{p} - \mathbf{p}_1 - \mathbf{p}_2) \delta(E_p - E_1 - E_2) \right. \\
& - 2 \left((u_1 - v_1)(u_p u_2 + v_p v_2) + (u_p - v_p)(u_1 u_2 + v_1 v_2) \right. \\
& \left. \left. - (u_2 - v_2)(u_p v_1 + u_1 v_p) \right)^2 \delta(\mathbf{p} + \mathbf{p}_1 - \mathbf{p}_2) \delta(E_p + E_1 - E_2) \right]. \quad (77)
\end{aligned}$$

The first term in (77) describes the decay of an excitation with momentum \mathbf{p} into two excitations with momenta \mathbf{p}_1 and \mathbf{p}_2 . At $T=0$, this is the only scattering process possible since there are no thermal excitations. The second term describes an excitation of momentum \mathbf{p} absorbing a thermal excitation of momentum \mathbf{p}_1 , leaving an excitation with momentum $\mathbf{p}_2 = \mathbf{p} + \mathbf{p}_1$.

This form of the collision integral in (77) was first written down by Eckern,¹⁹ and shortly after Kirkpatrick and Dorfman⁹ gave a more detailed derivation. Here, we have used the Kadanoff–Baym approach to give what we believe is a cleaner derivation of C_{12} , in a form which is also valid for a trapped Bose-condensed gas.

After some algebra, one can also rewrite (77) in the following more compact form⁹

$$\begin{aligned}
 C_{12}[f] = & 2g^2 n_c(\mathbf{R}, T) \int \frac{d\mathbf{p}_1 d\mathbf{p}_2 d\mathbf{p}_3}{(2\pi)^2} |A(2, 3; 1)|^2 \delta(\mathbf{p}_1 - \mathbf{p}_2 - \mathbf{p}_3) \\
 & \times \delta(E_1 - E_2 - E_3) [\delta(\mathbf{p} - \mathbf{p}_1) - \delta(\mathbf{p} - \mathbf{p}_2) - \delta(\mathbf{p} - \mathbf{p}_3)] \\
 & \times [(1 + f_1) f_2 f_3 - f_1 (1 + f_2)(1 + f_3)]. \quad (78)
 \end{aligned}$$

Here the scattering amplitude in $|A|^2$ is given in term of the Bose coherence factors u and v

$$\begin{aligned}
 A(2, 3; 1) \equiv & (u_3 - v_3)(u_1 u_2 + v_1 v_2) + (u_2 - v_2)(u_1 u_3 + v_1 v_3) \\
 & - (u_1 - v_1)(u_2 v_3 + v_2 u_3). \quad (79)
 \end{aligned}$$

The first term in (78) is equivalent to the first term in (77), while the other term in (77) is equivalent to the second and third terms in (78).

In conclusion, the kinetic equation we have derived for thermally excited quasiparticles is given by

$$\left[\frac{\partial}{\partial T} + \nabla_{\mathbf{p}}(E_p + \mathbf{v}_s \cdot \mathbf{p}) \cdot \nabla_{\mathbf{R}} - \nabla_{\mathbf{R}}(E_p + \mathbf{v}_s \cdot \mathbf{p}) \cdot \nabla_p \right] f_{qp}(\mathbf{p}, \mathbf{R}, T) = C_{12}[f_{qp}], \quad (80)$$

with C_{12} given explicitly by (77), or equivalently, (78). The derivation of this equation using the KB formalism is the main result of this paper. As we have noted, a kinetic equation of this kind has been derived by other authors^{13, 19, 9, 5} using a variety of techniques and approximations. The earliest microscopic derivation was given in Kane's thesis,¹³ but he did not evaluate explicitly the collision integral to obtain results such as (78) and (77).

To remove the rapidly varying phase of the order parameter real, we have gauge transformed to the local rest frame where the condensate is at

rest. Hence, the energy of the thermally excited quasiparticles is measured relative to this local frame. Since the thermal excitations are moving with the superfluid velocity \mathbf{v}_s relative to the condensate, the energy of quasiparticles measured relative to the condensate is $E_p + \mathbf{v}_s \cdot \mathbf{p}$. Therefore, the expression $E_p + \mathbf{v}_s \cdot \mathbf{p}$ in the streaming term on the left-hand side of (80) is expected. Similarly, if we denote the quasiparticle distribution in the coordinate system where the quasiparticles are at rest by $f(\mathbf{p}, \omega; \mathbf{R}, T)$, the quasiparticles moving with the velocity \mathbf{v}_s relative to the condensate will be described by the distribution function $f(\mathbf{p}, \omega = E_p + \mathbf{v}_s \cdot \mathbf{p}; \mathbf{R}, T) \equiv f_{qp}(\mathbf{p}, \mathbf{R}, T)$, as occurs in (80).^{32, 33}

If we use the frame of reference where the quasiparticles are at rest, the streaming term will include the energy of the quasiparticles only (i.e., the $\mathbf{v}_s \cdot \mathbf{p}$ term in (80) will not be present). This lab frame of reference is used in the work of Zaremba, Nikuni, and Griffin.³

To understand the C_{12} collision integral in (77) better and the corresponding scattering processes that it describes, it is useful to consider a few limiting cases for a uniform gas. We define $p_0^2 \equiv 2mgn_c$ as the characteristic momentum for the crossover between the linear and the quadratic part of the quasiparticle spectrum ($p_0 = \hbar k_0 \equiv \hbar \zeta^{-1}$, where ζ is the healing length). We then consider the following special cases:

(1) If all momenta $p_i \gg p_0$, then the quasiparticle spectrum E_p defined by (57) is equal to a single-particle spectrum $\tilde{\epsilon}_p$ given in (42). Moreover in this limit, it follows from (55) that $u \rightarrow 1$ and $v \rightarrow 0$. Hence, the scattering amplitude A in (79) becomes unity and the collision integral in (78) then reduces to the one recently derived by Zaremba, Nikuni, and Griffin^{4, 3} using a different approach (see also Ref. 5). Clearly, this approximation is only valid at finite temperatures where the dominant excitation spectrum is described by the Hartree–Fock single-particle spectrum in (42).

