

KADANOFF-BAYM KINETIC THEORY FOR A TRAPPED
BOSE-CONDENSED GAS

by

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Abstract

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Using the Kadanoff-Baym non-equilibrium Green's function formalism, we derive the equations of motion for the dynamics of a trapped dilute Bose-condensed atomic gas at finite temperatures. These include an equation of motion for the macroscopic wavefunction describing the atoms in the condensate, as well as a Boltzmann-like kinetic equation for the single-particle distribution function describing the atoms thermally excited out of the condensate. We derive these equations using several self-energy approximations for the non-condensate atoms. At high enough temperatures, one can work within the first order Hartree-Fock-Bogoliubov approximation for the Beliaev self-energy functions and derive a "collisionless" kinetic equation. We then extend this analysis to include the second-order self-energy contributions associated with two-body collisions, leading to a Boltzmann-like equation for Bose atoms, with the collision integrals included. This equation, combined with the generalized Gross-Pitaevskii equation of motion for the condensate atoms, has been extensively used by Zaremba, Nikuni and Griffin. The role of collisions which transfers atoms between the condensate and non-condensate component is emphasized. The major result of this thesis is a generalization of previous studies to very low temperatures, where the non-condensate thermal cloud atoms are treated in the so-called Bogoliubov-Popov approximation. We derive a kinetic equation for the resulting quasiparticles (instead of atoms) and show how the collision integrals are modified by Bogoliubov coherence factors. A detailed comparison is made with previous discussion

of the non-equilibrium behaviour of uniform interacting Bose gases, including the work of Kirkpatrick and Dorfman as well as Eckern.

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Chapter 1

Introduction

The achievement of Bose-Einstein condensation (BEC) in dilute gases of alkali atoms in 1995 [1] verified the correctness of Einstein's prediction in 1925 that a non-interacting gas of bosons would undergo a phase transition, characterized by a macroscopic number of atoms occupying the lowest energy single-particle state. The atoms that are in this quantum state are coherent (i.e. have the same phase) and can be described by a single, macroscopic wavefunction: a coherent matter wave or Bose condensate. Below the transition temperature T_{BEC} , the thermal de Broglie wavelength of the atoms in a gas at given temperature becomes comparable or larger than the average distance between the atoms. Clearly the wavefunctions of individual atoms overlap and the gas has to be treated as a highly correlated quantum gas. The macroscopic wavefunction of the Bose condensate is best thought of as the order parameter of the new quantum phase of matter, a view that goes back to the pioneering work of Fritz London in 1938 but was only formally defined in the late 1950's.

Beautiful images of Bose condensates in trapped atomic gases, which have become a trademark for the field, clearly show that a Bose gas cooled down to low temperatures exhibits a phase transition at the predicted value of the BEC transition temperature T_{BEC} . For an ideal Bose gas of N atoms in an isotropic harmonic potential of frequency ω_0 , the-

ory predicts that $k_B T_{BEC} \simeq \hbar\omega_0 N^{1/3}$. This phase transition is demonstrated through the dramatic appearance of a narrow central peak describing Bose condensed atoms, relative to a broader background distribution associated with the thermally excited atoms. For an excellent recent review of the theory of Bose-Einstein condensation in trapped gases, see Ref. [2].

The experimental realization of BEC produced an explosion of both theoretical and experimental interest and brought together physicists working in condensed-matter physics, quantum optics and fundamental quantum mechanics [3, 4]. For condensed-matter physicists, the achievement was particularly exciting because BEC in a weakly interacting uniform Bose gas was a subject of extensive many-body calculations in 1950's and 1960's as a model for the superfluid behaviour of liquid ^4He . Although theoretical studies assuming a weakly interacting Bose gas are not applicable to strongly interacting liquid helium, these studies played a major role in developing the theoretical framework for describing Bose superfluids. As noted above, Fritz London first suggested in 1938 that the superfluidity in liquid helium could have its explanation as a visible quantum effect related to the existence of a Bose condensate. He introduced a “macroscopic wavefunction” as the microscopic foundation for understanding superfluidity. In his seminal 1947 paper, Bogoliubov showed how such a Bose condensate could still exist in an interacting gas, although it changes the excitation spectrum from single-particle-like at high momenta to phonon-like at low momenta. Bogoliubov's work was one of the first treatments of a broken Bose symmetry and provided a microscopic model for understanding the kind of quasiparticle spectrum needed in Landau's phenomenological theory of superfluid helium. An extensive review of the theory for interacting Bose liquids can be found in Ref. [5] and a review of more recent work for trapped atomic Bose gases is given in Refs. [2, 6, 3].

Landau's well-known theory of superfluid helium is a very successful, but ultimately a phenomenological theory. It describes the low-frequency motion of two fluids (the superfluid and the normal-fluid components) by a coupled set of hydrodynamic equations.

Landau also introduced the idea that the normal fluid could be described as a gas of weakly interacting quasiparticles [7]. However, in deriving his two-fluid theory, Landau argued that the Bose statistics obeyed by the ^4He atoms is irrelevant and that his theory was applicable to any low-temperature quantum fluid. This obscured the role of a Bose condensate as the underlying basis for superfluid phenomena, and for many years this role remained controversial within the liquid ^4He community [5].

For a discussion of the properties of a dilute Bose gas at low temperatures (but not in superfluid Helium), the inter-atomic interaction $v(\mathbf{r} - \mathbf{r}')$ can be treated using the simple pseudopotential [8]

$$v(\mathbf{r} - \mathbf{r}') = g\delta(\mathbf{r} - \mathbf{r}'), \quad g = \frac{4\pi\hbar^2 a}{m} \quad (1.1)$$

where a is the s -wave scattering length for the real potential. The Born approximation for this contact interaction gives the same cross-section as the low energy s -wave ($l = 0$) scattering contribution of the real two-body potential and therefore one can use this pseudopotential to describe a trapped dilute Bose gas at the very low temperatures of interest in BEC studies. For this s -wave approximation to be applicable, we require that a be much smaller than the average distance between atoms, or $na^3 \ll 1$. This condition is very well satisfied in current trapped Bose gas experiments. In ^{87}Rb , for example, $a = 58 \text{ \AA}$ while the typical density at the centre of the trap is $\sim 10^{14} \text{ cm}^{-3}$, and so na^3 is less than 10^{-4} [2].

For a trapped, weakly interacting, dilute Bose gas, one can derive the microscopic equations of motion for both the condensate and the thermally excited (non-condensate) atoms. The crucial step in our theoretical treatment of Bose condensation is the concept of broken Bose symmetry [9, 10, 11], in which the thermal expectation value of the quantum field operator can be non-zero, $\langle \hat{\psi}(\mathbf{r}, t) \rangle = \Phi(\mathbf{r}, t)$. Here the time dependence of the quantum field operators is given by $\psi(\mathbf{r}, t) = U^\dagger(t)\psi(\mathbf{r})U(t)$, where $U(t)$ is the evolution operator of the perturbed system. The condensate is described by this macroscopic wave

function $\Phi(\mathbf{r}, t)$. Therefore, one can naturally decompose the quantum field operator

$$\psi(\mathbf{r}, t) = \Phi(\mathbf{r}, t) + \tilde{\psi}(\mathbf{r}, t), \quad (1.2)$$

where $\tilde{\psi}(\mathbf{r}, t)$ describes the atoms “outside the condensate”. The density of condensed atoms at point (\mathbf{r}, t) is given by $n_c(\mathbf{r}, t) = |\Phi(\mathbf{r}, t)|^2$.

At $T = 0$, almost all atoms in the atomic Bose gases currently studied are in the condensate. In technical terms, one says that the “quantum depletion” due to interactions can be neglected. It is estimated [12] to be less than 0.5%. The equation of motion for the macroscopic wavefunction $\Phi(\mathbf{r}, t)$ then completely describes the trapped gas. This is the famous *Gross-Pitaevskii* (GP) equation

$$i \frac{\partial \Phi(\mathbf{r}, t)}{\partial t} = \left[-\frac{\nabla^2}{2m} + U_{ext}(\mathbf{r}) + g |\Phi(\mathbf{r}, t)|^2 \right] \Phi(\mathbf{r}, t), \quad (1.3)$$

where $U_{ext}(\mathbf{r}) = m\omega_0^2 r^2/2$ is the isotropic parabolic trapping potential; in most of this thesis, we set $\hbar = 1$. Thus, at $T=0$, an atom in the condensate moves in the effective Hartree mean-field $gn_c(\mathbf{r}, t)$ produced by all other atoms in the condensate. This is a closed non-linear differential equation with vast amount of physics behind it, as thousands of papers have shown in the last 5 years [2, 3].

At finite temperatures, however, a trapped Bose gas is much more complex. The complete description of the system is now given by the GP-like equation for the macroscopic wavefunction $\Phi(\mathbf{r}, t)$ coupled to a kinetic equation for the density distribution function $f(\mathbf{p}, \mathbf{r}, t)$ for the thermally excited atoms with momentum \mathbf{p} at (\mathbf{r}, t) . The generalization of the Gross-Pitaevskii equation must include the effect of the non-condensate atoms on the condensate atoms. That is, at finite temperature, atoms that are thermally-excited give rise to a dynamical Hartree-Fock mean field that acts on the condensate atoms. More generally, the “exact” GP-like equation of motion for condensate atoms is given by

$$i \frac{\partial \Phi(\mathbf{r}, t)}{\partial t} = \left[-\frac{\nabla^2}{2m} + U_{ext}(\mathbf{r}) \right] \Phi(\mathbf{r}, t) + g \langle \psi^\dagger(\mathbf{r}, t) \psi(\mathbf{r}, t) \psi(\mathbf{r}, t) \rangle. \quad (1.4)$$

As we will see later in this thesis, various kinds of equations of motion for the macroscopic wavefunction $\Phi(\mathbf{r}, t)$ can be classified depending on the approximation used for the three-field correlation function on the right-hand side in (1.4).

A general form of the kinetic equation for the single-particle distribution function $f(\mathbf{p}, \mathbf{r}, t)$ describing the non-condensate atoms is given by the following expression

$$\left[\frac{\partial}{\partial t} + \nabla_{\mathbf{p}} E_p \cdot \nabla_{\mathbf{r}} - \nabla_{\mathbf{r}} E_p \cdot \nabla_{\mathbf{p}} \right] f(\mathbf{p}, \mathbf{r}, t) = \left[\frac{\partial f}{\partial t} \right]_{coll}. \quad (1.5)$$

Here $E_p(\mathbf{r}, t)$ is the excitation energy at (\mathbf{r}, t) ; its explicit form depends on the approximation we work in. The collision integral on the right-hand side of (1.5) describes the effect of collisions between atoms on the evolution of the distribution function $f(\mathbf{p}, \mathbf{r}, t)$. In a trapped Bose-condensed gas, it has two parts which play quite different roles

$$\left[\frac{\partial f}{\partial t} \right]_{coll} = C_{12}[f] + C_{22}[f]. \quad (1.6)$$

Here C_{22} denotes the collision integral that describes collisions between non-condensate atoms. Above T_{BEC} , this is the only term present. Below T_{BEC} , we also have the C_{12} term which describes collisions between condensate and non-condensate atoms. The role of C_{12} is crucial because it couples the condensate and non-condensate degrees of freedom. In the thesis, we present a detailed derivation of kinetic equations like (1.5) using various approximations. Landau and Khalatnikov first used a Boltzmann-like kinetic equation like (1.5) for a dilute gas of quasiparticle excitations in describing superfluid ^4He . However, its relation to an underlying Bose condensate was not spelled out (or even mentioned).

The dynamics of a trapped, Bose-condensed gas described by coupled equations of motion for the condensate and non-condensate components can be investigated in two different regimes, the ‘‘collisionless’’ regime and the ‘‘hydrodynamic’’ regime. If the density of atoms is sufficiently low that the collisions between atoms are infrequent and hence not very important, the dynamics of a gas is dominated by the Hartree-Fock mean fields; one refers to this regime as the collisionless regime [13]. In this case, one can neglect the

collision integrals on the right-hand side of (1.5). In contrast, in the collision-dominated “hydrodynamic” regime, frequent collisions establish a “local equilibrium state” of the condensate and non-condensate atoms. Such a local equilibrium distribution function $f_{eq}(\mathbf{p}, \mathbf{r}, t)$ is defined by the condition for collision integrals to vanish, i.e.,

$$\left[\frac{\partial f_{eq}(\mathbf{p}, \mathbf{r}, t)}{\partial t} \right]_{coll} = C_{12}[f_{eq}(\mathbf{p}, \mathbf{r}, t)] + C_{22}[f_{eq}(\mathbf{p}, \mathbf{r}, t)] = 0. \quad (1.7)$$

The complete description of the dynamics in the hydrodynamic regime can be given in terms of a few “local variables” for the condensate and non-condensate components, such as the local density and velocity. The famous Landau two-fluid hydrodynamics equations describe this local “equilibrium region” in superfluid ^4He .

The general theory for non-equilibrium phenomena in a trapped Bose gas is fully defined once we have equations such as (1.4) and (1.5). Taking moments of the kinetic equation and linearizing them, and the equations for a Bose order parameter, one can derive a set of closed hydrodynamic equations that can be shown to reduce to Landau’s two-fluid equations in an appropriate limit. Thus, at least for a trapped, dilute Bose gas, one can derive the two-fluid equations starting from a fundamental microscopic theory of Bose condensation.

To derive a microscopic theory of the non-equilibrium behaviour of a dilute, weakly interacting Bose-condensed gas at finite temperatures, in this thesis we use the Kadanoff-Baym non-equilibrium Green’s functions method [14]. The generalization to Bose-condensed systems was first considered by Kane and Kadanoff in 1965 [15, 16], with the specific goal of deriving the Landau two-fluid hydrodynamic equations of motion. We extend this work but concentrate on deriving an equation of motion for a macroscopic Bose wavefunction, coupled to a kinetic equation for a distribution function for thermally excited atoms. We do not use our equations to derive the two-fluid equations but simply refer to recent work by the Toronto-Queen’s group [17, 18].

Our general problem is to calculate the non-equilibrium response of a system induced by an external (space- and time-dependent) disturbance. External fields may be rep-

resented by an additional term in the Hamiltonian of the system. In the response to external disturbances, many interesting physical phenomena may appear. These include excitation of collective modes and various transport processes.

Kadanoff and Baym have formulated the method of calculating the non-equilibrium response of a system evolving under external time-dependent perturbations in a systematic way. To the lowest order in the external field, they show how one can formulate the method so that it is closely tied to the structure of the usual equilibrium Green's function method (discussed in standard many-body theory). In the limit when the external time-dependent fields are turned off, the KB non-equilibrium theory reduces to the usual equilibrium theory. Building on the basis of well-understood equilibrium Green's functions, one constructs fictitious, time-ordered, Green's functions defined for imaginary times. These fictitious Green's functions obey the same boundary conditions as equilibrium Green's functions do and, formally, the same equations of motion (but now defined for imaginary times). The physical response functions measured experimentally are related to the Green's functions for real times. However, the latter can be obtained from the non-equilibrium, imaginary-time Green's functions. This KB procedure will be reviewed in detail in Chapter 2 for normal (non Bose-condensed) systems.

In the KB approach, one first calculates the equations of motion for non-equilibrium, real-time single-particle Green's functions involving the quantum field operators in the presence of the external perturbing fields. The single-particle Green's function is a two-field correlation function; it describes the propagation of disturbances in which a single atom is either added to or removed from the many-particle system, namely

$$G(1, 1') = \frac{1}{i} \langle T \tilde{\psi}^\dagger(1) \tilde{\psi}(1') \rangle \quad (1.8)$$

where $1 \equiv \mathbf{r}_1, t_1$; $1' \equiv \mathbf{r}'_1, t'_1$ and T is the time ordering operator (to be defined in Chapter 2). Similarly, the two-particle Green's function can be introduced; this describes disturbances produced by the removal or addition of *two* atoms.

An equation of motion for an interacting single-particle Green's function G is given

(schematically) by Dyson's equation

$$G = G_0 + G_0 \Sigma G. \quad (1.9)$$

Here G_0 is the non-interacting single-particle Green's function and Σ is the single-particle self-energy function which contains all the effects associated with the two-particle interactions. The specific approximation for the single-particle self-energy determines the resulting microscopic theory. This is the same in both normal and Bose-condensed systems, although in the latter one must deal with a 2×2 matrix single-particle Green's functions because of new 'off-diagonal' correlation functions which arise when $\Phi(\mathbf{r}, t) \neq 0$ (see (1.2)).