(2) In the opposite limit, when all three momenta p_i are small, one can expand the Bose coherence factors u and v in the following way

$$\begin{aligned} u_p &\simeq \left(\frac{gn_c}{2E_p}\right)^{1/2} + \frac{1}{2} \left(\frac{E_p}{2gn_c}\right)^{1/2} \\ v_p &\simeq \left(\frac{gn_c}{2E_p}\right)^{1/2} - \frac{1}{2} \left(\frac{E_p}{2gn_c}\right)^{1/2}, \end{aligned} \quad (81)$$

where $E_p \simeq cp$ and $c = \sqrt{gn_c/m}$ is the speed of Bogoliubov sound. The sign of v_p in (81) is opposite from the one given in Refs. 30 and 31 because we

have defined $u_p v_p$ in (55) to be positive. In this limit, one obtains for the scattering amplitude the following expression¹⁹

$$A(1; 2, 3) \simeq \frac{3}{2^{7/4}} \sqrt{\frac{p_1 p_2 p_3}{p_0^3}}. \quad (82)$$

This approximation is valid at low temperatures, where only low-momentum excitations are relevant.

(3) Finally, one can consider the scattering of phonons (low-momentum excitations) with momentum $p_3 \ll p_0$ with particles (high-momentum excitations) with momenta $p_1, p_2 \gg p_0$. The corresponding amplitude for this process is given by¹⁹

$$A(1; 2, 3) \simeq 2^{3/4} (p_3/p_0)^{1/2}. \quad (83)$$

Therefore, in the case of a sound wave scattering with particle-like excitations the scattering amplitude only depends on the wavevector of the sound wave. It is independent of the momenta of the scattering particles.

5. LOCAL EQUILIBRIUM SOLUTION

To describe the thermalization of quasiparticles, it is sufficient to consider the C_{22} collision integral. From (48), (52) and (62), one obtains

$$\begin{aligned} C_{22}[f] = & -\frac{1}{2} g^2 \int \frac{d\mathbf{p}_i d\omega_i}{(2\pi)^8} \delta(E_p + \omega_1 - \omega_2 - \omega_3) \delta(\mathbf{p} + \mathbf{p}_1 - \mathbf{p}_2 - \mathbf{p}_3) \\ & \times [ff_1(1+f_2)(1+f_3) - (1+f)(1+f_1)f_2f_3] \\ & \times \left[\text{Tr} \left(\hat{a}(\mathbf{p}_2, \omega_2) \hat{a}(\mathbf{p}, E_p) \right) \text{Tr}(\hat{a}(\mathbf{p}_1, \omega_1) \hat{a}(\mathbf{p}_3, \omega_3)) \right) \\ & + \text{Tr} \left(2\hat{a}(\mathbf{p}_2, \omega_2) \hat{a}(\mathbf{p}_1, \omega_1) \hat{a}(\mathbf{p}_3, \omega_3) \hat{a}(\mathbf{p}, E_p) \right) \right]. \quad (84) \end{aligned}$$

The local equilibrium distribution function for quasiparticles $f_0(\mathbf{p}, \omega; \mathbf{R}, T)$ is determined by the requirement that $C_{22}[f_0] = 0$. One can see from (84) that one doesn't have to specify some specific approximation for the single-particle spectral densities. We only need a solution for f such that the expression in (84) containing the f 's vanishes. One can verify that $C_{22}[f_0] = 0$ if $f_0(\mathbf{p}, \omega; \mathbf{R}, T)$ has the following form

$$f_0(\mathbf{p}, \omega; \mathbf{R}, T) = \frac{1}{e^{\beta(\omega - \mathbf{p} \cdot (\mathbf{v}_n - \mathbf{v}_s) - \mu_{qp}(\mathbf{R}, T))} - 1}. \quad (85)$$

The vector $\mathbf{v}_n - \mathbf{v}_s$ describes the mean drift velocity of the quasiparticle gas in the local rest frame of the superfluid and for small velocities we have^{9, 14}

$$\rho_n(\mathbf{v}_n - \mathbf{v}_s) = \int d\mathbf{p} \mathbf{p} f_0(\mathbf{p}, \omega; \mathbf{R}, T). \quad (86)$$

which defines the normal fluid density ρ_n . This is consistent with the usual Landau definition. The distribution function given in (85) differs from the usual equilibrium quasiparticle distribution function discussed in the standard literature for phonons and rotons in liquid helium.¹⁴ Since the number of quasiparticles is not conserved, the usual form for the equilibrium quasiparticle distribution function has no chemical potential (i.e., the chemical potential is zero). Here, we have introduced a chemical potential in (85) to allow for the possibility that the condensate atoms and the thermally excited quasiparticles are *not* in the diffusive equilibrium with each other.