In the Kadanoff-Baym procedure [14, 19], one rewrites the exact equations of motion in terms of relative and center-of-mass space-time coordinates, defined by

$$\mathbf{R} = \frac{1}{2}(\mathbf{r}_1 + \mathbf{r}'_1), \quad \mathbf{r} = \mathbf{r}_1 - \mathbf{r}'_1 \quad T = \frac{1}{2}(t_1 + t'_1), \quad t = \frac{1}{2}(t_1 - t'_1). \quad (1.10)$$

In the recent literature, this is sometimes known as the Wigner representation [20, 21]. In thermal equilibrium, the Green's functions are functions only of the relative space-time coordinates \mathbf{r} and t , and moreover are sharply peaked about $\mathbf{r} \simeq 0$ and $t \simeq 0$ [14]. This is simply because correlations between atoms rapidly decay in both space and time. Since we assume that the external disturbances are slowly varying in space and time (with a wavelength much larger than the thermal deBroglie wavelength), we expect that these slowly varying external disturbances will not change this dependence of $G(\mathbf{r}, t; \mathbf{R}, T)$ on small values of \mathbf{r} and t . Therefore, all non-equilibrium correlation functions (like G, Σ , etc.) are assumed to be always dominated by the small values of relative coordinates (\mathbf{r}, t) (equivalently, by high momenta and frequencies in the Fourier transforms), but vary slowly as functions of the center-of-mass coordinates (\mathbf{R}, T) . Fourier transforming the function $G(\mathbf{r}, t; \mathbf{R}, T)$ with respect to \mathbf{r} and t , $G(\mathbf{p}, \omega; \mathbf{R}, T)$ describes the elementary excitations of momenta \mathbf{p} and energy ω at point (\mathbf{R}, T) [14]. These assumptions are central to how one uses the KB method to derive generalized Boltzmann equation from

exact quantum equations of motion. They agree with our extensive knowledge of how correlations behave in many-body systems. In this thesis, we will not critically examine these assumptions but refer to the literature [20].

The distribution function for thermally-excited atoms is the single-particle Wigner distribution function $f_W(\mathbf{p}, \mathbf{R}, T)$ given by

$$f_W(\mathbf{p}, \mathbf{R}, T) = \int d\mathbf{r} e^{-i\mathbf{p}\cdot\mathbf{r}} \langle \tilde{\psi}^\dagger(\mathbf{R} - \frac{\mathbf{r}}{2}, T) \tilde{\psi}(\mathbf{R} + \frac{\mathbf{r}}{2}, T) \rangle \quad (1.11)$$

and it is related to the Green's functions describing the thermally excited atoms. The Wigner function is the quantum generalization of the classical distribution function giving the number of atoms with momentum \mathbf{p} at point (\mathbf{R}, T) .

Bose broken symmetry [9, 11, 22, 23] plays a crucial role in Bose-condensed fluids because it leads to a coupling of the single-particle and density fluctuation excitations, namely the poles of single-particle Green's function are the same as the poles of the density correlation function below the superfluid transition temperature. This equivalence is unique for the Bose-condensate phase and it explains why one can measure the elementary excitation spectrum (given by the poles of single-particle Green's function) by measuring the density fluctuation modes that are the poles of the density response function. In normal fluids (non-Bose condensed, above the transition temperature) the density response spectrum is quite different from the elementary excitation spectrum.

The calculation of the elementary excitation spectrum and/or the density fluctuation spectrum is, in general, very complex in many-body systems. However, in a Bose-condensed fluid, one can take advantage of the above-mentioned equivalence of the density fluctuation excitations and elementary excitations. One can proceed directly by calculating the poles of equilibrium Green's functions within some given approximation for the self-energies. One can also proceed indirectly by calculating the density fluctuation spectrum from the density response function and then use the above equivalence.

As summarized in the classic paper by Hohenberg and Martin [11], there are two traditional types of approximations that one can use in Bose superfluids, gapless approxima-

tions and conserving approximations. The advantage of gapless approximations is that the non-condensate single-particle Green's function exhibits the correct spectrum (in a uniform gas, the quasiparticles are phonon-like in the long-wavelength, as required in a Bose-condensed phase). In homogeneous systems, a gapless spectrum for long-wavelength excitations is guaranteed if the self-energies satisfy the so-called Hugenholtz-Pines theorem [5, 11, 24], namely if the chemical potential μ at given temperature is equal to the difference of the diagonal and off-diagonal matrix elements of the Beliaev self-energy function $\Sigma_{\alpha\beta}$, i.e.,

$$\mu = \Sigma_{11}(\mathbf{q} = 0, \omega = 0) - \Sigma_{12}(\mathbf{q} = 0, \omega = 0). \quad (1.12)$$

In contrast, “conserving approximations” (often also called Φ -derivable approximations) are based on the introduction of a functional, from which the self-energy function $\Sigma_{\alpha\beta}$ can be derived by functional differentiation. Using the self-energy, the resulting single-particle Green's function $G_{\alpha\beta}$ can then be used to generate a density response function whose spectrum is guaranteed to satisfy all the usual conservation laws. Using the equivalence of the poles of the single-particle and density fluctuation spectrum, the spectrum of the density response function so generated must correspond to the single-particle spectrum given by some improved approximation for the Beliaev self-energies.

We now give a brief summary of the contents of the chapters in this thesis. In Ch. 2, for the convenience of the reader, we give a detailed summary of the Kadanoff-Baym method for normal (non Bose-condensed) systems. In Ch. 3, this method is generalized for Bose-condensed systems and we discuss the Beliaev self-energy functions in both conserving and gapless approximations. These approximations are the building blocks of the KB formalism.

In Ch. 4, we derive the self-consistent Hartree-Fock-Bogoliubov (HFB) collisionless kinetic equations and the associated equation of motion for the condensate wavefunction for a trapped Bose-condensed gas. Our work generalizes the pioneering work by Kane and Kadanoff (KK) for a uniform Bose gas [15]. We include the off-diagonal (anomalous)

pair correlations, and thus we are led to introduce an off-diagonal distribution function in addition to the normal (diagonal) distribution function. This results in two coupled kinetic equations. If the off-diagonal distribution function can be neglected as a higher-order contribution, our result reduces to the semi-classical kinetic equation derived and used by Zaremba, Griffin and Nikuni [25]. We discuss the static “local equilibrium” solution of our coupled HFB kinetic equations, within a semi-classical approximation. We also verify the rigid in-phase oscillation of the equilibrium condensate and non-condensate density profiles, each oscillating with the trap frequency, is a solution of our equations. This is the famous sloshing mode, often called the Kohn mode. This chapter is based on Ref. [26].

In Ch. 5, we use the Kadanoff-Baym non-equilibrium Green’s function formalism to derive the kinetic equation for the non-condensate atoms at finite temperatures but, in contrast with Ch. 4, we now include the effect of binary collisions between the atoms. The effect of collisions is included using the second-order self-energy given by the Beliaev (gapless) approximation. We limit our discussion to finite temperatures where the kinetic equation for excited atoms can be treated in Hartree-Fock (HF) spectrum approximation. In this limit, we can neglect the off-diagonal propagators. As expected, this leads to the kinetic equations and collision integrals used in work by Zaremba, Nikuni, and Griffin (ZNG) [17]. We also derive a consistent equation of motion for the condensate wavefunction, involving a finite-temperature generalization of the well-known Gross-Pitaevskii equation. This includes a dissipative term from collisions with the thermally excited atoms, as well as the Hartree-Fock mean field of the non-condensate. This work is published in Ref. [19].

In Ch. 6, we present the generalization of the results of Ch. 5 that is valid at low temperatures, where the input single-particle spectrum is now described by the Bogoliubov-Popov approximation, instead of the high-energy HF particle spectrum. We derive a kinetic equation for the quasiparticle distribution function, with collision integrals de-

scribing scattering between quasiparticles and the condensate atoms. We also derive a generalized Gross-Pitaevskii equation for the condensate wavefunction that also includes the damping effects due to collisions between atoms in the condensate and the thermally excited quasiparticles. For a uniform Bose gas, our kinetic equation for the thermally-excited quasiparticles reduces to that found many years ago by Eckern [27], as well as by Kirkpatrick and Dorfman [28]. These results are published in Ref. [29].

In Ch. 7, we make some brief concluding remarks and suggest some possible future extensions and applications. Taken together, the results of this thesis give a microscopic derivation of kinetic equations for the non-condensate atoms in conjunction with a generalized version of the GP equation for the condensate. They provide a sound basis for discussion of a wide class of non-equilibrium phenomena in trapped Bose gases. Although such applications are not treated in this thesis, a few are briefly mentioned in the concluding Ch. 7.

Chapter 2

The Kadanoff-Baym method for normal systems: a review

This chapter summarizes the Kadanoff-Baym formalism for normal (non Bose-condensed) systems following the discussion in their famous monograph [14]. This allows one to understand the key concepts, and is more transparent than its generalization to Bose-condensed gases described in later chapters. An excellent review of the nonequilibrium real-time Green's functions and the generalized kinetic equation for the normal systems (non Bose-condensed) can also be found in the recent book by Zubarev, Morozov and Röpke [20]. This chapter is included for the convenience of the reader and contains nothing original. However, it introduces the notation and ideas on which Ch.3 - Ch.6 are based.

We first review the theory of equilibrium Green's function because the Kadanoff-Baym formalism is built on this. This includes standard many-body theory [10, 30], where one can develop a diagrammatic representation of the self-energy functions, introduce Wick's theorem and so forth. In contrast, when one deals with the time-dependent perturbations and hence the system is not in equilibrium, it is very difficult to develop a useful systematic theory to calculate properties of the perturbed system. The power

of the Kadanoff-Baym method is that it builds a systematic theory for non-equilibrium systems on the underlying equilibrium approximation. In this chapter, we review how this formalism works.

2.1 Equilibrium Green's functions

The Green's functions, which are the basis of this thesis, are thermodynamic averages of the quantum field operators $\psi(\mathbf{r}, t)$ and $\psi^\dagger(\mathbf{r}', t')$ at two different space-time points. We first review the theory of equilibrium Green's functions as an introduction to the non-equilibrium Green's functions. In terms of Bose quantum field operators, the many-body Hamiltonian $\hat{K}_0 = \hat{H} - \mu_0 \hat{N}$ describing Bosons interacting through a two-body potential $v(\mathbf{r})$ is given by

$$\hat{K}_0 = \int d\mathbf{r} \psi^\dagger(\mathbf{r}) \left(-\frac{1}{2m} \nabla_{\mathbf{r}}^2 - \mu_0 \right) \psi(\mathbf{r}) + \frac{1}{2} \int d\mathbf{r} d\mathbf{r}' \psi^\dagger(\mathbf{r}) \psi^\dagger(\mathbf{r}') v(\mathbf{r} - \mathbf{r}') \psi(\mathbf{r}) \psi(\mathbf{r}'). \quad (2.1)$$

The equation of motion for an operator $\psi(\mathbf{r}, t)$ is given in the Heisenberg representation by [10, 31]

$$i \frac{\partial \psi(\mathbf{r}, t)}{\partial t} = [\psi(\mathbf{r}, t), K_0]. \quad (2.2)$$

The solution of (2.2) can be written in the following form

$$\psi(\mathbf{r}, t) = e^{iK_0 t} \psi(\mathbf{r}) e^{-iK_0 t}. \quad (2.3)$$

This simple time-dependence of the quantum field operators given by (2.3) is only valid when the “grand-canonical” Hamiltonian K_0 is time-independent.

The single-particle equilibrium Green's function is defined as the time-ordered expectation value of the quantum field operators at two different space-time points

$$G(1, 1') = -i \langle T(\psi(1) \psi^\dagger(1')) \rangle, \quad (2.4)$$

while the two-particle Green's function is defined by

$$G_2(12, 1'2') = \frac{1}{i^2} \langle T(\psi(1) \psi(2) \psi^\dagger(2') \psi^\dagger(1')) \rangle. \quad (2.5)$$

Here T represents the usual Dyson time-ordering operation and 1 is the abbreviated notation for (\mathbf{r}, t) and $1'$ for (\mathbf{r}', t') . The time-ordering operator T , when applied to a product of operators, arranges them in the chronological order so that operator with the earliest time argument acts first (i.e. to the right), namely

$$\begin{aligned} T[\psi(1)\psi(1')] &\equiv \psi^\dagger(1')\psi(1) & t_1 < t_{1'} \\ &\equiv \psi(1)\psi^\dagger(1') & t_1 > t_{1'}. \end{aligned} \quad (2.6)$$

The thermal expectation values in (2.4) and (2.5) are computed using the grand-canonical ensemble

$$\langle X \rangle = \frac{\text{Tr} [e^{-\beta K_0} X]}{\text{Tr} [e^{-\beta K_0}]}. \quad (2.7)$$

The single-particle Green's function $G(1, 1')$ in (2.4) describes the propagation of an atom which is either added to or removed from the many particle system in equilibrium. For example, when $t_1 > t_{1'}$, the creation operator creates an atom at the point $(\mathbf{r}_{1'}, t_{1'})$. An atom propagates to the later time t_1 , when an atom is removed at point \mathbf{r}_1 , and the system returns to its equilibrium state. Similarly, the two-particle Green's function $G_2(12, 1'2')$ in (2.5) describes, for various time orderings, disturbances produced by the removal or addition of two atoms. For example, when t_1 and $t_{1'}$ are later than t_2 and $t_{2'}$, the two-particle Green's function describes the disturbance produced by the addition of one atom and the removal of one atom, and the subsequent return to equilibrium by the removal of an atom and the addition of an atom [14]. In addition to the one-particle Green's function in (2.4), we define the correlation functions [14]

$$\begin{aligned} G^>(1, 1') &\equiv -i\langle \psi(1)\psi^\dagger(1') \rangle \\ G^<(1, 1') &\equiv -i\langle \psi^\dagger(1')\psi(1) \rangle. \end{aligned} \quad (2.8)$$

The notation $>$ denotes that $G = G^>$ for $t > t'$ and $<$ is denotes that $G = G^<$ for $t > t'$.

The time-evolution operator $e^{-iK_0 t}$ in the Heisenberg representation and the weighting factor $e^{-\beta K_0}$ that appears in the expression for the grand-canonical average are the same

for the imaginary time $t = -i\beta$. This equivalence can be used to derive the famous Martin-Schwinger boundary condition obeyed by the equilibrium Green's functions,

$$G^<(11')|_{t_1=0} = e^{\beta\mu} G^>(11')|_{t_1=-i\beta}. \quad (2.9)$$

This boundary condition is crucial in the KB formalism, because it allows one to express Green's functions $G(t - t')$ for imaginary times $0 < t - t' < i\beta$ as a Fourier series over imaginary frequencies, and solve for the Green's functions.

One can interpret (2.2) as the equation of motion for the field operators (and hence Green's functions as well) for imaginary times in the strip $0 \leq \tau \equiv it \leq \beta$. We also generalize the definition for the time-ordering operator T to imaginary times. Later times are defined to be the farther down the imaginary axis. Therefore, in this imaginary-time representation, $G(1, 1')$ in (2.4) becomes

$$G(1, 1') = \begin{cases} G^>(1, 1') & \text{for } it_1 > it_{1'} \\ G^<(1, 1') & \text{for } it_1 < it_{1'}. \end{cases}$$

The equation of motion for the quantum field operator $\psi(1)$ can be found using (2.2) with K_0 given by (2.1), namely

$$\left(i \frac{\partial}{\partial t} + \frac{\nabla^2}{2m} + \mu_0 \right) \psi(\mathbf{r}, t) = \int d\mathbf{r}_1 v(\mathbf{r} - \mathbf{r}_1) \psi^\dagger(\mathbf{r}_1, t) \psi(\mathbf{r}_1, t) \psi(\mathbf{r}, t). \quad (2.10)$$

Multiplying (2.10) with $\psi^\dagger(1')$, applying the T -operator and taking the expectation value of the resulting equation, one obtains

$$\left\langle T \left(i \frac{\partial}{\partial t} + \frac{\nabla^2}{2m} + \mu_0 \right) \psi(\mathbf{r}, t) \psi^\dagger(1') \right\rangle = \int d\mathbf{r}_2 v(\mathbf{r}_1 - \mathbf{r}_2) \left\langle T \left[\psi(1) \psi(2) \psi^\dagger(2^+) \psi^\dagger(1') \right] \right\rangle \Big|_{t_2=t_1}. \quad (2.11)$$

The notation 2^+ means that the time argument of $\psi^\dagger(2)$ is infinitesimally larger than the time arguments of other ψ 's. To take the time derivative outside the T -operator, we need to consider terms containing it carefully. One can show that

$$i \frac{\partial}{\partial t_1} \left\langle T \left(\psi(1) \psi^\dagger(1') \right) \right\rangle - \left\langle T \left(i \frac{\partial}{\partial t_1} \psi(1) \psi^\dagger(1') \right) \right\rangle = \delta(1 - 1'). \quad (2.12)$$

Using this in (2.11), we finally obtain the following desired equation of motion for $G(1, 1')$, namely

$$\left(i \frac{\partial}{\partial t_1} + \frac{\nabla_1^2}{2m} + \mu_0\right) G(1, 1') = \delta(1 - 1') + i \int d\mathbf{r}_2 v(\mathbf{r}_1 - \mathbf{r}_2) G_2(12; 1'2^+) \Big|_{t_2=t_1}. \quad (2.13)$$

Using the same method, one can also obtain the adjoint equation of motion for $G(1, 1')$ containing the derivatives with respect to $1'$ variables,

$$\left(-i \frac{\partial}{\partial t_{1'}} + \frac{\nabla_{1'}^2}{2m} + \mu_0\right) G(1, 1') = \delta(1 - 1') + i \int d\mathbf{r}_2 G_2(12^-; 1'2) v(\mathbf{r}_1 - \mathbf{r}_2). \quad (2.14)$$

We emphasize that (2.13) and (2.14) are for imaginary times and satisfy the boundary condition (2.9).