To understand the physics of the quasiparticle chemical potential in (85) better, let us first consider the high temperature case. At high temperatures, the particle and the quasiparticle excitation spectrum are equivalent and the the local equilibrium distribution function in the lab frame is given by³

$$f_0(\mathbf{p}, \mathbf{R}, T) = \frac{1}{e^{\beta(((\mathbf{p} - m\mathbf{v}_n)^2/2m) + U(\mathbf{R}, T) - \tilde{\mu}(\mathbf{R}, T))} - 1}, \quad (87)$$

with $U(\mathbf{R}, T) = U_{\text{ext}}(\mathbf{R}) + 2gn(\mathbf{R}, T)$ and $\tilde{\mu}(\mathbf{R}, T)$ is the local chemical potential of the non-condensate atoms. If we transform (87) to the local rest frame (where $\mathbf{v}_s = 0$), then $\mathbf{p}' \equiv \mathbf{p} - m\mathbf{v}_s$ is the momentum in the local rest frame and (87) becomes

$$f_0(\mathbf{p}, \mathbf{R}, T) = \frac{1}{e^{\beta[E_{p'} - \mathbf{p}' \cdot (\mathbf{v}_n - \mathbf{v}_s) - (\tilde{\mu} - \mu_c) + (1/2)m(\mathbf{v}_n - \mathbf{v}_s)^2]} - 1}. \quad (88)$$

Here $E_{p'} \equiv (p'^2/2m) + gn_c(\mathbf{R}, T)$ is the excitation energy in the local rest frame. Therefore, if we define³

$$\mu_{\text{diff}} \equiv \tilde{\mu} - \mu_c - \frac{1}{2}m(\mathbf{v}_n - \mathbf{v}_s)^2, \quad (89)$$

we see the “quasiparticle chemical potential” μ_{qp} introduced in (85) can be identified with μ_{diff} discussed at length by Zaremba, Nikuni, and Griffin.³ Hence, we see that in the quasiparticle description in a local frame in which the condensate is stationary, the difference (μ_{diff}) between the chemical

potentials of the condensate and non-condensate that was introduced by ZNG to describe the non-diffusive equilibrium of these two components appears very naturally as the quasiparticle chemical potential. The standard case discussed in the superfluid helium ^4He literature¹⁴ corresponds to $\mu_{qp} = 0$ (see, however, the discussion¹⁴ of the second viscosity coefficients in superfluid ^4He).

To summarize, we can distinguish two distribution functions f which satisfy $C_{22}[f_0] = 0$:

(1) The condensate atoms and the quasiparticle excitations are in diffusive thermal equilibrium, i.e., $\mu_{qp} = 0$ and hence $C_{12}[f] = 0$.

(2) The condensate atoms and the quasiparticle excitations are not in the diffusive thermal equilibrium, i.e., $\mu_{qp} \neq 0$. In this case, one finds that $C_{12}[f]$ is proportional to $[1 - e^{\beta\mu_{qp}}]$, as in ZNG.³

6. GENERALIZED GROSS-PITAEVSKII EQUATION

As we noted in the Introduction, the dynamics of a trapped Bose-condensed gas is usually described in the literature by the Gross-Pitaevskii equation of motion (1) for the condensate order parameter. Linearizing this equation, one obtains the collective mode frequencies that have been confirmed in many experiments.¹ However, at finite temperatures, the simple GP equation is insufficient since it does not describe the thermally excited atoms. Moreover, even at $T \ll T_{BEC}$, in recent experiments at JILA on ^{85}Rb ,³⁴ the dimensionless gas parameter can be as large as $\sqrt{n_c a^3} \sim 10^{-1}$, i.e., the quantum depletion of the condensate is not negligible. The simplest generalization of the zero temperature GP equation of motion is usually done by including an additional self-consistent Hartree-Fock mean field $2g\tilde{n}$ produced by the thermally excited atoms.^{1, 18} The condensate atoms described by $\Phi(\mathbf{R}, T)$ move in this mean-field, in addition to the field produced by the condensate. However it is clear that the second-order collisions which we have included in deriving the quasiparticle kinetic equation in Sec. 4 must also be included in a generalized GP equation for $\Phi(\mathbf{R}, T)$. Technically, this arises from the three-field correlation function given in (23).

Zaremba, Nikuni, and Griffin³ have evaluated this three-field correlation function for the thermally excited atoms following the method of Kirkpatrick and Dorfman,⁹ a method which we do not find very transparent (see Appendix A of Ref. 3). ZNG obtained a generalized Gross-Pitaevskii equation with a dissipative term associated with the C_{12} collisions. This new dissipative term is, as expected, proportional to the collision integral for scattering between atoms in the condensate and thermal

atoms, since such collisions change the number of atoms in the condensate. The ZNG work was limited to finite temperatures where the thermal atoms can be described as free atoms moving in the dynamic Hartree–Fock mean-field produced by all other atoms (both those in the condensate and in the thermal cloud). Recently, we have derived⁴ the same generalized GP equation as ZNG using the powerful KB method method. In this section, we now extend this kind of calculation so that is valid at low temperatures. The new equation of motion for the order parameter will be shown to be identical to that obtained in Refs. 4 and 3, apart from the fact that C_{12} is now given by the expression in (77). That is, it now involves the Bogoliubov quasiparticles and collision cross-section is renormalized by various Bose-coherence factors involving the u 's and v 's.