The single-particle Green's function contains very useful dynamical and statistical mechanical information of the system. For example, it is easily seen that $G^<(\mathbf{r}, t; \mathbf{r}, t)$ is proportional to the equilibrium expectation value of the density of particles. Similarly, we can express the total energy (i.e., the expectation value of the Hamiltonian) in terms of $G^<$ in the following way (for more details, please see pg. 14 in [14])

$$\langle H \rangle = \frac{i}{4} \int d\mathbf{r} \left[\left(i \frac{\partial}{\partial t} - i \frac{\partial}{\partial t'} + \frac{\nabla \cdot \nabla'}{m} \right) G^<(\mathbf{r}t, \mathbf{r}'t') \right]_{\mathbf{r}'=\mathbf{r}, t'=t}. \quad (2.15)$$

The two-particle Green's function G_2 defined in (2.5) and appearing in (2.13) and (2.14) describes the propagation of two atoms added to the gas. Higher-order Green's functions, defined similarly to G and G_2 , describe the effect of adding or removing more than two atoms.

2.2 Non-equilibrium Green's functions

In Section 2.1, we have been discussing the equilibrium Green's functions. However, if we are interested in disturbances generated by external time-dependent fields, we must work with the non-equilibrium Green's functions. The system in the presence of the external perturbing field can be described by adding the additional term $H'(t)$ to the Hamiltonian

$$K(t) = K_0 + H'(t). \quad (2.16)$$

For the external perturbing field, we consider external time-dependent disturbances which couple to the local density of atoms $n(\mathbf{r}, t)$,

$$H'(t) \equiv \int d\mathbf{r}_1 U(\mathbf{r}_1, t) n(\mathbf{r}_1, t), \quad (2.17)$$

where the density of atoms is given by

$$n(\mathbf{r}, t) = \psi^\dagger(\mathbf{r}, t) \psi(\mathbf{r}, t). \quad (2.18)$$

The external scalar field $U(\mathbf{r}, t)$ has a given time-dependence. We assume that this non-equilibrium perturbation vanishes for times $t < t_0$, i.e., the system is in thermal equilibrium before the time t_0 . In the equilibrium case, the time-dependence of the operators in the Heisenberg representation is given by (2.3). In the non-equilibrium case, we have to include the time-dependent external potential $H'(t)$. Fortunately, one can generalize the well-established equilibrium formalism in such a way that it acts as the basis for the non-equilibrium theory. Using the imaginary-time non-equilibrium Green's functions, one can obtain results which can be brought into contact with the real-time physical response functions needed to describe non-equilibrium experimental probes.

Physical response functions are defined for real times. However, these are difficult to obtain *directly* since they do not satisfy a simple boundary condition [14]. Instead of working with real-time Green's functions, the Kadanoff and Baym formalism works with Green's functions defined on the imaginary time domain ($0 \leq \tau = it \leq \beta$) which satisfy the same boundary condition at 0 and $\tau = \beta$ as equilibrium Green's functions do. Once we obtain the equations of motion for these imaginary-time Green's functions we can, by a unique analytic continuation, obtain equations of motion for the real-time response functions of physical interest. This is the famous Martin-Schwinger approach [32], which is developed in detail by Kadanoff and Baym [14].

In the non-equilibrium case, the Schrödinger equation for the evolution of system under the perturbation (for $t > t_0$) is given by

$$i \frac{\partial}{\partial t} |\phi_t\rangle = [K_0 + H'(t)] |\phi_t\rangle. \quad (2.19)$$

We first introduce the usual interaction representation by writing the time dependence of a wavefunction as

$$|\phi_t\rangle \equiv e^{-iK_0(t-t_0)} |\phi_t\rangle_I. \quad (2.20)$$

where the subscript I labels the interaction representation. Substituting (2.20) back into (2.19), one obtains

$$i \frac{\partial}{\partial t} |\phi_t\rangle_I \equiv H'_I(t) |\phi_t\rangle_I. \quad (2.21)$$

Here, the time evolution of the external perturbation is given by

$$H'_I(t) \equiv e^{iK_0(t-t_0)} H'(t) e^{-iK_0(t-t_0)}. \quad (2.22)$$

If we integrate both sides of (2.21) from t_0 to t , we find

$$|\phi_t\rangle_I = |\phi_{t_0}\rangle_I + \frac{1}{i} \int_{t_0}^t dt' H'_I(t') |\phi_{t'}\rangle_I \quad (2.23)$$

To the first order in H' , this gives

$$|\phi_t\rangle_I = |\phi_{t_0}\rangle_I + \frac{1}{i} \int_{t_0}^t dt' H'_I(t') |\phi_{t_0}\rangle_I. \quad (2.24)$$

The second-order correction can be found by replacing the right-hand side of (2.24) into (2.23) and so on by iteration. The general solution of (2.23) can be written as

$$|\phi_t\rangle_I = T \left(e^{-i \int_{t_0}^t dt' H'_I(t')} \right) |\phi_{t_0}\rangle_I \quad (2.25)$$

or

$$|\phi_t\rangle_I = v(t, t_0) |\phi_{t_0}\rangle_I, \quad (2.26)$$

where we have defined the evolution operator

$$v(t, t_0) \equiv T \left[e^{-i \int_{t_0}^t dt' H'_I(t')} \right]. \quad (2.27)$$

The time evolution of an observable A evolving under the external disturbances $H'(t)$ is, in the interaction representation, given by

$$\langle \phi_t | A | \phi_t \rangle = \langle \phi_{t_0} | A_U(t) | \phi_{t_0} \rangle, \quad (2.28)$$

where

$$A_U(t) = \mathcal{U}^\dagger(t, t_0) A \mathcal{U}(t, t_0) \quad (2.29)$$

with the total evolution operator $\mathcal{U}(t, t_0)$

$$\mathcal{U}(t, t_0) = T \left[e^{-i \int_{t_0}^t K(t') dt'} \right] = e^{-i K_0(t-t_0)} v(t, t_0). \quad (2.30)$$

Here the subscript U denotes the external scalar potential. One easily verifies that $A_U(t)$ satisfies

$$i \frac{\partial A_U(t)}{\partial t} = [A_U(t), K_U(t)]. \quad (2.31)$$

Since we are interested in external disturbances of the type given in (2.17), we label all the quantities developing under H' with U . The time evolution of the operator $A_U(t)$ can be expressed in terms of the operator A_I in the interaction representation

$$A_U(t) = v^\dagger(t, t_0) A_I(t) v(t, t_0), \quad (2.32)$$

where, as before,

$$A_I(t) = e^{i K_0(t-t_0)} A e^{-i K_0(t-t_0)}. \quad (2.33)$$

One can show that the evolution operator $v(t_0, t_1)$ satisfies

$$i \frac{\partial}{\partial t_1} v(t_0, t_1) = H'(t_1) v(t_0, t_1), \quad (2.34)$$

$$v(t_0, t_0) = 1, \quad (2.35)$$

$$\lim_{t_0 \rightarrow -\infty} v(t_0, t_1) \equiv v(t_1), \quad (2.36)$$

$$\lim_{t_0 \rightarrow -\infty} v(t_0, t_0 - i\beta) = 1. \quad (2.37)$$

The thermal expectation value of an local observable $A(\mathbf{r})$ evolving in the presence of external time-dependent perturbation $H'(t)$ is given by

$$\begin{aligned} \langle A_U(\mathbf{r}, t) \rangle &\equiv \sum_i \frac{e^{-\beta(E_i - \mu N_i)}}{Z} \langle E_i | A_U(\mathbf{r}, t) | E_i \rangle \\ &= \sum_i \frac{e^{-\beta(E_i - \mu N_i)}}{Z} \langle E_i | \mathcal{U}^\dagger(t, t_0) A(\mathbf{r}, t) \mathcal{U}(t, t_0) | E_i \rangle \end{aligned}$$

$$\begin{aligned}
 &= \sum_i \frac{e^{-\beta(E_i - \mu N_i)}}{Z} \langle E_i | v^\dagger(t, t_0) e^{iK_0(t-t_0)} A(\mathbf{r}) e^{-iK_0(t-t_0)} v(t, t_0) | E_i \rangle \\
 &= \sum_i \frac{e^{-\beta(E_i - \mu N_i)}}{Z} \langle E_i | v^\dagger(t, t_0) A_I(\mathbf{r}, t) v(t, t_0) | E_i \rangle \\
 &= \sum_i \frac{e^{-\beta(E_i - \mu N_i)}}{Z} \langle \phi_{it} | A_I(\mathbf{r}, t) | \phi_{it} \rangle \\
 &\equiv \langle A_I(\mathbf{r}, t) \rangle_U.
 \end{aligned} \tag{2.38}$$

where the expectation value written without the subscript U denotes the equilibrium expectation value, and $| E_i \rangle$ is the initial state at t_0 when the perturbation is turned on. A state with energy E_i evolves in time under the effect of external perturbation $H'(t)$ as $|\phi_i(t)\rangle = v(t, t_0) | E_i \rangle$. Therefore the thermal expectation value of an observable $A(\mathbf{r})$ with respect to states evolving in the presence of an external time-dependent potential can be calculated as the equilibrium thermal expectation value $A_U(\mathbf{r}, t)$.

We now introduce a new, fictitious Green's function in the imaginary time interval $[t_0, t_0 - i\beta]$ in the following way

$$G(1, 1'; U; t_0) \equiv -i \frac{\langle T[S\psi(1)\psi^\dagger(1')] \rangle}{\langle T[S] \rangle}, \tag{2.39}$$

where

$$S \equiv \exp \left[-i \int_{t_0}^{t_0 - i\beta} dt_2 H'(t_2) \right]. \tag{2.40}$$

We note that $TS = v(t_0, t_0 - i\beta)$. In the interaction representation, all the U dependence is explicit in the S factor (2.40), and the time-dependence of the field operators is the same as in the absence of the $H'(t)$ (see (2.33)). This generalized Green's function $G(1, 1'; U; t_0)$ satisfies the boundary condition

$$G(1, 1'; U; t_0) \Big|_{t_1=t_0} = e^{\beta\mu} G(1, 1'; U; t_0) \Big|_{t_1=t_0 - i\beta}, \tag{2.41}$$

compared with that for an equilibrium Green's function

$$G(1, 1') \Big|_{t_1=0} = e^{\beta\mu} G(1, 1') \Big|_{t_1=-i\beta}. \tag{2.42}$$

Response functions are defined for real times and described by the real-time Greens' functions $g(1, 1'; U)$ is given by

$$g(1, 1'; U) = -i \langle T[\psi_U(1)\psi_U^\dagger(1')] \rangle = -i \langle T[v^\dagger(t_1)\psi(1)v(t_1)v^\dagger(t_{1'})\psi^\dagger(1')v(t_{1'})] \rangle \quad (2.43)$$

where

$$v(t_1) \equiv T \exp \left[-i \int_{-\infty}^{t_1} dt_2 H'(t_2) \right]. \quad (2.44)$$

To see how the imaginary-time Green's functions defined in (2.39) and real-time Green's functions defined in (2.43) are related, it is useful to write $G(1, 1'; U; t_0)$ in the following the form [assuming $i(t_1 - t_0) < i(t_{1'} - t_0)$]

$$\begin{aligned} G(1, 1'; U; t_0) &= G^<(1, 1'; U; t_0) = \frac{1}{i} \langle v(t_0, t_0 - i\beta)[v^\dagger(t_0, t_{1'})\psi^\dagger(1')v(t_0, t_{1'})] \\ &\quad v^\dagger(t_0, t_1)\psi(1)v(t_0, t_1) \rangle / \langle v(t_0, t_0 - i\beta) \rangle, \end{aligned} \quad (2.45)$$

and compare this with the expression for $g^<(1, 1'; U)$, namely

$$g^<(1, 1'; U) = -i \langle v^\dagger(t_1)\psi(1)v(t_1)v^\dagger(t_{1'})\psi^\dagger(1')v(t_{1'}) \rangle. \quad (2.46)$$

One sees directly that the real-time Green's function $g(1, 1'; U)$ are identical to $G(1, 1'; U; t_0)$ in the limit when $t_0 \rightarrow -\infty$, i.e.,

$$\begin{aligned} \lim_{t_0 \rightarrow -\infty} G^<(1, 1'; U; t_0) &= g^<(1, 1'; U) \\ \lim_{t_0 \rightarrow -\infty} G^>(1, 1'; U; t_0) &= g^>(1, 1'; U). \end{aligned} \quad (2.47)$$

This is the key formula of the whole KB formalism for going from an imaginary time Green's function to a real time response function. Hence, to find equations of motion for response functions defined by real time Green's functions, one can use the equations of motion for the imaginary-time Green's functions. The real-time Green's functions are then obtained by analytic continuation. One writes an imaginary-time Green's function as a Fourier series over the discrete Matsubara frequencies. The analytical continuation (which is unique) goes from the Fourier coefficients defined on a discrete set of imaginary frequencies to an analytic function for all frequencies.

Having outlined the basic “building blocks” underlying the Kadanoff-Baym non-equilibrium functions method, we are now ready to derive the equations of motion for the real-time non-equilibrium Green’s functions $g(1, 1'; U)$. We first derive the equations of motion for the imaginary-time non-equilibrium Green’s functions. As before we start from the Heisenberg equation of motion (2.2). We first consider the term $T[S\psi(1)]$. It can be written in the following form

$$\begin{aligned} T[S\psi(1)] &= T \left[\exp \left(-i \int_{t_0}^{t_0-i\beta} d2U(2)n(2) \right) \psi(1) \right] \\ &= T \left[\exp \left(-i \int_{t_1}^{t_0-i\beta} d2U(2)n(2) \right) \right] \psi(1) T \left[\exp \left(-i \int_{t_0}^{t_1} d2U(2)n(2) \right) \right]. \end{aligned} \quad (2.48)$$

Therefore the time derivative of (2.48) can be shown to reduce to

$$i \frac{\partial}{\partial t_1} T[S\psi(1)] = iT \left[S \frac{\partial \psi(1)}{\partial t_1} \right] + T[S\psi(1)] U(1). \quad (2.49)$$

Hence we obtain for the equation of motion the following equation

$$\begin{aligned} \left(i \frac{\partial}{\partial t_1} + \frac{\nabla_1^2}{2m} - U(1) + \mu_0 \right) G(1, 1'; U; t_0) &= \delta(1 - 1') \\ &+ i \int d\mathbf{r}_2 v(\mathbf{r}_1 - \mathbf{r}_2) G_2(12; 1'2^+; U; t_0) \Big|_{t_2=t_1}, \end{aligned} \quad (2.50)$$

where G_2 is defined in (2.5). Furthermore, since we want to derive the equation of motion for the non-equilibrium Green’s functions for a system that is disturbed by an external, time-dependent potential, we need to find what the change in G is resulting from an infinitesimal change in the external potential $U(2) \rightarrow U(2) + \delta U(2)$. First the evolution operator S changes as follows

$$\delta S = \delta \left[\exp \left(-i \int_{t_0}^{t_0-i\beta} d2U(2)n(2) \right) \right] = S \frac{1}{i} \int_{t_0}^{t_0-i\beta} d2\delta U(2)n(2). \quad (2.51)$$

Using (2.51), one can show that the infinitesimal change in G resulting from the infinitesimal change in the external potential U is given by

$$\delta G(1, 1'; U; t_0) = \int_{t_0}^{t_0-i\beta} d2 \left[G_2(12, 1'2^+; U; t_0) - G(1, 1'; U; t_0) G(2, 2^+; U; t_0) \right] \delta U(2). \quad (2.52)$$

Thus the functional derivative of $G(1, 1'; U; t_0)$ with respect to $U(2)$ gives

$$\frac{\delta G(1, 1'; U; t_0)}{\delta U(2)} = \left[G_2(12, 1'2^+; U; t_0) - G(1, 1'; U; t_0)G(2, 2^+; U; t_0) \right], \quad (2.53)$$

which is a nice way of generating the two-particle correlation function. Using (2.52), we can rewrite the equation of motion in (2.50) in the form

$$\left(i \frac{\partial}{\partial t_1} + \frac{\nabla_1^2}{2m} - U(1) + \mu_0 - i \int d\mathbf{r}_2 v(\mathbf{r}_1 - \mathbf{r}_2) \right. \\ \left. \times \left[G(\mathbf{r}_2 t_1; \mathbf{r}_2 t_1^+; U; t_0) + \frac{\delta}{\delta U(\mathbf{r}_2 t_1)} \right] \right) G(1, 1'; U; t_0) = \delta(1 - 1'). \quad (2.54)$$

We define the self-energy function Σ in the presence of the external potential U in the following form

$$\int d\bar{1} \left[G_0^{-1}(1, \bar{1}; U; t_0) - \Sigma(1, \bar{1}; U; t_0) \right] G(\bar{1}, 1'; U; t_0) = \delta(1 - 1'), \quad (2.55)$$

and the inverse propagator $G^{-1}(1, 1'; U; t_0)$ is defined as

$$\int_{t_0}^{t_0 - i\beta} d\bar{1} G^{-1}(1, \bar{1}; U; t_0) G(\bar{1}, 1'; U; t_0) = \delta(1 - 1'). \quad (2.56)$$

G_0^{-1} in (2.55) is the inverse of the non-interacting propagator. In the above equations and elsewhere, integration over $d\bar{1}$ means integration over the coordinates (\mathbf{r}_1, t_1) and $\delta(1 - 1') \equiv \delta(\mathbf{r} - \mathbf{r}')\delta(t - t')$. Using (2.54), (2.55) and (2.56), one can show that the self-energy function in (2.55) can be written as

$$\Sigma(1, 1'; U; t_0) = i \int d\bar{1} v(1 - \bar{1}) G(\bar{1}, \bar{1}^+; U; t_0) \delta(1 - 1') \\ + \int d\bar{2} \bar{1} v(1 - \bar{2}) \frac{\delta G(1, \bar{1}; U; t_0)}{\delta U(\bar{2})} G^{-1}(\bar{1}, 1'; U; t_0). \quad (2.57)$$

Using $\delta G G^{-1} + G \delta G^{-1} = 0$ (which follows directly from (2.56)), one can rewrite this self-energy function as

$$\Sigma(1, 1'; U; t_0) = \delta(1 - 1') i \int d\bar{1} v(1 - \bar{1}) G(\bar{1}, \bar{1}^+; U; t_0) \\ + i v(1 - 1') G(1, 1'; U; t_0) + \int d\bar{2} \bar{1} v(1 - \bar{2}) G(1, \bar{1}; U; t_0) \frac{\delta \Sigma(\bar{1}, 1'; U; t_0)}{\delta U(\bar{2})}. \quad (2.58)$$

The Hartree-Fock approximation is obtained by neglecting the $\delta\Sigma/\delta U$ term in (2.58), giving

$$\begin{aligned}\Sigma_{HF}(1, 1'; U; t_0) &= \delta(1 - 1')i \int d\bar{1}v(1 - \bar{1})G(\bar{1}, \bar{1}^+; U; t_0) \\ &+ iv(1 - 1')G(1, 1'; U; t_0).\end{aligned}\quad (2.59)$$

The next approximation is obtained by replacing $\delta\Sigma/\delta U$ by $\delta\Sigma_{HF}/\delta U$ and contains product of three G 's

$$\begin{aligned}\Sigma(1, 1'; U; t_0) - \Sigma_{HF}(1, 1'; U; t_0) &= i^2 \int d\bar{1}d\bar{2}v(1 - \bar{1})v(\bar{2} - 1') \\ &\times [G(1, 1'; U; t_0)G(\bar{2}, \bar{1}; U; t_0)G(\bar{1}, \bar{2}; U; t_0) \\ &+ G(1, \bar{2}; U; t_0)G(\bar{2}, \bar{1}; U; t_0)G(\bar{1}, 1'; U; t_0)].\end{aligned}\quad (2.60)$$

These terms contribute to the collision integrals in a Boltzmann equation. Eq. (2.58) thus allows one to generate increasingly higher order self-consistent approximations by a simple iteration.

Using the same method, one can obtain the equation of motion for the single-particle Green's function containing derivatives with the other variables $1'$. Multiplying (2.55) by G_0 , one easily finds the equations of motion for the imaginary-time Green's function $G(1, 1'; U; t_0)$ in the presence of the slowly varying external field U to be the Dyson equations

$$\begin{aligned}\left[i \frac{\partial}{\partial t_1} + \frac{\nabla_1^2}{2m} - U(1) + \mu_0 \right] G(1, 1'; U; t_0) &- \int_{t_0}^{t_0 - i\beta} d\bar{1} \Sigma(1, \bar{1}; U; t_0) G(\bar{1}, 1'; U; t_0) \\ &= \delta(1, 1')\end{aligned}\quad (2.61)$$

and

$$\begin{aligned}\left[-i \frac{\partial}{\partial t_{1'}} + \frac{\nabla_{1'}^2}{2m} - U(1') + \mu_0 \right] G(1, 1'; U; t_0) &- \int_{t_0}^{t_0 - i\beta} d\bar{1} G(1, \bar{1}; U; t_0) \Sigma(\bar{1}, 1'; U; t_0) \\ &= \delta(1, 1').\end{aligned}\quad (2.62)$$

We split the self-energy function $\Sigma(1, 1'; U; t_0)$ into two parts

$$\Sigma(1, 1'; U; t_0) = \Sigma_{HF}(1, 1'; U; t_0) + \Sigma_c(1, 1'; U; t_0),\quad (2.63)$$

where the first-order Hartree-Fock contribution Σ_{HF} in (2.59) describes the mean-field effects of an interaction and Σ_c denotes the ‘‘collisional contributions’’ which are second and higher order. We denote this collisional part as Σ_c , and define

$$\begin{aligned}\Sigma_c(1, 1'; U; t_0) &= \Sigma_c^>(1, 1'; U; t_0) && \text{for } i(t_1 - t_{1'}) > 0 \\ &= \Sigma_c^<(1, 1'; U; t_0) && \text{for } i(t_1 - t_{1'}) < 0.\end{aligned}\quad (2.64)$$

Physical response functions measured in experiments are related to the correlation functions such as in (2.8). Therefore we write (2.61) for the case $i(t_1 - t_0) < i(t_{1'} - t_0)$, to obtain

$$\begin{aligned}\left[i \frac{\partial}{\partial t_1} + \frac{\nabla_1^2}{2m} - U_{eff}(1; t_0) \right] G^<(1, 1'; U; t_0) &= \int_{t_0}^{t_1} d\bar{1} \Sigma_c^>(1, \bar{1}; U; t_0) G^<(\bar{1}, 1'; U; t_0) \\ + \int_{t_1}^{t_{1'}} d\bar{1} \Sigma_c^<(1, \bar{1}; U; t_0) G^<(\bar{1}, 1'; U; t_0) &+ \int_{t_{1'}}^{t_0 - i\beta} d\bar{1} \Sigma_c^<(1, \bar{1}; U; t_0) G^>(\bar{1}, 1'; U; t_0).\end{aligned}\quad (2.65)$$

To obtain the real time correlation function in (2.43), we take the limit $t_0 \rightarrow -\infty$ of (2.65) and use (2.47),

$$\begin{aligned}\left[i \frac{\partial}{\partial t_1} + \frac{\nabla_1^2}{2m} - U_{eff}(1) \right] g^<(1, 1'; U) &= \int_{-\infty}^{t_1} d\bar{1} [\Sigma_c^>(1, \bar{1}; U) - \Sigma_c^<(1, \bar{1}; U)] g^<(\bar{1}, 1'; U) \\ - \int_{-\infty}^{t_{1'}} d\bar{1} \Sigma_c^<(1, \bar{1}; U) [g^>(\bar{1}, 1'; U) - g^<(\bar{1}, 1'; U)].\end{aligned}\quad (2.66)$$

Here the effective potential U_{eff} and the second-order approximation to the self-energy Σ_c^{\gtrless} are given by

$$\begin{aligned}U_{eff}(1) &\equiv U_{eff}(1; -\infty) = U(\mathbf{r}, t) - \mu_0 + i \int d\bar{1} v(1 - \bar{1}) g^<(\bar{1}, \bar{1}^+; U) \\ \Sigma_c^{\gtrless}(1, 1'; U) &\equiv \Sigma_c^{\gtrless}(1, \bar{1}; U; -\infty) = - \int d2d3 v(13)v(21') \\ &\times \left[g^{\gtrless}(1, 1'; U) g^{\gtrless}(2, 3; U) g^{\lessgtr}(3, 2; U) + g^{\lessgtr}(1, 3; U) g^{\gtrless}(2, 1'; U) g^{\lessgtr}(3, 2; U) \right]_{t_2=t_1, t_3=t_1}.\end{aligned}\quad (2.67)$$

Considering (2.62) for the case $i(t_1 - t_0) < i(t_{1'} - t_0)$, one finds an analogous equation of motion for $g^<(1, 1')$ with the respect to other variables $1'$, namely

$$\left[-i \frac{\partial}{\partial t_{1'}} + \frac{\nabla_{1'}^2}{2m} - U_{eff}(1') \right] g^<(1, 1'; U) = \int_{-\infty}^{t_1} d\bar{1} [g^>(1, \bar{1}; U) - g^<(1, \bar{1}; U)] \Sigma_c^<(\bar{1}, 1; U)$$

$$- \int_{-\infty}^{t_1'} d\bar{1} g^<(1, \bar{1}; U) [\Sigma_c^>(\bar{1}, 1'; U) - \Sigma_c^<(\bar{1}, 1'; U)]. \quad (2.68)$$

Applying the same arguments as above, one can also find equation of motion for $g^>$ in the case $i(t_1 - t_0) > i(t_1' - t_0)$, i.e.

$$\begin{aligned} \left[i \frac{\partial}{\partial t_1} + \frac{\nabla_1^2}{2m} - U_{eff}(1) \right] g^>(1, 1'; U) &= \int_{-\infty}^{t_1} d\bar{1} [\Sigma_c^>(1, \bar{1}; U) - \Sigma_c^<(1, \bar{1}; U)] g^>(\bar{1}, 1'; U) \\ &- \int_{-\infty}^{t_1'} d\bar{1} \Sigma_c^>(1, \bar{1}; U) [g^>(\bar{1}, 1'; U) - g^<(\bar{1}, 1'; U)]. \end{aligned} \quad (2.69)$$

and

$$\begin{aligned} \left[-i \frac{\partial}{\partial t_1'} + \frac{\nabla_{1'}^2}{2m} - U_{eff}(1') \right] g^>(1, 1'; U) &= \int_{-\infty}^{t_1} d\bar{1} [g^>(1, \bar{1}; U) - g^<(1, \bar{1}; U)] \Sigma_c^>(\bar{1}, 1; U) \\ &- \int_{-\infty}^{t_1'} d\bar{1} g^>(1, \bar{1}; U) [\Sigma_c^>(\bar{1}, 1'; U) - \Sigma_c^<(\bar{1}, 1'; U)]. \end{aligned} \quad (2.70)$$

The equations of motion in (2.66)-(2.70) are formally exact and can be used to derive a generalized Boltzmann equation. They are extremely complicated. However, in the limit when the external perturbing potential $U(\mathbf{R}, T)$ varies slowly in space in time, one can separate out fast and slow processes by transforming the equations of motion into the so-called Wigner representation, that is one writes all physical quantities in terms of the center-of-mass (\mathbf{R}, T) and relative coordinates (\mathbf{r}, t) given by

$$\begin{aligned} \mathbf{R} &= \frac{1}{2}(\mathbf{r}_1 + \mathbf{r}_{1'}), & T &= \frac{1}{2}(t_1 + t_{1'}) \\ \mathbf{r} &= \mathbf{r}_1 - \mathbf{r}_{1'}, & t &= t_1 - t_{1'}. \end{aligned} \quad (2.71)$$

In thermal equilibrium, the Green's functions and the self-energy function depend on the relative coordinates (\mathbf{r}, t) only, and moreover these are sharply peaked about $\mathbf{r} = 0$ and $t = 0$. We use the different way that non-equilibrium functions depend on (\mathbf{r}, t) and on (\mathbf{R}, T) in deriving a generalized Boltzmann equation. We thus assume that the system is only slightly perturbed out of thermal equilibrium and all physical quantities depend on small values of relative coordinates (\mathbf{r}, t) only. This is a crucial physical assumption in the derivation of a generalized Boltzmann equation using the Kadanoff-Baym formalism.

To derive a generalized Boltzmann equation, one needs to rewrite the equation for the real time non-equilibrium Green's function $g^<(1, 1'; U)$ in the Wigner representation. To this end, we first subtract (2.68) from (2.66) to obtain

$$\begin{aligned}
 & \left[i \frac{\partial}{\partial t_1} + i \frac{\partial}{\partial t_{1'}} + \frac{\nabla_1^2}{2m} - \frac{\nabla_{1'}^2}{2m} - U_{eff}(1) + U_{eff}(1') \right] g^<(1, 1'; U) \\
 &= \int_{-\infty}^{t_1} d\bar{1} [\Sigma_c^>(1, \bar{1}; U) - \Sigma_c^<(1, \bar{1}; U)] g^<(\bar{1}, 1'; U) \\
 &+ \int_{-\infty}^{t_{1'}} d\bar{1} g^<(1, \bar{1}; U) [\Sigma_c^>(\bar{1}, 1'; U) - \Sigma_c^<(\bar{1}, 1'; U)] \\
 &- \int_{-\infty}^{t_{1'}} d\bar{1} \Sigma_c^<(1, \bar{1}; U) [g^>(\bar{1}, 1'; U) - g^<(\bar{1}, 1'; U)] \\
 &- \int_{-\infty}^{t_1} d\bar{1} [g^>(1, \bar{1}; U) - g^<(1, \bar{1}; U)] \Sigma_c^<(\bar{1}, 1; U). \tag{2.72}
 \end{aligned}$$

We now rewrite (2.72) in terms of the center-of-mass and relative coordinates $(\mathbf{r}, t; \mathbf{R}, T)$. Using $g^>(\mathbf{r}, t; \mathbf{R}, T) \equiv g^>(1, 1'; U)$, the left hand side of (2.72) becomes

$$\left[i \frac{\partial}{\partial T} + \frac{\nabla_{\mathbf{R}} \cdot \nabla_{\mathbf{r}}}{m} - U_{eff}(\mathbf{R} + \mathbf{r}/2, T + t/2) + U_{eff}(\mathbf{R} - \mathbf{r}/2, T - t/2) \right] g^<(\mathbf{r}, t; \mathbf{R}, T) \tag{2.73}$$

Since we limit ourselves to disturbances that perturb the system only slightly out of equilibrium, and only small values of (\mathbf{r}, t) are important, we can expand the difference in the effective potential in (2.73) in powers of \mathbf{r} and t , and keep only the linear terms,

$$\left[i \frac{\partial}{\partial T} + \frac{\nabla_{\mathbf{R}} \cdot \nabla_{\mathbf{r}}}{m} - \left[\left(\mathbf{r} \cdot \nabla_{\mathbf{R}} + t \frac{\partial}{\partial T} \right) U_{eff}(\mathbf{R}, T) \right] \right] g^<(\mathbf{r}, t; \mathbf{R}, T). \tag{2.74}$$