To derive an equation of motion for the condensate order parameter, we first write equation (30) for $\Phi(\mathbf{r}, t)$ in the local rest frame. As before, under the gauge transformation (32), the only change is that the non-interacting propagator is now given by (34). The equation of motion in the new local frame is (see (27))

$$\begin{aligned} & \left[i \frac{\partial}{\partial t} - \frac{\partial \theta(1)}{\partial t} + \frac{1}{2m} [\nabla_{\mathbf{r}} + im \mathbf{v}_s(1)]^2 + \mu_0 - U_{\text{ext}}(\mathbf{r}) - g(2\tilde{n}(1) + n_c(1)) \right] \Phi(1) \\ &= \int_{-\infty}^t d\bar{t} [S_{11}^> - S_{11}^<] \left(\mathbf{r} - \bar{\mathbf{r}}, t - \bar{t}; (\mathbf{r} + \bar{\mathbf{r}})/2, (t + \bar{t})/2 \right) \Phi(\bar{\mathbf{r}}, \bar{t}) \\ &+ \int_{-\infty}^t d\bar{t} [S_{12}^> - S_{12}^<] \left(\mathbf{r} - \bar{\mathbf{r}}, t - \bar{t}; (\mathbf{r} + \bar{\mathbf{r}})/2, (t + \bar{t})/2 \right) \Phi^*(\bar{\mathbf{r}}, \bar{t}). \end{aligned} \quad (90)$$

Here, we have rewritten the condensate self-energy in the center-of-mass and relative coordinates and, as usual, set $\tilde{m} = 0$ (the Popov approximation). We recall that in the local rest frame, the order parameter phase is removed and hence $\Phi(\mathbf{r}, t) = \sqrt{n_c(\mathbf{r}, t)}$.

We assume, as usual, that the S correlation function (29) is dominated by small values of the relative space-time coordinates $(\mathbf{r} - \bar{\mathbf{r}}, t - \bar{t})$. Hence we can approximate S_{11}^{\gtrless} in (90) by $S^{\gtrless}(\mathbf{r} - \bar{\mathbf{r}}, t - \bar{t}; \mathbf{r}, t)$. For the same reason, we can also approximate the macroscopic wavefunction $\Phi(\bar{\mathbf{r}}, \bar{t})$ in the integrand of (90) by $\Phi(\mathbf{r}, t) \equiv \sqrt{n_c(\mathbf{r}, t)}$. Hence, (90) simplifies to

$$\begin{aligned} & \left[i \frac{\partial}{\partial t} - \frac{\partial \theta(1)}{\partial t} + \frac{1}{2m} [\nabla_{\mathbf{r}} + im \mathbf{v}_s(1)]^2 + \mu_0 - U_{\text{ext}}(\mathbf{r}) - g(2\tilde{n}(1) + n_c(1)) \right] \Phi(1) \\ &= \int_{-\infty}^t d\bar{\mathbf{r}} d\bar{t} [(S_{11}^> - S_{11}^<)(\mathbf{r} - \bar{\mathbf{r}}, t - \bar{t}; \mathbf{r}, t) \Phi(\mathbf{r}, t) \\ &+ (S_{12}^> - S_{12}^<)(\mathbf{r} - \bar{\mathbf{r}}, t - \bar{t}; \mathbf{r}, t) \Phi^*(\mathbf{r}, t)]. \end{aligned} \quad (91)$$

We can rewrite (91) (labeling $(\mathbf{r}, t) \rightarrow (\mathbf{R}, T)$) as follows

$$\begin{aligned}
 & \left[i \frac{\partial}{\partial T} - \frac{\partial \theta(\mathbf{R}, T)}{\partial T} + \frac{1}{2m} [\nabla_{\mathbf{R}} + im \mathbf{v}_s(\mathbf{R}, T)]^2 + \mu_0 - U_{\text{ext}}(\mathbf{R}) \right. \\
 & \quad \left. - g(2\tilde{n}(\mathbf{R}, T) + n_c(\mathbf{R}, T)) \right] \Phi(\mathbf{R}, T) \\
 & = \Phi(\mathbf{R}, T) \int \frac{d\mathbf{p} d\omega}{(2\pi)^4} [S_{11}^> - S_{11}^< + S_{12}^> - S_{12}^<](\mathbf{p}, \omega; \mathbf{R}, T) \\
 & \quad \times \int_{-\infty}^T d\bar{\mathbf{r}} d\bar{t} e^{i\mathbf{p}(\mathbf{R}-\bar{\mathbf{r}}) - i\omega(T-\bar{t})}. \tag{92}
 \end{aligned}$$

In the second-order Beliaev approximation, the condensate self-energy is given by (29). The Fourier transform of this is

$$\begin{aligned}
 & \hat{S}^{\cong}(\mathbf{p}, \omega; \mathbf{R}, T) \\
 & = -\frac{1}{2} g^2 \int \frac{d\mathbf{p}_i d\omega_i}{(2\pi)^8} \delta(\omega + \omega_1 - \omega_2 - \omega_3) \delta(\mathbf{p} + \mathbf{p}_1 - \mathbf{p}_2 - \mathbf{p}_3) \\
 & \quad \times \left[\hat{g}^{\cong}(\mathbf{p}_2, \omega_2; \mathbf{R}, T) \text{Tr}[\hat{g}^{\cong}(\mathbf{p}_1, \omega_1; \mathbf{R}, T) \hat{g}^{\cong}(\mathbf{p}_3, \omega_3; \mathbf{R}, T)] \right. \\
 & \quad \left. + 2\hat{g}^{\cong}(\mathbf{p}_2, \omega_2; \mathbf{R}, T) \hat{g}^{\cong}(\mathbf{p}_1, \omega_1; \mathbf{R}, T) \hat{g}^{\cong}(\mathbf{p}_3, \omega_3; \mathbf{R}, T) \right]. \tag{93}
 \end{aligned}$$