Similarly, if one expresses the right-hand side of (2.72) in terms of $(\mathbf{r}, t; \mathbf{R}, T)$ and expands it the same way, (2.72) finally becomes

$$\begin{aligned}
 & \left\{ i \frac{\partial}{\partial T} + \frac{\nabla_{\mathbf{R}} \cdot \nabla_{\mathbf{r}}}{m} - \left[\left(\mathbf{r} \cdot \nabla_{\mathbf{R}} + t \frac{\partial}{\partial T} \right) U_{eff}(\mathbf{R}, T) \right] \right\} g^<(\mathbf{r}, t; \mathbf{R}, T) \\
 &= \int d\bar{\mathbf{r}} d\bar{t} [g^<(\bar{\mathbf{r}}, \bar{t}; \mathbf{R}, T) \Sigma_c^>(\mathbf{r} - \bar{\mathbf{r}}, t - \bar{t}; \mathbf{R}, T) - g^>(\bar{\mathbf{r}}, \bar{t}; \mathbf{R}, T) \Sigma_c^<(\mathbf{r} - \bar{\mathbf{r}}, t - \bar{t}; \mathbf{R}, T)]. \tag{2.75}
 \end{aligned}$$

Fourier transforming (2.75), one obtains

$$\begin{aligned}
 & \left[\frac{\partial}{\partial T} + \frac{\mathbf{p} \cdot \nabla_{\mathbf{R}}}{m} - \nabla_{\mathbf{R}} U_{eff}(\mathbf{R}, T) \cdot \nabla_{\mathbf{p}} + \frac{\partial U_{eff}(\mathbf{R}, T)}{\partial T} \frac{\partial}{\partial \omega} \right] g^<(\mathbf{p}, \omega; \mathbf{R}, T) \\
 &= g^<(\mathbf{p}, \omega; \mathbf{R}, T) \Sigma_c^>(\mathbf{p}, \omega; \mathbf{R}, T) - g^>(\mathbf{p}, \omega; \mathbf{R}, T) \Sigma_c^<(\mathbf{p}, \omega; \mathbf{R}, T). \tag{2.76}
 \end{aligned}$$

The Fourier transform $g^{\lessgtr}(\mathbf{p}, \omega; \mathbf{R}, T)$ is defined by

$$g^{\lessgtr}(\mathbf{p}, \omega; \mathbf{R}, T) = \int_{-\infty}^{\infty} dt \int d\mathbf{r} e^{i(\omega t - \mathbf{p} \cdot \mathbf{r})} i g^{\lessgtr}(\mathbf{r}, t; \mathbf{R}, T), \quad (2.77)$$

where

$$g^<(\mathbf{r}, t; \mathbf{R}, T) = i \left\langle \psi^\dagger \left(\mathbf{R} - \frac{\mathbf{r}}{2}, T - \frac{t}{2} \right) \psi \left(\mathbf{R} + \frac{\mathbf{r}}{2}, T + \frac{t}{2} \right) \right\rangle. \quad (2.78)$$

The Fourier transform of the self-energy function is given as

$$\Sigma^{\lessgtr}(\mathbf{p}, \omega; \mathbf{R}, T) = \int_{-\infty}^{\infty} dt \int d\mathbf{r} e^{i(\omega t - \mathbf{p} \cdot \mathbf{r})} i \Sigma^{\lessgtr}(\mathbf{r}, t; \mathbf{R}, T). \quad (2.79)$$

We note that $g^<(\mathbf{p}, \omega; \mathbf{R}, T)$ gives the density of atoms with momentum \mathbf{p} and energy ω at the space-time point \mathbf{R}, T .

One can see that the equation of motion for $g^<(\mathbf{p}, \omega; \mathbf{R}, T)$ given in (2.76) depends on $g^<(\mathbf{p}, \omega; \mathbf{R}, T)$ and $g^>(\mathbf{p}, \omega; \mathbf{R}, T)$ as well. Therefore, one also needs to derive equation of motion for $g^>(\mathbf{p}, \omega; \mathbf{R}, T)$. Applying the same analysis we used to derive (2.76), one obtains

$$\begin{aligned} & \left[\frac{\partial}{\partial T} + \frac{\mathbf{p} \cdot \nabla_{\mathbf{R}}}{m} - \nabla_{\mathbf{R}} U_{eff}(\mathbf{R}, T) \cdot \nabla_{\mathbf{p}} + \frac{\partial U_{eff}(\mathbf{R}, T)}{\partial T} \frac{\partial}{\partial \omega} \right] g^>(\mathbf{p}, \omega; \mathbf{R}, T) \\ &= g^<(\mathbf{p}, \omega; \mathbf{R}, T) \Sigma_c^>(\mathbf{p}, \omega; \mathbf{R}, T) - g^>(\mathbf{p}, \omega; \mathbf{R}, T) \Sigma_c^<(\mathbf{p}, \omega; \mathbf{R}, T). \end{aligned} \quad (2.80)$$

It is useful to introduce the non-equilibrium spectral density $a(\mathbf{p}, \omega; \mathbf{R}, T)$ defined as

$$a(\mathbf{p}, \omega; \mathbf{R}, T) \equiv g^>(\mathbf{p}, \omega; \mathbf{R}, T) - g^<(\mathbf{p}, \omega; \mathbf{R}, T). \quad (2.81)$$

One can show, using (2.76) and (2.80), that the collision terms cancel out in the equation of motion for the non-equilibrium spectral density $a(\mathbf{p}, \omega; \mathbf{R}, T)$, and we obtain

$$\left[\frac{\partial}{\partial T} + \frac{\mathbf{p} \cdot \nabla_{\mathbf{R}}}{m} - \nabla_{\mathbf{R}} U_{eff} \cdot \nabla_{\mathbf{p}} + \frac{\partial U_{eff}}{\partial T} \frac{\partial}{\partial \omega} \right] a(\mathbf{p}, \omega; \mathbf{R}, T) = 0. \quad (2.82)$$

The usual Boltzmann equation is expressed in terms of a distribution function $f(\mathbf{r}, \mathbf{p}, t)$.

KB introduce this by writing the correlation functions in the following form

$$\begin{aligned} g^<(\mathbf{p}, \omega; \mathbf{R}, T) &= a(\mathbf{p}, \omega; \mathbf{R}, T) f(\mathbf{p}, \mathbf{R}, T) \\ g^>(\mathbf{p}, \omega; \mathbf{R}, T) &= a(\mathbf{p}, \omega; \mathbf{R}, T) (1 + f(\mathbf{p}, \mathbf{R}, T)). \end{aligned} \quad (2.83)$$

To illustrate this approach, we consider the so-called “quasiparticle approximation” for the non-equilibrium spectral density

$$a(\mathbf{p}, \omega, \mathbf{R}, T) = 2\pi\delta(\omega - E_p(\mathbf{R}, T)), \quad (2.84)$$

where $E_p(\mathbf{R}, T) = p^2/2m + U_{eff}(\mathbf{R}, T)$. This can be verified to be a solution of (2.82). Using (2.83), (2.82) and (2.84) in (2.76), after considerable algebra one finally obtains the Boltzmann equation

$$\begin{aligned} & \left[\frac{\partial}{\partial T} + \frac{\mathbf{p} \cdot \nabla_{\mathbf{R}}}{m} - \nabla_{\mathbf{R}} U_{eff}(\mathbf{R}, T) \cdot \nabla_{\mathbf{p}} \right] f(\mathbf{p}, \mathbf{R}, T) \\ &= - \int \frac{d\mathbf{p}_1}{(2\pi)^3} \frac{d\mathbf{p}_2}{(2\pi)^3} \frac{d\mathbf{p}_3}{(2\pi)^3} (2\pi)^4 \delta(\mathbf{p} + \mathbf{p}_1 - \mathbf{p}_2 - \mathbf{p}_3) \\ & \times \delta(\varepsilon_p + \varepsilon_{p_1} - \varepsilon_{p_2} - \varepsilon_{p_3}) (1/2) [v(\mathbf{p} - \mathbf{p}_2) - v(\mathbf{p} - \mathbf{p}_3)]^2 \\ & \times [f f_1 (1 + f_2)(1 + f_3) - (1 + f)(1 + f_1) f_2 f_3], \end{aligned} \quad (2.85)$$

where $\varepsilon_{p_i} = p_i^2/2m$ and $f_i = f(\mathbf{p}_i, \mathbf{R}, T)$. Here, we have used the Fourier transform of the expression for Σ_c^{\gtrless} given in (2.67). We have thus obtained the usual Boltzmann equation for interacting Bose atoms (see Ch. 9 of Ref. [14]). More specifically, we see how the C_{22} collision integral describing collisions between atoms naturally arises from the KB formalism. As expected, creation of a Boson in a binary collision involves the statistical factor $1 + f$, while the destruction of a Boson involves the factor f . In later chapters, we will give more details of the calculations leading to kinetic equations like (2.85).

This concludes our review of the Kadanoff-Baym formalism for normal (non Bose-condensed) systems. We have shown how one can, starting from the non-equilibrium Green’s function defined on the imaginary time domain, derive a quantum Boltzmann equation for real times. The major feature of the KB formalism is that any specific equilibrium self-energy approximation can be generalized as the basis for non-equilibrium problems.

Chapter 3

Kadanoff-Baym method for Bose-condensed gases

In this chapter, we apply the Kadanoff-Baym method to Bose-condensed systems. Kane and Kadanoff (KK) [15, 16] first generalized the KB method, with the specific goal of deriving the phenomenological two-fluid hydrodynamical equations of Landau [7]. In the thesis, we use the KK generalization to derive a quantum Boltzmann-like equation appropriate to a trapped, dilute Bose condensed gas. We show how the KB formalism can be used to derive a kinetic equation for a trapped Bose gas, in a way that sets the stage for further extensions. Our goals are different from KK, who were interested in giving a microscopic basis for two-fluid hydrodynamics. We are interested in deriving a kinetic equation for the non-condensate atoms. One can derive two-fluid hydrodynamics from this kinetic equation, but this is not done in this thesis [18].

3.1 Generalization of the KB method for Bose-condensed gases

A Bose-condensed fluid is conveniently described in terms of the spinor quantum field operators

$$\hat{\Psi}(1) = \begin{pmatrix} \psi(1) \\ \psi^\dagger(1) \end{pmatrix}, \quad \hat{\Psi}^\dagger(1) = (\psi^\dagger(1), \psi(1)). \quad (3.1)$$

The many-body Hamiltonian describing N interacting bosons confined by an external potential $U_{ext}(\mathbf{r})$ is given by

$$\begin{aligned} K = H - \mu N &= \int d\mathbf{r} \psi^\dagger(\mathbf{r}) \left[-\frac{\hbar}{2m} \nabla_{\mathbf{r}}^2 + U_{ext}(\mathbf{r}) - \mu_0 \right] \psi(\mathbf{r}) \\ &+ \frac{1}{2} \int d\mathbf{r} d\mathbf{r}' \psi^\dagger(\mathbf{r}) \psi^\dagger(\mathbf{r}') v(\mathbf{r} - \mathbf{r}') \psi(\mathbf{r}') \psi(\mathbf{r}), \end{aligned} \quad (3.2)$$

where $v(\mathbf{r} - \mathbf{r}')$ is the two-body interatomic potential. For ultra-cold Bose gases, this potential can be expressed in terms of the s -wave scattering length as given in (1.1). In this chapter, we work with a general two-body potential since various contributions are more clearly shown.

We separate out the condensate part of the field operator in the usual fashion [10]

$$\psi(\mathbf{r}, t) = \langle \psi(\mathbf{r}, t) \rangle + \tilde{\psi}(\mathbf{r}, t), \quad (3.3)$$

where $\langle \tilde{\psi}(\mathbf{r}) \rangle = 0$ and $\langle \psi(\mathbf{r}) \rangle = \Phi(\mathbf{r})$ is the Bose macroscopic wavefunction. The non-condensate (or excited atom component) field operators $\tilde{\psi}(\mathbf{r})$ and $\tilde{\psi}^\dagger(\mathbf{r})$ satisfy the usual Bose commutation relations. In a Bose-condensed fluid, the finite value of $\Phi(\mathbf{r}, t)$ leads to finite values of the off-diagonal (or anomalous) propagators $\langle \tilde{\psi}(1) \tilde{\psi}(1') \rangle$ and $\langle \tilde{\psi}^\dagger(1) \tilde{\psi}^\dagger(1') \rangle$. These must be dealt with on an equal basis with the diagonal (or normal) propagators; many-body theory is most easily formulated in terms of a 2×2 matrix single-particle Green's function defined by [15, 11]

$$\hat{g}(1, 1'; U) = -i \langle T \Psi(1) \Psi^\dagger(1') \rangle = -i \begin{pmatrix} \langle T \psi(1) \psi^\dagger(1') \rangle & \langle T \psi(1) \psi(1') \rangle \\ \langle T \psi^\dagger(1) \psi^\dagger(1') \rangle & \langle T \psi^\dagger(1) \psi(1') \rangle \end{pmatrix}. \quad (3.4)$$

Here, as before, T represents the time-ordering operation and we use the usual KB abbreviated notation, $1 \equiv (\mathbf{r}, t)$ and $1' \equiv (\mathbf{r}', t')$. The real time response functions $\hat{g}^<$ and $\hat{g}^>$ are defined as in the case of normal system (see Ch. 2), but we must keep in mind that these are now 2×2 matrix propagators.

Using (3.3), the matrix propagator in (3.4) splits into two parts

$$\hat{g}(1, 1'; U) = \hat{\hat{g}}(1, 1'; U) + \hat{h}(1, 1'; U). \quad (3.5)$$

Here $\hat{\hat{g}}$ is identical to (3.4) except that it involves the non-condensate part of the field operators, and the condensate part is given by

$$\hat{h}(1, 1'; U) \equiv -i \begin{pmatrix} \Phi(1)\Phi^*(1') & \Phi(1)\Phi(1') \\ \Phi^*(1)\Phi^*(1') & \Phi^*(1)\Phi(1') \end{pmatrix}, \quad (3.6)$$

with $\langle \psi^\dagger(\mathbf{r}, t) \rangle \equiv \Phi^*(\mathbf{r}, t)$.

We see that the dynamics of a Bose-condensed system can formally be separated into two parts: the dynamics of atoms that are in the condensate and described by a macroscopic wavefunction Φ , and the dynamics of atoms that are not Bose-condensed, which are described by the 2×2 matrix propagator $\hat{\hat{g}}$. However the equations of motion describing these two components are coupled and one has to solve them self-consistently. The description of the dynamics of non-condensate atoms is similar to the description that we have reviewed for normal systems in Ch. 2. The main difference is that now one has non-zero anomalous averages which have to be included.

A very useful and elegant way of generating the equations of motion for both $\hat{\hat{g}}$ and Φ is to use functional derivatives with respect to weak external fields [11, 14, 33],

$$H'(t_1) = \frac{1}{2} \int d\mathbf{r}_1 d2 \psi^\dagger(1) U(1, 2) \psi(2) + \int d\mathbf{r}_1 [\psi^\dagger(1) \eta_{ext}(1) + \psi(1) \eta_{ext}^*(1)]. \quad (3.7)$$

Here we generalize the external perturbation given in (2.17) to allow the thermal expectation values of the quantum field operators to be non-zero. $U(1, 2)$ is an external generating scalar field non-local in space and time. It represents a perturbation in which an atom is

removed from the system at point 1 and added at 2. The symmetry-breaking fields η_{ext} and η_{ext}^* describe particle creation and destruction [9, 11]. All higher-order Green's functions can be neatly expressed as functional derivatives of single-particle Green's functions with respect to such generating fields, as we saw in Ch. 2. More details on the derivation of the self-energy functions for a Bose-condensed gas will be given in the next section.

The equation for the condensate can be written in terms of the 2-component order parameter $\hat{G}_{1/2}(1) \equiv \sqrt{-i}\langle\Psi(1)\rangle$, where Ψ is given by (3.1). One finds [11, 19, 33]

$$\int d\bar{1}\hat{g}_0^{-1}(1,\bar{1})G_{1/2}(\bar{1}) = \sqrt{-i}\hat{\eta}(1) + \sqrt{-i}\hat{\eta}_{ext}(1), \quad (3.8)$$

where the ‘‘condensate source function’’ η is defined in terms of the three-field correlation function

$$\sqrt{-i}\hat{\eta}(1) \equiv \frac{1}{2} \int d\bar{2}\sqrt{-i}v(1\bar{2})\langle T\Psi(1)\Psi^\dagger(\bar{2})\Psi(\bar{2})\rangle. \quad (3.9)$$

The inverse of the non-interacting 2×2 matrix Bose gas propagator $\hat{g}_0(1,1')$ in (3.8) is defined by

$$\hat{g}_0^{-1}(1,1') = \left[i\hat{\tau}_3 \frac{\partial}{\partial t_1} + \frac{\nabla_1^2}{2m} - U_{ext}(\mathbf{r}_1) - U(1) + \mu_0 \right] \hat{I} \delta(1,1'), \quad (3.10)$$

where $\hat{\tau}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and \hat{I} is the 2×2 identity matrix. The external particle-source fields in (3.7) are

$$\hat{\eta}_{ext}(1) \equiv \begin{pmatrix} \eta_{ext}(1) \\ \eta_{ext}^*(1) \end{pmatrix}. \quad (3.11)$$

Approximations for the macroscopic wavefunction are determined by the choice we make for the condensate source function $\hat{\eta}$ in (3.9).