In evaluating the right-hand side of (92), we use the identity

$$\lim_{\delta \rightarrow 0^+} \int_{-\infty}^T d\bar{t} e^{-i(\omega + i\delta)(T-\bar{t})} \simeq \pi \delta(\omega) + iP \left(\frac{1}{\omega} \right), \tag{94}$$

and only keep the delta function part, to obtain

$$\begin{aligned}
 & \left[i \frac{\partial}{\partial T} - \frac{\partial \theta(\mathbf{R}, T)}{\partial T} + \frac{1}{2m} [\nabla_{\mathbf{R}} + im \mathbf{v}_s(\mathbf{R}, T)]^2 \right. \\
 & \quad \left. + \mu_0 - U_{\text{ext}}(\mathbf{R}) - g(2\tilde{n}(\mathbf{R}, T) + n_c(\mathbf{R}, T)) \right. \\
 & \quad \left. - (S_{11}^> - S_{11}^< + S_{12}^> - S_{12}^<)(\mathbf{p} = 0, \omega = 0; \mathbf{R}, T) \right] \Phi(\mathbf{R}, T) = 0. \tag{95}
 \end{aligned}$$

Using (48) in the condensate self-energy S given in (93), the second-order terms appearing in (95) reduce to

$$\begin{aligned}
& (S_{11}^> - S_{11}^< + S_{12}^> - S_{12}^<)(\mathbf{p} = 0, \omega = 0; \mathbf{R}, T) \\
&= i \frac{1}{2} g^2 \int \frac{d\mathbf{p}_i d\omega_i}{(2\pi)^8} \delta(\omega_1 - \omega_2 - \omega_3) \delta(\mathbf{p}_1 - \mathbf{p}_2 - \mathbf{p}_3) \\
&\quad \times [f_1(1 + f_2)(1 + f_3) - (1 + f_1) f_2 f_3] \left[(a_{11} + a_{12})(\mathbf{p}_2, \omega_2) \right. \\
&\quad \times \text{Tr} \left(\hat{a}(\mathbf{p}_1, \omega_1) \hat{a}(\mathbf{p}_3, \omega_3) \right) + 2 \left(\hat{a}(\mathbf{p}_2, \omega_2) \hat{a}(\mathbf{p}_1, \omega_1) \hat{a}(\mathbf{p}_3, \omega_3) \right)_{11} \\
&\quad \left. + 2 \left(\hat{a}(\mathbf{p}_2, \omega_2) \hat{a}(\mathbf{p}_1, \omega_1) \hat{a}(\mathbf{p}_3, \omega_3) \right)_{12} \right]. \tag{96}
\end{aligned}$$

Recalling that in the local rest frame we have $\Phi(\mathbf{R}, T) = \sqrt{n_c(\mathbf{R}, T)}$, with no phase, we finally obtain a generalized Gross–Pitaevskii equation in the following form

$$\begin{aligned}
i \frac{\partial \sqrt{n_c(\mathbf{R}, T)}}{\partial T} = & \left[\frac{\partial \theta(\mathbf{R}, T)}{\partial T} - \frac{1}{2m} [\nabla_{\mathbf{R}} + im\mathbf{v}_s(\mathbf{R}, T)]^2 - \mu_0 \right. \\
& \left. + U_{\text{ext}}(\mathbf{R}) + g[2\tilde{n}(\mathbf{R}, T) + n_c(\mathbf{R}, T)] - iR(\mathbf{R}, T) \right] \sqrt{n_c(\mathbf{R}, T)}. \tag{97}
\end{aligned}$$

The new dissipative term R in the GP equation is clearly related to the C_{12} collision term in the kinetic equation (80), namely^{3,4}

$$R(\mathbf{R}, T) \equiv \int \frac{d\mathbf{p}}{(2\pi)^3} \frac{C_{12}[f(\mathbf{p}, \mathbf{R}, T)]}{2n_c(\mathbf{R}, T)}. \tag{98}$$

This term describes the damping of condensate amplitude fluctuations due to collisions with the thermal excitations. The appearance of the dissipative term in (97) is expected since the C_{12} collisions change the number of atoms in the condensate and hence can modify the magnitude of the condensate macroscopic wavefunction. We note that since we ignore the real part of the second-order self-energies, the condensate chemical potential in (43) is not modified.

For consistency, since we have used the Thomas–Fermi approximation in defining our Bogoliubov quasiparticle energies in (57), we should neglect the $\nabla_{\mathbf{R}}^2$ term in the chemical potential in (43), as well as in our equation of motion for the condensate, as given by (90) and the final result (97).

If we transform back into the lab frame (where we have $\Phi = \sqrt{n_c} e^{i\theta}$), (97) reduces to the time-dependent generalized Gross–Pitaevskii equation

for $\Phi(\mathbf{R}, T)$ discussed by ZNG.³ However, C_{12} now involves the Bogoliubov quasiparticle spectrum in place of the HF particle-like spectrum used in Ref. 3, and in addition, the collision integral matrix elements involve the characteristic Bose coherence factors u and v .

7. KOHN MODE

In this section, we show that the non-condensate and condensate both exhibit the rigid in-phase oscillations, the Kohn mode. This mode is discussed in detail in Sec. 6 of Ref. 3, and the analysis there is easily generalized to the more general equations we are discussing. The center-of-mass oscillation of the non-condensate and condensate density profiles corresponding to the Kohn mode is given by

$$\begin{aligned} n_c(\mathbf{R}, T) &\equiv n_{c0}(\mathbf{R} - \boldsymbol{\eta}(T)) \\ \tilde{n}(\mathbf{R}, T) &\equiv \tilde{n}_0(\mathbf{R} - \boldsymbol{\eta}(T)). \end{aligned} \quad (99)$$

Here, the center-of-mass displacement $\boldsymbol{\eta}(T)$ (with $\mathbf{v}_s = \dot{\boldsymbol{\eta}}$) satisfies the harmonic oscillator equation of motion

$$\frac{\partial^2 \eta_\alpha}{\partial T^2} = -\omega_\alpha^2 \eta_\alpha, \quad (100)$$

where ω_α is the trap frequency in the α th direction. The quasiparticle distribution function $f(\mathbf{p}, \mathbf{R}, T)$ corresponds to the equilibrium density profile oscillating around its center of mass with the trap frequency, i.e.,

$$f(\mathbf{p}, \mathbf{R}, T) \equiv f_0(\mathbf{p}, \mathbf{R} - \boldsymbol{\eta}(T)). \quad (101)$$

To prove (101), we note that with (99), the expression for the Bogoliubov excitation energy in (57) reduces to

$$E_p(\mathbf{R}, T) = \sqrt{\varepsilon_p^2 + 2gn_{c0}(\mathbf{R} - \boldsymbol{\eta}) \varepsilon_p} \equiv E_{p0}(\mathbf{R} - \boldsymbol{\eta}). \quad (102)$$

Therefore, the kinetic equation for the quasiparticle distribution function in (80) is

$$\begin{aligned} &\left[\frac{\partial}{\partial T} + \nabla_{\mathbf{p}}(E_{p0}(\mathbf{R} - \boldsymbol{\eta}(T)) + \dot{\boldsymbol{\eta}} \cdot \mathbf{p}) \cdot \nabla_{\mathbf{R}} - \nabla_{\mathbf{R}} E_{p0}(\mathbf{R} - \boldsymbol{\eta}(T)) \cdot \nabla_{\mathbf{p}} \right] f_0(\mathbf{p}, \mathbf{R} - \boldsymbol{\eta}(T)) \\ &= C_{12}[f_0(\mathbf{p}, \mathbf{R} - \boldsymbol{\eta}(T))]. \end{aligned} \quad (103)$$

If we expand $f_0(\mathbf{p}, \mathbf{R} - \boldsymbol{\eta}(T))$ around $\boldsymbol{\eta} = 0$,

$$f_0(\mathbf{p}, \mathbf{R} - \boldsymbol{\eta}(T)) = f_0(\mathbf{p}, \mathbf{R}) - \boldsymbol{\eta} \cdot \nabla_{\mathbf{R}} f_0(\mathbf{p}, \mathbf{R}), \quad (104)$$

and neglect the quadratic terms in η , (103) simplifies to

$$\begin{aligned} & \nabla_{\mathbf{p}} E_{p_0}(\mathbf{R} - \boldsymbol{\eta}(T)) \cdot \nabla_{\mathbf{R}} f_0(\mathbf{p}, \mathbf{R} - \boldsymbol{\eta}(T)) \\ & \quad - \nabla_{\mathbf{R}} E_{p_0}(\mathbf{R} - \boldsymbol{\eta}(T)) \cdot \nabla_{\mathbf{p}} f_0(\mathbf{p}, \mathbf{R} - \boldsymbol{\eta}(T)) \\ & = C_{12}[f_0(\mathbf{p}, \mathbf{R} - \boldsymbol{\eta}(T))]. \end{aligned} \quad (105)$$

The left-hand side of (105) is seen to be the kinetic equation for the equilibrium distribution function. To prove that the Kohn mode is a solution, one only has to show $C_{12}[f_0(\mathbf{p}, \mathbf{R} - \boldsymbol{\eta}(T))] = 0$. Assuming the equilibrium quasiparticle distribution function f_0 is given by (85) with $\mu_{qp} = 0$, and using the identity for the Bose distribution function

$$1 + f(x) = e^x f(x), \quad (106)$$

one obtains following expression for the $C_{12}[f_0]$

$$\begin{aligned} & C_{12}[f_0(\mathbf{p}, \mathbf{R} - \boldsymbol{\eta}(T))] \\ & = 2g^2 n_c(\mathbf{R} - \boldsymbol{\eta}(T)) \int \frac{d\mathbf{p}_1 d\mathbf{p}_2 d\mathbf{p}_3}{(2\pi)^2} |A_0(2, 3; 1)|^2 \delta(\mathbf{p}_1 - \mathbf{p}_2 - \mathbf{p}_3) \\ & \quad \times \delta[E_{10}(\mathbf{R} - \boldsymbol{\eta}(T)) - E_{20}(\mathbf{R} - \boldsymbol{\eta}(T)) - E_{30}(\mathbf{R} - \boldsymbol{\eta}(T))] \\ & \quad \times [\delta(\mathbf{p} - \mathbf{p}_1) - \delta(\mathbf{p} - \mathbf{p}_2) - \delta(\mathbf{p} - \mathbf{p}_3)] (1 + f_{10}) f_{20} f_{30} \\ & \quad \times [1 - e^{-\beta[E_{10} - E_{20} - E_{30} - (\mathbf{p}_1 - \mathbf{p}_2 - \mathbf{p}_3)(\mathbf{v}_n - \mathbf{v}_s)}]]. \end{aligned} \quad (107)$$

Using the delta functions in (107) corresponding to the conservation of energy and momentum, it immediately follows that

$$C_{12}[f_0(\mathbf{p}, \mathbf{R} - \boldsymbol{\eta}(T))] = 0. \quad (108)$$

This proves that the non-condensate exhibits a rigid simple harmonic displacement with the trap frequency.

Since we have proven that the collision integral vanishes for the Kohn mode type of oscillations, the dissipative term R in the generalized GP equation in (97) will vanish as well. Therefore, it follows that the equilibrium condensate profiles oscillate with the trap frequency (for the analogous calculation based on the HF and HFB single-particle spectrum, see Refs. 3 and 23).