Using (3.3), one can also decompose the three-field correlation function involved in (3.9). For example, one has

$$\begin{aligned} \langle T\psi(1)\psi^\dagger(2)\psi(2) \rangle &= \Phi(1)n_c(2) + \Phi(2)\langle T\tilde{\psi}(1)\tilde{\psi}^\dagger(2) \rangle \\ &+ \Phi^*(2)\langle T\tilde{\psi}(1)\tilde{\psi}(2) \rangle + \langle T\tilde{\psi}(1)\tilde{\psi}^\dagger(2)\tilde{\psi}(2) \rangle. \end{aligned} \quad (3.12)$$

In the first order Hartree-Fock-Bogoliubov (HFB) approximation, one neglects the three-field correlation function $\langle T\tilde{\psi}\tilde{\psi}^\dagger\tilde{\psi} \rangle$ for the non-condensate atoms. In this approximation, (3.12) only involves the condensate density $n_c = |\Phi|^2$ and the non-condensate correlation functions $\langle \tilde{\psi}^\dagger\tilde{\psi} \rangle$ and $\langle \tilde{\psi}\tilde{\psi} \rangle$.

Equation (3.8) for the order parameter can be rewritten in terms of a condensate self-energy function S defined by

$$\int d\bar{1} \hat{S}(1, \bar{1}) \hat{h}(\bar{1}, 1') \equiv \sqrt{-i} \eta(1) \hat{G}_{1/2}^\dagger(1'), \quad (3.13)$$

with the condensate propagator h given by (3.6). In place of (3.8), we have the new equation of motion

$$\int d\bar{1} \left[\hat{g}_0^{-1}(1, \bar{1}) - \hat{S}^{HF}(1, \bar{1}) \right] h(\bar{1}, 1') = \int_{-\infty}^{t_1} d\bar{1} (\hat{S}^>(1, \bar{1}) - \hat{S}^<(1, \bar{1})) \hat{h}(\bar{1}, 1'). \quad (3.14)$$

Here the mean-field contributions are included in the Hartree-Fock part of the condensate self-energy S^{HF} , while S^{\gtrless} involves contributions that are second order or higher in the interaction and are defined in the same way as Σ^{\gtrless} in (2.64).

To describe the dynamics of the non-condensate, one needs to derive a quantum Boltzmann equation for the distribution function of the thermally excited atoms. Using the method described in Ch. 2, one can write the equations of motion for the non-equilibrium real-time non-condensate 2×2 propagators $\hat{g}(1, 1')$,

$$\hat{g}(1, 1'; U) = -i \begin{pmatrix} \langle T\tilde{\psi}(1)\tilde{\psi}^\dagger(1') \rangle & \langle T\tilde{\psi}(1)\tilde{\psi}(1') \rangle \\ \langle T\tilde{\psi}^\dagger(1)\tilde{\psi}^\dagger(1') \rangle & \langle T\tilde{\psi}^\dagger(1)\tilde{\psi}(1') \rangle \end{pmatrix} \quad (3.15)$$

in the following 2×2 matrix form

$$\begin{aligned} & \int d\bar{1} \left[\hat{g}_0^{-1}(1, \bar{1}) - \hat{\Sigma}^{HF}(1, \bar{1}) \right] \hat{g}^{\gtrless}(\bar{1}, 1') \\ &= \int_{-\infty}^{t_1} d\bar{1} \hat{\Gamma}(1, \bar{1}) \hat{g}^{\gtrless}(\bar{1}, 1') - \int_{-\infty}^{t_1'} d\bar{1} \hat{\Sigma}_c^{\gtrless}(1, \bar{1}) \hat{a}(\bar{1}, 1'), \end{aligned} \quad (3.16)$$

and

$$\begin{aligned} & \int d\bar{1} \hat{g}^{\gtrless}(1, \bar{1}) \left[\hat{g}_0^{-1}(\bar{1}, 1') - \hat{\Sigma}^{HF}(\bar{1}, 1') \right] \\ &= \int_{-\infty}^{t_1} d\bar{1} \hat{a}(1, \bar{1}) \hat{\Sigma}_c^{\gtrless}(\bar{1}, 1') - \int_{-\infty}^{t_1'} d\bar{1} \hat{g}^{\gtrless}(1, \bar{1}) \hat{\Gamma}(\bar{1}, 1'). \end{aligned} \quad (3.17)$$

Here $\hat{a}(1, 1')$ and $\hat{\Gamma}(1, 1')$ are defined by the matrix elements

$$\begin{aligned} a_{\alpha\beta}(1, 1') &\equiv \tilde{g}_{\alpha\beta}^>(1, 1') - \tilde{g}_{\alpha\beta}^<(1, 1') \\ \Gamma_{\alpha\beta}(1, 1') &\equiv \Sigma_{\alpha\beta}^>(1, 1') - \Sigma_{\alpha\beta}^<(1, 1'). \end{aligned} \quad (3.18)$$

The *non-equilibrium* single-particle spectral density $a_{\alpha\beta}(1, 1')$ plays a crucial role in this thesis. The single-particle self-energy which is involved in (3.16) and (3.17) has been, as usual, split into two parts [15, 11] (recall that these functions are now 2×2 matrices)

$$\hat{\Sigma}(1, 1') = \hat{\Sigma}^{HF}(1, 1') + \hat{\Sigma}_c(1, 1'). \quad (3.19)$$

We have thus obtained the non-equilibrium form of the Dyson-Beliaev equations of motion for the non-condensate atoms. These involve the non-equilibrium condensate wavefunction, which is governed by its own equation of motion. In thermal equilibrium, these equations reduce to the usual Beliaev equations discussed in most many-body theory books [10, 30]. These will be the starting point for our derivation of a Boltzmann-like equation, generalizing the kind of analysis given in Ch. 2. However, we first must discuss what kind of self-energies $\Sigma_{\alpha\beta}$ we will work with.

3.2 Choice of self-energy approximations

In this section, we review two types of approximation that one uses to derive the self-energy functions. In the so-called “ Φ -derivable” (or conserving) approximation [11, 24, 34, 35, 36], one assumes that one can find a functional Φ such that one can generate the self-energy function Σ and the condensate source function η by functional differentiation of it with the respect to single-particle propagators \tilde{G} and $G_{1/2}$. If such a functional exists, one can show that the quantities calculated in this approximation will satisfy all conservation laws. The problem that one might arrive at using a conserving approximation is that the excitation spectrum may have a gap in the long wavelength limit. In contrast, in a “gapless approximation”, one is guaranteed to have phonons in

the long-wavelength (low-momentum) limit, but the approximation may not be conserving. In this section, we use the imaginary-time Green's function to derive the self-energy functions.

3.2.1 Conserving or “ Φ -derivable” approximation

In this section, for clarity, we will use a general interatomic potential v , rather than the s -wave approximation given by (1.1). To derive the self-energy and the source function in the so-called “ Φ -derivable” approximation, we use the following expressions [11, 24, 34, 35, 36]

$$\begin{aligned} \left. \frac{\delta}{\delta \tilde{G}(11')} \Phi [\tilde{G}, G_{1/2}] \right|_{G_{1/2}} &\equiv \Sigma(11') \\ \frac{1}{2\sqrt{-i}} \left. \frac{\delta}{\delta G_{1/2}(1)} \Phi [\tilde{G}, G_{1/2}] \right|_{\tilde{G}} &\equiv \eta(1) \end{aligned} \quad (3.20)$$

where Φ is a functional of the Green's functions \tilde{G} and $G_{1/2}$. If such a functional Φ exists, the density response generated will satisfy conservation laws, but in general the long-wavelength single-particle excitation spectrum will have a gap (no phonons for low k).

In the “ Φ -derivable” approximation, \tilde{G} and $G_{1/2}$ are independent functions. The relation between $G_{1/2}$ and $G_{1/2}^\dagger$ is given by the following identities (subscript represents spinor components)

$$\begin{aligned} G_{1/2,\alpha}^\dagger(1) &= \tau_{\alpha\beta}^1 G_{1/2,\beta}(1) \\ \tau_{\alpha\beta}^1 G_{1/2,\beta}^\dagger(1) &= G_{1/2,\alpha}(1). \end{aligned} \quad (3.21)$$

Therefore

$$\frac{\delta G_{1/2,\alpha}(1)}{\delta G_{1/2,\sigma}^\dagger(1)} = \tau_{\alpha\beta}^1 \frac{\delta G_{1/2,\beta}^\dagger(1)}{\delta G_{1/2,\sigma}^\dagger(1)} = \delta_{\beta,\sigma} \tau_{\alpha\beta}^1 = \tau_{\alpha\sigma}^1. \quad (3.22)$$

In the Hartree-Fock-Bogoliubov (HFB) approximation, the HFB functional Φ is given by the following expression [11]

$$\Phi = \frac{i}{4} \tilde{G}(11)v(12)\tilde{G}(22) + \frac{i}{2} G_{1/2}(1)G_{1/2}^\dagger(1)v(12)\tilde{G}(22)$$

$$\begin{aligned}
& + \frac{i}{4} G_{1/2}(1) G_{1/2}^\dagger(1) v(12) G_{1/2}(2) G_{1/2}^\dagger(2) + \frac{i}{2} \tilde{G}(12) v(12) \tilde{G}(21) \\
& + i G_{1/2}(1) G_{1/2}^\dagger(2) v(12) \tilde{G}(21),
\end{aligned} \tag{3.23}$$

where the repeated arguments are integrated. From now on, the matrix nature of the various functions is left implicit. Using (3.20), one obtains for η and Σ in the HFB approximation generated by (3.23), the following expressions

$$\begin{aligned}
\Sigma^{HF}(11') & = \frac{i}{2} v(12) [G_{1/2}(2) G_{1/2}^\dagger(2) + \tilde{G}(22)] \delta(11') \\
& + i v(11') [G_{1/2}(1) G_{1/2}^\dagger(1') + \tilde{G}(11')]
\end{aligned} \tag{3.24}$$

and

$$\begin{aligned}
\sqrt{-i}\eta(1) & = \frac{i}{2} v(12) [G_{1/2}(1) G_{1/2}(2) G_{1/2}^\dagger(2) G_{1/2}(1) + \tilde{G}(22)] \\
& + i v(12) G_{1/2}(2) \tilde{G}(12).
\end{aligned} \tag{3.25}$$

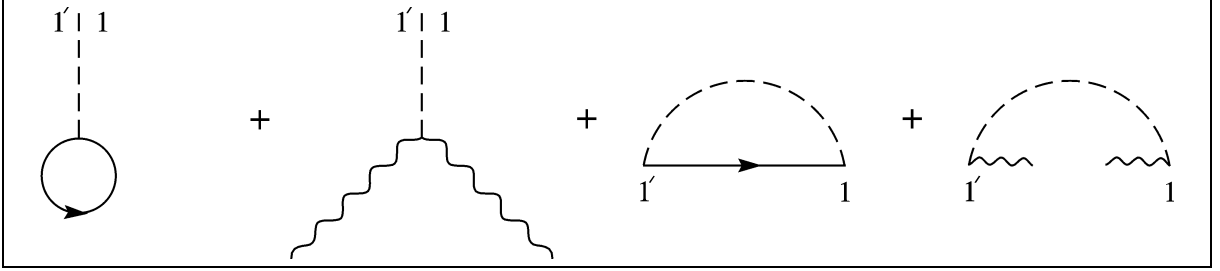


Figure 3.1: Self-energy diagrams Σ^{HF} in the first-order Hartree-Fock-Bogoliubov approximation.

Feynman diagrams representing the first order HFB approximation are shown in Fig. 3.1. The solid line represents the non-condensate propagator \tilde{G} , the wiggly line represents the condensate propagator h and the dashed line represents the two-body interaction v . We can rewrite (3.25) in terms of the condensate self-energy S , giving

$$S^{HF}(11') = \frac{i}{2} v(12) [G_{1/2}(2) G_{1/2}^\dagger(2) + \tilde{G}(22)] \delta(11') + i v(11') \tilde{G}(11'). \tag{3.26}$$

The diagrams representing the first order condensate self-energies S^{HF} are shown in Fig. 3.2.

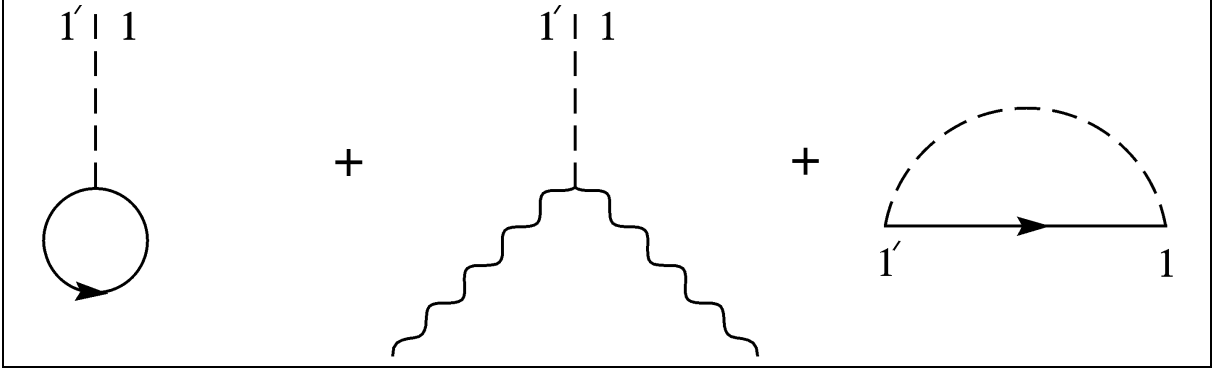


Figure 3.2: Condensate self-energy S^{HF} diagrams in the first-order Hartree-Fock-Bogoliubov approximation.

To obtain a functional up to second-order terms, we generalize the first order expression in (3.23)

$$\begin{aligned}
\Phi &= \frac{i}{4}\tilde{G}(11)v(12)\tilde{G}(22) + \frac{i}{2}G_{1/2}(1)G_{1/2}^\dagger(1)v(12)\tilde{G}(22) \\
&+ \frac{i}{4}G_{1/2}(1)G_{1/2}^\dagger(1)v(12)G_{1/2}(2)G_{1/2}^\dagger(2) + \frac{i}{2}\tilde{G}(12)v(12)\tilde{G}(21) \\
&+ iG_{1/2}(1)G_{1/2}^\dagger(2)v(12)\tilde{G}(21) - \frac{1}{4}\tilde{G}(21)\tilde{G}(12)v(24)v(31)\tilde{G}(43)\tilde{G}(34) \\
&- \frac{1}{4}\tilde{G}(21)\tilde{G}(12)v(24)v(31)\tilde{G}(43)G_{1/2}(3)G_{1/2}^\dagger(4). \tag{3.27}
\end{aligned}$$

Using (3.20), this gives Σ and η up to second order in v , with

$$\begin{aligned}
\Sigma_c(11') &= -\frac{1}{2}v(31)v(21')\tilde{G}(11') \left[\tilde{G}(23)\tilde{G}(32) + \tilde{G}(23)h(32) + h(23)\tilde{G}(32) \right] \\
&- \frac{1}{2}v(31)v(21')h(11')\tilde{G}(23)\tilde{G}(32), \tag{3.28}
\end{aligned}$$

and

$$\sqrt{-i\eta_c}(1) = -\frac{1}{2}v(31)v(24)\tilde{G}(32)\tilde{G}(23)\tilde{G}(14)G_{1/2}(4). \tag{3.29}$$

In terms of the condensate self-energy function S , this corresponds to

$$S_c(1,1') = -\frac{1}{2}v(13)v(21')\tilde{G}(11')\tilde{G}(23)\tilde{G}(32). \tag{3.30}$$

The first-order contribution to the self-energy and the source function (the Hartree-Fock part) is the same as in the HFB approximation given in (3.24) and (3.25).

Kane and Kadanoff used this conserving approximation in their work, and the self-energies given in (3.24), (3.26), (3.28) and (3.30). However, as they emphasized, this approximation does not satisfy the Hugenholtz-Pines theorem, and the single-particle Green's function \tilde{G} does not give a phonon-like spectrum at long wavelengths.

3.2.2 Gapless approximation

The self-energy in the Beliaev (gapless) approximation is *defined* by [11]

$$\sqrt{-i} \frac{\delta \hat{\eta}(1)}{\delta \hat{G}_{1/2}(1')} \equiv \hat{\Sigma}(11'). \quad (3.31)$$

Thus, to derive the second-order Beliaev approximation for the self-energy, we need first find the second-order expression for the source function $\hat{\eta}$. We express the three-point function in (3.9) in terms of functional derivatives in the following way [11, 34]

$$\sqrt{-i} \hat{\eta}(1) = iv(12) \left[\frac{1}{2} \left[\tilde{G}(22) + \hat{G}_{1/2}(2) \hat{G}_{1/2}^\dagger(2) \right] \hat{G}_{1/2}(1) + i \frac{\delta \hat{G}_{1/2}(1)}{\delta U(22)} \right]. \quad (3.32)$$

$\hat{G}(1, 1')$ is the imaginary-time non-condensate propagator, from which one obtains the real-time propagator $\hat{g}(1, 1')$ used in the KB formalism. Equation (3.32) can also be written as:

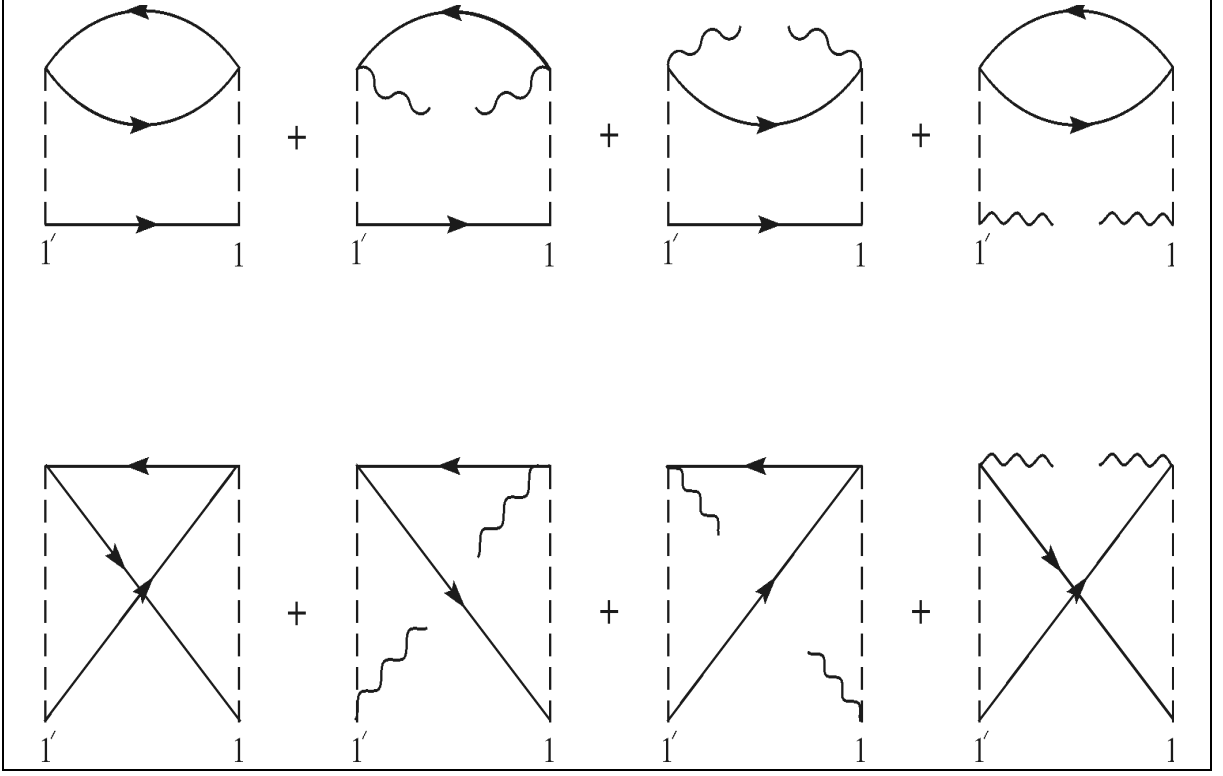
$$\begin{aligned} \sqrt{-i} \eta(1) &= iv(12) \left[\frac{1}{2} \left[\tilde{G}(22) + G_{1/2}(2) G_{1/2}^\dagger(2) \right] G_{1/2}(1) + \tilde{G}(12) G_{1/2}(2) \right] \\ &+ \frac{i}{2} v(12) \tilde{G}(14) \tilde{G}(23) \frac{\delta \Sigma(35)}{\delta G_{1/2}^\dagger(4)} \tilde{G}(52). \end{aligned} \quad (3.33)$$

We approximate Σ in (3.33) by the full Hartree-Fock-Bogoliubov (HFB) first-order self-energy (denoted as HF for simplicity)

$$\begin{aligned} \Sigma^{HF}(11') &= \frac{i}{2} v(12) \left[G_{1/2}(2) G_{1/2}^\dagger(2) + \tilde{G}(22) \right] \delta(11') \\ &+ iv(11') \left[G_{1/2}(1) G_{1/2}^\dagger(1') + \tilde{G}(11') \right]. \end{aligned} \quad (3.34)$$

Using (3.34) in (3.31), we obtain the desired second-order expression for the source function

$$\sqrt{-i} \eta_c(1) = \frac{i}{2} v(12) \left[G_{1/2}(1) G_{1/2}(2) G_{1/2}^\dagger(2) + G_{1/2}(1) \tilde{G}(22) \right]$$

Figure 3.3: Second-order self-energy Σ_c diagrams in the Beliaev approximation.

$$\begin{aligned}
& + iv(12)G_{1/2}(2)\tilde{G}(12) - \frac{1}{2}v(13)v(24)\tilde{G}(32)\tilde{G}(23)\tilde{G}(14)G_{1/2}(4) \\
& - v(13)v(24)\tilde{G}(14)\tilde{G}(43)\tilde{G}(32)G_{1/2}(2), \tag{3.35}
\end{aligned}$$

and for the second-order contribution for the condensate self-energy S_c

$$\begin{aligned}
S_c(1, 1') = & - \frac{1}{2}v(13)v(21')\tilde{G}(11') [\tilde{G}(23)\tilde{G}(32)] \\
& - v(13)v(21')\tilde{G}(12) [\tilde{G}(23)\tilde{G}(31')]. \tag{3.36}
\end{aligned}$$

Finally, using (3.35) in (3.31), we obtain the Beliaev self-energy to second order in v . In this gapless approximation, we note that \tilde{G} and $G_{1/2}$ are not independent:

$$\frac{\delta\tilde{G}}{\delta G_{1/2}} = \tilde{G} \frac{\delta\Sigma}{\delta G_{1/2}} \tilde{G}. \tag{3.37}$$

Using (3.31), the second order collisional part of the Beliaev self-energy is given by

$$\hat{\Sigma}_c(1, 1') = - \frac{1}{2}v(13)v(21')\tilde{G}(11') [\tilde{G}(23)\tilde{G}(32) + \tilde{G}(23)h(32) + h(23)\tilde{G}(32)]$$

$$\begin{aligned}
& - v(13)v(21')\tilde{G}(12) \left[\tilde{G}(23)h(31') + h(23)\tilde{G}(31') + \tilde{G}(23)\tilde{G}(31') \right] \\
& - \frac{1}{2}v(13)v(21') \left[h(11')\tilde{G}(23)\tilde{G}(32) + 2h(12)\tilde{G}(23)\tilde{G}(31') \right]. \quad (3.38)
\end{aligned}$$

The diagrams representing the second-order Beliaev self-energies Σ_c are shown in Fig. 3.3. The first four diagrams in Fig. 3.3 correspond to the second-order contributions in the conserving approximation discussed in the previous section [15].

We note that the expression for the self-energy in (3.38) goes past what Beliaev included in 1957. Above T_{BEC} , the original Beliaev approximation reduces to the Hartree-Fock approximation, i.e., it does not include the normal second-order self-energy contributions $\tilde{G}\tilde{G}\tilde{G}$ associated with collisions. In this thesis, we include such terms and therefore, above T_{BEC} , the kinetic equations we derive in this thesis reduce to the usual Boltzmann equation.

In this section, we have shown how one can derive the self-energy functions in the both conserving and gapless approximations. We will use these results in the rest of the thesis when we derive Boltzmann-like equations and a generalized Gross-Pitaevskii equation for the order parameter within a given self-energy approximation.

Chapter 4

Coupled Hartree-Fock-Bogoliubov kinetic equations

4.1 Introduction

In this chapter, we use the KK approach [15, 16] to deal with a trapped Bose-condensed gas. When we limit ourselves to Hartree-Fock (HF) self-energies (see Section 3.2), the kinetic equations we obtain are in the collisionless approximation [14]. Because we include both normal and anomalous pair correlations, our self-energies give the so-called Hartree-Fock-Bogoliubov (HFB) approximation, as reviewed by Griffin [24]. While the HFB approximation in a uniform gas is known to lead to a single-particle spectrum with an energy gap in the long wavelength limit, it can be shown to be a “conserving” approximation that generates two-particle response functions which satisfy all conservation laws [11].

Our main formal result is given in Section 4.3, namely two coupled kinetic equations for the diagonal (f_1) and off-diagonal (f_2) distribution functions for the excited atoms, in addition to an equation of motion for the condensate order parameter. If f_2 can be ignored, we obtain the semi-classical collisionless kinetic equation for f_1 which has been

used in recent discussions of the non-equilibrium properties of the non-condensed atoms [37, 38]. In order to gain some insight into our HFB kinetic equations, in Section 4.4 we discuss the static local equilibrium solutions within the semi-classical approximation and verify that these satisfy our equations. In Section 4.5, we show that our equations have a solution which corresponds to a rigid oscillation of the equilibrium density profiles (normal and anomalous), with a frequency equal to the parabolic trap frequency. This generalized Kohn mode is generic in trapped Bose gases, but how it arises as a solution of our HFB kinetic equations gives one insight into their structure.

4.2 Derivation of HFB equations

In Ch. 3, we derived the non-equilibrium Dyson-Beliaev equations of motion for the non-condensate atoms given by (3.16) and (3.17), and the equation of motion for the condensate in (3.8). In this chapter, we derive the collisionless Boltzmann equation, keeping only the first-order HFB self-energies on the left-hand side of (3.16) and (3.17). Therefore, the HFB equations of motion can be conveniently written in the following 2×2 matrix form

$$\int d\bar{1} \left[\hat{g}_0^{-1}(1, \bar{1}) - \hat{\Sigma}^{HF}(1, \bar{1}) \right] \hat{g}^{\lessgtr}(\bar{1}, 1') = 0, \quad (4.1)$$

$$\int d\bar{1} \hat{g}^{\lessgtr}(1, \bar{1}) \left[\hat{g}_0^{-1}(\bar{1}, 1') - \hat{\Sigma}^{HF}(\bar{1}, 1') \right] = 0, \quad (4.2)$$

$$\int d\bar{1} \left[\hat{g}_0^{-1}(1, \bar{1}) - \hat{S}^{HF}(1, \bar{1}) \right] \hat{h}(\bar{1}, 1') = -i\hat{\eta}^{ext}(1)\langle \hat{\Psi}^\dagger(1') \rangle. \quad (4.3)$$

In the HFB approximation, the 2×2 self-energies in (4.1)-(4.3) are derived in Ch. 3 and for the contact interaction in (1.1), (3.24) and (3.26) reduce to

$$\begin{aligned} \hat{\Sigma}^{HF}(1, 1') &= g \begin{pmatrix} 2n(1), & m(1) \\ m^*(1), & 2n(1) \end{pmatrix} \delta(1, 1'), \\ \hat{S}^{HF}(1, 1') &= g \begin{pmatrix} 2\tilde{n}(1) + n_c(1), & \tilde{m}(1) \\ \tilde{m}^*(1), & 2\tilde{n}(1) + n_c(1) \end{pmatrix} \delta(1, 1'). \end{aligned} \quad (4.4)$$

In the above equations, $n(1)$ is the *total* local density given by

$$n(1) = i \left[\tilde{g}_{11}^{\leq}(1, 1^+) + h_{11}(1, 1) \right] = \langle \tilde{\psi}^\dagger(1) \tilde{\psi}(1) \rangle + |\Phi(1)|^2 \equiv \tilde{n}(1) + n_c(1), \quad (4.5)$$

where $\tilde{n}(1)$ and $n_c(1)$ are non-condensate and condensate density, respectively. Similarly, $m(1)$ is the “anomalous” local density defined by

$$\begin{aligned} m(1) &= i \left[\tilde{g}_{12}^{\leq}(1, 1) + h_{12}(1, 1) \right] = \langle \tilde{\psi}(1) \tilde{\psi}(1) \rangle + [\Phi(1)]^2 = \tilde{m}(1) + [\Phi(1)]^2 \\ m^*(1) &= i \left[\tilde{g}_{21}^{\leq}(1, 1) + h_{21}(1, 1) \right] = \langle \tilde{\psi}^\dagger(1) \tilde{\psi}^\dagger(1) \rangle + [\Phi^*(1)]^2 = \tilde{m}^*(1) + [\Phi^*(1)]^2. \end{aligned} \quad (4.6)$$

These HFB results are consistent with the equations of Kane and Kadanoff [15], although they left the “particle-source” fields η and η^* implicit. The associated equation of motion for the order parameter $\Phi(1)$ in this HFB approximation is given by

$$\left[i \frac{\partial}{\partial t_1} + \frac{\nabla_1^2}{2m} + \mu_0 - U_{ext}(\mathbf{r}_1) - U(1) - g[2\tilde{n}(1) + n_c(1)] \right] \Phi(1) = g\tilde{m}(1)\Phi^*(1) + \eta_{ext}(1), \quad (4.7)$$

and its complex conjugate. Equations (4.1)-(4.7) are a closed set of equations and define what is called the dynamic HFB approximation.

We now turn to solving these HFB equations of motion for atoms in the presence of a trapping potential. As we discussed in Ch. 2, if the external generating fields induce a disturbance with a wavelength much longer than thermal wavelengths and frequencies much smaller than characteristic particle energies, then the propagator $g(1, 1') = g(\mathbf{r}, t; \mathbf{R}, T)$ can be expected to vary slowly as a function of the center-of-mass coordinates

$$\mathbf{R} = \frac{1}{2}(\mathbf{r}_1 + \mathbf{r}_{1'}), \quad T = \frac{1}{2}(t_1 + t_{1'}), \quad (4.8)$$

and to be dominated by small values of the relative coordinates

$$\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_{1'}, \quad t = t_1 - t_{1'}. \quad (4.9)$$

More precisely, Fourier transforming with respect to \mathbf{r} and t , $g(\mathbf{p}, \omega; \mathbf{R}, T)$ describes the density of elementary excitations of momenta \mathbf{p} and energy ω at point (\mathbf{R}, T) [14]. These

quasiparticles are assumed to have high momentum and energy (relative to the collective modes we could obtain from the kinetic equations), which means that $g(\mathbf{r}, t; \mathbf{R}, T)$ is mainly weighted at small values of the relative coordinates \mathbf{r} and t .

If the Bose-condensate order parameter $\langle \psi(1) \rangle$ is written in terms of amplitude and phase variables

$$\langle \psi(1) \rangle = [n_c(1)]^{\frac{1}{2}} e^{i\theta(1)}, \quad (4.10)$$

we can generalize the usual *definitions* for *superfluid velocity* and *local chemical potential* to non-equilibrium systems by identifying [15]

$$\begin{aligned} \nabla_1 \theta(1) &\equiv m \mathbf{v}_s(1) \\ \frac{\partial \theta(1)}{\partial t_1} &\equiv - \left[\mu(1) - \mu_0 + \frac{1}{2} m v_s^2(1) \right]. \end{aligned} \quad (4.11)$$

The superfluid velocity \mathbf{v}_s enters as the gradient of the phase of the condensate wavefunction and the local chemical potential $\mu(1)$ is connected with the time-derivative of the phase.