8. CONCLUSIONS

Using the Kadanoff–Baym non-equilibrium Green’s function formalism^{10, 12} we have derived a kinetic equation for the quasiparticle distribution function and a generalized Gross–Pitaevskii equation valid at all temperatures. Our kinetic equation involves the Bogoliubov quasiparticle spectrum and Bose coherence factors involving the u and v functions. Our equations reduce to those obtained at high temperatures by ZNG.³ As we have emphasized in Sec. 2, the approximation that we have used in this paper is gapless and it gives the correct low-momentum (long-wavelength) limit. In contrast, the so-called “conserving approximations” are based on a functional from which both self-energy Σ and the source η functions can be derived by functional differentiation (see p. 338 ff of Ref. 20, as well as Sec. 3 in Ref. 4). The resulting single-particle Green’s function can be used to generate a density response function whose spectrum is guaranteed to satisfy conservation laws,^{18, 20, 35, 36} even though the generating Green’s functions have an energy gap in the long-wavelength limit. We note that Kane and Kadanoff¹² based their work on a simpler second-order approximation for the self-energies but which had the advantage of being “conserving.”

The simple quasiparticle approximation that we have used in this paper has allowed us to derive the kinetic equation in a “Boltzmann-like” form. The kinetic equation in the Bogoliubov–Popov approximation given in (80) is only valid in this quasiparticle approximation. In deriving these results, we have neglected the real part of the second-order self-energies that give rise to many-body corrections. Our discussion could be generalized to include these real parts, but this improved theory would be very complex. A first step would be to include some of these renormalization effects within a simple quasiparticle approximation to the spectral densities $a_{\alpha\beta}$. However, this ad-hoc procedure cannot be guaranteed to lead to a conserving approximation.^{12, 15}

Our kinetic equation for the quasiparticle distribution and the generalized Gross–Pitaevskii equation are coupled and have to be solved self-consistently. They should provide a sound basis for the future systematic study of the non-equilibrium response of a trapped Bose gas at low temperatures. We remark that we could use our results to derive the Landau–Khalatnikov two-fluid hydrodynamic equations, in the collision-dominated region. This derivation was also the goal of the work of Kane and Kadanoff.¹² We note that the approach developed in Ref. 14 is based on a quasiparticle kinetic equation which is precisely of the kind we have derived in (80). Thus one sees how (80) could be used to derive the Landau–Khalatnikov two-fluid equations which include damping from various transport processes; Kane and Kadanoff did not include these processes in their work. This would

generalize recent work²² deriving the Landau–Khalatnikov two-fluid equations with hydrodynamic damping in the high temperature region.

APPENDIX

For illustration, we give the detailed proof that the collision integral I in (52) conserves momentum, namely

$$\int d\mathbf{p} \mathbf{p} I[f(\mathbf{p}, \mathbf{R}, T)] = 0. \quad (109)$$

The proof is essentially the same as one uses in classical gases (See Chap. 5 of Ref. 37). Using the expression for the second-order Beliaev energy in (62) in conjunction with (48), one obtains

$$\begin{aligned} & \int d\mathbf{p} \mathbf{p} I[f(\mathbf{p}, \mathbf{R}, T)] \\ &= -\frac{1}{2} g^2 \int \frac{d\mathbf{p} d\omega}{(2\pi)^4} \frac{d\mathbf{p}_i d\omega_i}{(2\pi)^8} \delta(\omega + \omega_1 - \omega_2 - \omega_3) \delta(\mathbf{p} + \mathbf{p}_1 - \mathbf{p}_2 - \mathbf{p}_3) \\ & \times \mathbf{p} \left\{ [ff_1(1+f_2) - (1+f)(1+f_1)f_2f_3] \left[\text{Tr}(\hat{a}(\mathbf{p}_2, \omega_2) \hat{a}(\mathbf{p}, \omega)) \right. \right. \\ & \times \text{Tr}(\hat{a}(\mathbf{p}_1, \omega_1) \hat{a}(\mathbf{p}_3, \omega_3)) \\ & \left. \left. + 2 \text{Tr} \left(\hat{a}(\mathbf{p}_2, \omega_2) \hat{a}(\mathbf{p}_1, \omega_1) \hat{a}(\mathbf{p}_3, \omega_3) \hat{a}(\mathbf{p}, \omega) \right) \right] \right\} \quad (A2) \end{aligned}$$

$$\begin{aligned} & + [f(1+f_2)(1+f_3) - (1+f)f_2f_3] \left[\text{Tr}(\hat{a}(\mathbf{p}_2, \omega_2) \hat{a}(\mathbf{p}, \omega)) \right. \\ & \times \text{Tr}(\hat{h}(\mathbf{p}_1, \omega_1) \hat{a}(\mathbf{p}_3, \omega_3)) \\ & \left. \left. + 2 \text{Tr} \left(\hat{a}(\mathbf{p}_2, \omega_2) \hat{h}(\mathbf{p}_1, \omega_1) \hat{a}(\mathbf{p}_3, \omega_3) \hat{a}(\mathbf{p}, \omega) \hat{a}(\mathbf{p}, \omega) \right) \right] \right\} \quad (A3) \end{aligned}$$

$$\begin{aligned} & + [ff_1(1+f_2) - (1+f)(1+f_1)f_2] \left[\text{Tr}(\hat{a}(\mathbf{p}_2, \omega_2) \hat{a}(\mathbf{p}, \omega)) \right. \\ & \times \text{Tr}(\hat{a}(\mathbf{p}_1, \omega_1) \hat{h}(\mathbf{p}_3, \omega_3)) \\ & \left. \left. + 2 \text{Tr} \left(\hat{a}(\mathbf{p}_2, \omega_2) \hat{a}(\mathbf{p}_1, \omega_1) \hat{h}(\mathbf{p}_3, \omega_3) \hat{a}(\mathbf{p}, \omega) \hat{a}(\mathbf{p}, \omega) \right) \right] \right\} \quad (A4) \end{aligned}$$

$$\begin{aligned} & + [ff_1(1+f_3) - (1+f)(1+f_1)f_3] \left[\text{Tr}(\hat{h}(\mathbf{p}_2, \omega_2) \hat{a}(\mathbf{p}, \omega)) \right. \\ & \times \text{Tr}(\hat{a}(\mathbf{p}_1, \omega_1) \hat{a}(\mathbf{p}_3, \omega_3)) \\ & \left. \left. + 2 \text{Tr} \left(\hat{h}(\mathbf{p}_2, \omega_2) \hat{a}(\mathbf{p}_1, \omega_1) \hat{a}(\mathbf{p}_3, \omega_3) \hat{a}(\mathbf{p}, \omega) \hat{a}(\mathbf{p}, \omega) \right) \right] \right\} \quad (A5) \end{aligned}$$

Consider the (A2) term first. The change of dummy variables

$$(\mathbf{p}, \omega) \Leftrightarrow (\mathbf{p}_2, \omega_2) \quad (\mathbf{p}_1, \omega_1) \Leftrightarrow (\mathbf{p}_3, \omega_3) \quad (\text{A6})$$

doesn't change the delta functions. Because of the cyclic invariance of trace, the first term in (A2) is also unchanged—the only change is that \mathbf{p} in front of the integral becomes $-\mathbf{p}_2$. After the change of variables given by (A6), the second term in (A2) becomes

$$\begin{aligned} \text{Tr} \left(\hat{a}(\mathbf{p}, \omega) \hat{a}(\mathbf{p}_3, \omega_3) \hat{a}(\mathbf{p}_1, \omega_1) \hat{a}(\mathbf{p}_2, \omega_2) \right) \\ = \text{Tr} \left(\hat{a}(\mathbf{p}_2, \omega_2) \hat{a}(\mathbf{p}, \omega) \hat{a}(\mathbf{p}_3, \omega_3) \hat{a}(\mathbf{p}_1, \omega_1) \right) \end{aligned} \quad (\text{A7})$$

Transforming $(\mathbf{p}, \omega) \Leftrightarrow (\mathbf{p}_1, \omega_1)$ in (A7), doesn't change either the δ -functions or the product of f 's. Therefore the trace in second term in (A2) doesn't change. Thus (A2) after the transformations is unchanged, but \mathbf{p} is replaced with $-\mathbf{p}_2$. Using the transformation,

$$(\mathbf{p}, \omega) \Leftrightarrow (\mathbf{p}_1, \omega_1) \quad (\mathbf{p}_2, \omega_2) \Leftrightarrow (\mathbf{p}_3, \omega_3) \quad (\text{A8})$$

we would again obtain the same expression as (A2), but with \mathbf{p} replaced with \mathbf{p}_1 . Finally, if we make the transformation

$$(\mathbf{p}, \omega) \Leftrightarrow (\mathbf{p}_3, \omega_3) \quad (\mathbf{p}_2, \omega_2) \Leftrightarrow (\mathbf{p}_1, \omega_1) \quad (\text{A9})$$

we obtain the same expression as in (A2), but with \mathbf{p} replaced with $-\mathbf{p}_3$. We conclude that we can write (A2) as one fourth of the sum of four equivalent terms. The integrand of this new expression is thus seen to be proportional to

$$(\mathbf{p} + \mathbf{p}_1 - \mathbf{p}_2 - \mathbf{p}_3) \delta(\mathbf{p} + \mathbf{p}_1 - \mathbf{p}_2 - \mathbf{p}_3), \quad (\text{A10})$$

which clearly vanishes.

A similar discussion can be given of the other terms in (A3)–(A5). Making the change of variables

$$(\mathbf{p}, \omega) \Leftrightarrow (\mathbf{p}_1, \omega_1) \quad (\mathbf{p}_2, \omega_2) \Leftrightarrow (\mathbf{p}_3, \omega_3) \quad (\text{A11})$$

in (A4) first, and using the cyclic invariance of the trace, one can show that (A4) is the same as (A5), but with \mathbf{p}_1 instead of \mathbf{p} as an overall multiplying factor. Using

$$(\mathbf{p}, \omega) \Leftrightarrow (\mathbf{p}_3, \omega_3) \quad (\mathbf{p}_2, \omega_2) \Leftrightarrow (\mathbf{p}_2, \omega_2). \quad (\text{A12})$$

in (A3), and then relabeling $\mathbf{p} \rightarrow \mathbf{p}_1$ in the second term in (A3), we can reduce (A3) to the same expression as in (A5) but with \mathbf{p} replaced with

– \mathbf{p}_3 . Therefore the sum of the terms (A3), (A4), and (A5) has an integrand involving

$$(\mathbf{p} + \mathbf{p}_1 - \mathbf{p}_3) \delta(\mathbf{p} + \mathbf{p}_1 - \mathbf{p}_2 - \mathbf{p}_3) \delta(\mathbf{p}_2) \delta(\omega_2), \quad (\text{A13})$$

which clearly vanishes. This completes the proof of (A1) and hence of momentum conservation by collisions. A similar calculation is given in Kane's thesis.¹³

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