However, a problem arises in that the phase in (4.10) is a rapidly varying function of (\mathbf{R}, T) . This induces strong variations in the off-diagonal elements of \hat{h} in (3.6), and these are coupled to the components of the non-condensate propagator \tilde{g} . To remove this strong (\mathbf{R}, T) -dependence associated with the phase θ in (4.10), we apply the well-known [15] gauge transformations on $\hat{h}(1, 1')$ and $\hat{g}^{\lessgtr}(1, 1')$:

$$\begin{aligned} \hat{h}'(1, 1') &= e^{-i\theta(1)\tau^{(3)}} \hat{h}(1, 1') e^{i\theta(1')\tau^{(3)}} \\ \hat{g}'^{\lessgtr}(1, 1') &= e^{-i\theta(1)\tau^{(3)}} \hat{g}^{\lessgtr}(1, 1') e^{i\theta(1')\tau^{(3)}}, \end{aligned} \quad (4.12)$$

where $\tau^{(3)}$ is the Pauli spin matrix. The physical interpretation of (4.12) is that it involves a transformation to a coordinate system in which non-condensate atoms are moving with average velocity \mathbf{v}_s with respect to a stationary condensate. The gauge transformation (4.12) removes the strong (\mathbf{R}, T) -dependence associated with the order parameter phase

θ and leaves the equations of motion (4.1)-(4.3) invariant if we replace g_0^{-1} in (3.10) by

$$g_0'^{-1}(1, 1') = \left[i\tau^{(3)} \frac{\partial}{\partial t_1} - \frac{\partial\theta(1)}{\partial t_1} + \frac{1}{2}[\nabla_1 + i\nabla_1\theta(1)\tau^{(3)}]^2 - U_{ext}(\mathbf{r}_1) - U(1) + \mu_0 \right] \delta(1, 1'). \quad (4.13)$$

The gauge transformation changes the momentum $\mathbf{p} \rightarrow \mathbf{p} - m\mathbf{v}_s$, as expected for the momentum in the local rest frame. After carrying out this gauge transformation, the HFB equations for $\tilde{g}_{11}^{\geq}(1, 1'; U)$ and $\tilde{g}_{12}^{\geq}(1, 1'; U)$ given by (4.1)-(4.2) become

$$\begin{aligned} & \left[i\frac{\partial}{\partial t_1} - \frac{\partial\theta(1)}{\partial t_1} + \frac{1}{2m}[\nabla_1 + i\nabla_1\theta(1)]^2 + \mu_0 - U_{eff}(1) \right] \tilde{g}'_{11}^{\geq}(1, 1'; U) \\ & = gm'(1)\tilde{g}'_{21}^{\geq}(1, 1'; U) \\ & \left[i\frac{\partial}{\partial t_1} - \frac{\partial\theta(1)}{\partial t_1} + \frac{1}{2m}[\nabla_1 + i\nabla_1\theta(1)]^2 + \mu_0 - U_{eff}(1) \right] \tilde{g}'_{12}^{\geq}(1, 1'; U) \\ & = gm'(1)\tilde{g}'_{22}^{\geq}(1, 1'; U) \\ & \left[-i\frac{\partial}{\partial t_{1'}} - \frac{\partial\theta(1')}{\partial t_{1'}} + \frac{1}{2m}[\nabla_{1'} - i\nabla_{1'}\theta(1')]^2 + \mu_0 - U_{eff}(1') \right] \tilde{g}'_{11}^{\leq}(1, 1'; U) \\ & = gm'^*(1')\tilde{g}'_{12}^{\leq}(1, 1'; U) \\ & \left[i\frac{\partial}{\partial t_{1'}} - \frac{\partial\theta(1')}{\partial t_{1'}} + \frac{1}{2m}[\nabla_{1'} + i\nabla_{1'}\theta(1')]^2 + \mu_0 - U_{eff}(1') \right] \tilde{g}'_{12}^{\leq}(1, 1'; U) \\ & = gm'(1')\tilde{g}'_{11}^{\leq}(1, 1'; U). \end{aligned} \quad (4.14)$$

Here

$$U_{eff}(1) \equiv U_{ext}(\mathbf{r}_1) + U(1) + 2gn'(1) \quad (4.15)$$

is the effective self-consistent Hartree-Fock dynamic mean field. The condensate part $2gn_c(1)$ in (4.15) can be viewed as an additional ‘‘external field’’ acting on the non-condensate. Since we will work with these gauge-transformed correlation functions in the rest of the thesis, we shall drop the primes on $\tilde{g}_{\alpha\beta}$, n and m to simplify the notation.

The corresponding equation of motion for the condensate amplitude $\sqrt{n_c(1)}$ in the moving frame of reference can be written in the form:

$$\left[i\frac{\partial}{\partial t_1} - \frac{\partial\theta(1)}{\partial t_1} + \frac{\nabla_1^2}{2m} - \frac{1}{2}m\mathbf{v}_s^2 + \mu_0 - U_{ext}(\mathbf{r}_1) - U(1) - g[2\tilde{n}(1) + n_c(1)] + \right.$$

$$+i\mathbf{v}_s(1) \cdot \nabla_1 + \frac{i}{2}\nabla_1 \cdot \mathbf{v}_s(1) \Big] \sqrt{n_c(1)} = g\tilde{m}(1)\sqrt{n_c(1)} + \eta'(1). \quad (4.16)$$

Here $\eta'(1) \equiv \eta(1)e^{-i\theta(1)}$ is the symmetry-breaking source function in the moving frame of reference. We note that condensate and non-condensate mean-fields enter (4.14) and (4.16) with different weights (1 or 2). This is because atoms in the condensate are in the same quantum state and thus there is no exchange term. In the case of non-condensate atoms, both Hartree and Fock terms arise since we are dealing with atoms in different quantum states.

4.3 HFB kinetic equations

To rewrite the HFB equations of motion derived in Section 4.2 in the more useful form of kinetic equations, we recall the connection between the usual single-particle distribution function $f_1(\mathbf{p}, \mathbf{R}, T)$ and the diagonal Green's function $\tilde{g}_{\alpha\alpha}^<(1, 1')$. We define (see p. 67 of Ref. [14]):

$$\begin{aligned} f_1(\mathbf{p}, \mathbf{R}, T) &\equiv \int d\mathbf{r} e^{-i\mathbf{p}\cdot\mathbf{r}} [i\tilde{g}_{11}^<(\mathbf{r}, t=0; \mathbf{R}, T)] \\ &= \int d\mathbf{r} e^{-i\mathbf{p}\cdot\mathbf{r}} \langle \tilde{\psi}_U^\dagger(\mathbf{R} - \frac{\mathbf{r}}{2}, T) \tilde{\psi}_U(\mathbf{R} + \frac{\mathbf{r}}{2}, T) \rangle \end{aligned} \quad (4.17)$$

where, by definition, the non-condensate density is given by

$$\tilde{n}(\mathbf{R}, T) = \int \frac{d\mathbf{p}}{(2\pi)^3} f_1(\mathbf{p}, \mathbf{R}, T). \quad (4.18)$$

We can see that $f_1(\mathbf{p}, \mathbf{R}, T)$ corresponds to the well-known Wigner distribution function. In the classical limit, it reduces to the distribution function giving the number of atoms with momentum \mathbf{p} at point \mathbf{R} and time T . The symmetry-breaking terms in (3.7) give a finite value to anomalous Green's functions, and thus it is natural to introduce an additional distribution function for the non-condensate atoms which will give us the anomalous non-condensate density $\tilde{m}(1)$, namely

$$\begin{aligned} f_2(\mathbf{p}, \mathbf{R}, T) &\equiv \int d\mathbf{r} e^{-i\mathbf{p}\cdot\mathbf{r}} [i\tilde{g}_{12}^<(\mathbf{r}, t=0; \mathbf{R}, T)] \\ &= \int d\mathbf{r} e^{-i\mathbf{p}\cdot\mathbf{r}} \langle \tilde{\psi}_U(\mathbf{R} - \frac{\mathbf{r}}{2}, T) \tilde{\psi}_U(\mathbf{R} + \frac{\mathbf{r}}{2}, T) \rangle. \end{aligned} \quad (4.19)$$

Here, as we defined in Ch. 2, the subscript U denotes an operator evolving under the external scalar potential U and the time evolution is given by (2.29). One can easily verify that the pair correlation function \tilde{m} is given by

$$\tilde{m}(\mathbf{R}, T) = \int \frac{d\mathbf{p}}{(2\pi)^3} f_2(\mathbf{p}, \mathbf{R}, T). \quad (4.20)$$

It is important to remember that, as defined, the distribution functions f_1 and f_2 describe the behaviour of the atoms. They should not be confused with the distribution function for the quasiparticle excitations, such as discussed in Ref. [28].

To obtain kinetic equations, we rewrite the equations for $\tilde{g}_{11}^<(1, 1')$ and $\tilde{g}_{12}^<(1, 1')$ in the relative and center-of-mass coordinates $(\mathbf{r}, t; \mathbf{R}, T)$. We could obtain equations for $g(\mathbf{r}, t; \mathbf{R}, T)$ as was done in Ref.[15], but for the simple HFB self-energies of interest (which are frequency independent), it is sufficient to consider $t = 0$, *i.e.*, set $t_{1'} = t_1^+ = T$ (see Ch. 7 of [14]). Using (4.14), this procedure gives

$$\begin{aligned} & \left[i \frac{\partial}{\partial T} + \Delta U_{eff}^\mu(\mathbf{r}, \mathbf{R}, T) + \frac{1}{m} \nabla_{\mathbf{r}} \cdot \nabla_{\mathbf{R}} + i \mathbf{v}_s^-(\mathbf{r}, \mathbf{R}, T) \cdot \nabla_{\mathbf{r}} + \frac{i}{2} \mathbf{v}_s^+(\mathbf{r}, \mathbf{R}, T) \cdot \nabla_{\mathbf{R}} \right. \\ & \left. + \frac{i}{2} \nabla_{\mathbf{r}} \cdot \mathbf{v}_s^-(\mathbf{r}, \mathbf{R}, T) + \frac{i}{4} \nabla_{\mathbf{R}} \cdot \mathbf{v}_s^+(\mathbf{r}, \mathbf{R}, T) \right] \tilde{g}_{11}^<(\mathbf{r}, \mathbf{R}, T) \\ & = g \left[m(\mathbf{R} + \frac{\mathbf{r}}{2}, T) \tilde{g}_{21}^<(\mathbf{r}, \mathbf{R}, T) - m^*(\mathbf{R} - \frac{\mathbf{r}}{2}, T) \tilde{g}_{12}^<(\mathbf{r}, \mathbf{R}, T) \right], \end{aligned} \quad (4.21)$$

where we have introduced the abbreviations

$$\Delta U_{eff}^\mu(1, 1') \equiv \mu(1) - \mu(1') - U_{eff}(1) + U_{eff}(1'), \quad (4.22)$$

and

$$\mathbf{v}_s^\pm(\mathbf{r}, \mathbf{R}, T) \equiv \mathbf{v}_s(\mathbf{R} + \frac{\mathbf{r}}{2}, T) \pm \mathbf{v}_s(\mathbf{R} - \frac{\mathbf{r}}{2}, T). \quad (4.23)$$

Similarly, the equation of motion for $\tilde{g}_{12}^<(1, 1')$ expressed in the $(\mathbf{r}, \mathbf{R}; t, T)$ variables gives

$$\begin{aligned} & \left[i \frac{\partial}{\partial T} + \mu(\mathbf{R} + \frac{\mathbf{r}}{2}, T) + \mu(\mathbf{R} - \frac{\mathbf{r}}{2}, T) - U_{eff}(\mathbf{R} + \frac{\mathbf{r}}{2}, T) - U_{eff}(\mathbf{R} - \frac{\mathbf{r}}{2}, T) + \frac{\nabla_{\mathbf{R}}^2}{4m} \right. \\ & \left. + \frac{1}{m} \nabla_{\mathbf{r}}^2 + i \mathbf{v}_s^-(\mathbf{r}, \mathbf{R}, T) \cdot \nabla_{\mathbf{r}} + \frac{i}{2} \mathbf{v}_s^+(\mathbf{r}, \mathbf{R}, T) \cdot \nabla_{\mathbf{R}} + \frac{i}{2} \nabla_{\mathbf{r}} \cdot \mathbf{v}_s^-(\mathbf{r}, \mathbf{R}, T) + \right. \\ & \left. + \frac{i}{4} \nabla_{\mathbf{R}} \cdot \mathbf{v}_s^+(\mathbf{r}, \mathbf{R}, T) \right] \tilde{g}_{12}^<(\mathbf{r}, \mathbf{R}, T) \end{aligned}$$

$$= g \left[m(\mathbf{R} + \frac{\mathbf{r}}{2}, T) \tilde{g}_{22}^{\leq}(\mathbf{r}, \mathbf{R}, T) + m(\mathbf{R} - \frac{\mathbf{r}}{2}, T) \tilde{g}_{11}^{\leq}(\mathbf{r}, \mathbf{R}, T) \right]. \quad (4.24)$$

Finally, the equation of motion (4.16) for the *amplitude* of the order parameter can be written in the (\mathbf{R}, T) coordinates as

$$\begin{aligned} & \left[i \frac{\partial}{\partial T} - \frac{\partial \theta(\mathbf{R}, T)}{\partial T} + \frac{\nabla_{\mathbf{R}}^2}{2m} - \frac{1}{2} m \mathbf{v}_s^2(\mathbf{R}, T) + \mu_0 - U_{ext}(\mathbf{R}) - U(\mathbf{R}, T) \right. \\ & \left. - g [2\tilde{n}(\mathbf{R}, T) + n_c(\mathbf{R}, T)] + i \mathbf{v}_s(\mathbf{R}, T) \cdot \nabla_{\mathbf{R}} + \frac{i}{2} \nabla_{\mathbf{R}} \cdot \mathbf{v}_s(\mathbf{R}, T) \right] \sqrt{n_c(\mathbf{R}, T)} \\ & = g \tilde{m}(\mathbf{R}, T) \sqrt{n_c(\mathbf{R}, T)} + \eta'(\mathbf{R}, T). \end{aligned} \quad (4.25)$$

The coupled set of HFB equations given by (4.21), (4.24) and (4.25) are the main formal results of this chapter. These are a straightforward generalization of the analogous equations in the Hartree-Fock approximation in normal Bose gases discussed by Kadanoff and Baym (see Eq. (7.7) of Ref.[14]). These results are important since they allow us to go beyond the simple Hartree-Fock-Popov (HFP) approximation which has been the basis of recent work on the non-equilibrium properties of a trapped Bose-condensed gas (see, for example, Refs. [37, 38]). In the HFP approximation, we recall that the anomalous density \tilde{m} is neglected [2, 24].

We next proceed to use (4.21) and (4.24) to derive self-consistent equations for the atomic distribution functions $f_1(\mathbf{p}, \mathbf{R}, T)$ and $f_2(\mathbf{p}, \mathbf{R}, T)$ defined in (4.17) and (4.19) for the case when the external perturbation varies slowly in space and time. In this case, we expect that physical quantities \mathbf{v}_s , μ , U , U_{ext} , $n(1)$, and $m(1)$ all vary slowly as functions of the center-of-mass coordinates (\mathbf{R}, T) . Thus, in the lowest approximation, and using the fact that small values of \mathbf{r} are most important, we can limit ourselves to

$$\mathbf{v}_s(\mathbf{R} \pm \frac{\mathbf{r}}{2}, T) = \mathbf{v}_s(\mathbf{R}, T) \pm \left[\frac{\mathbf{r}}{2} \cdot \nabla_{\mathbf{R}} \right] \mathbf{v}_s(\mathbf{R}, T), \quad (4.26)$$

and hence

$$\begin{aligned} \mathbf{v}_s^+(\mathbf{r}, \mathbf{R}, T) &\simeq 2\mathbf{v}_s(\mathbf{R}, T), & \mathbf{v}_s^-(\mathbf{r}, \mathbf{R}, T) &\simeq [\mathbf{r} \cdot \nabla_{\mathbf{R}}] \mathbf{v}_s(\mathbf{R}, T), \\ \Delta U_{eff}^\mu(\mathbf{r}, \mathbf{R}, T) &\simeq -\mathbf{r} \cdot \nabla_{\mathbf{R}} [U_{eff}(\mathbf{R}, T) - \mu(\mathbf{R}, T)]. \end{aligned} \quad (4.27)$$

If we rewrite (4.21) using (4.17) and (4.19), and Fourier transform it, we obtain after some algebra

$$\begin{aligned}
 & \left[\frac{\partial}{\partial T} - \nabla_{\mathbf{R}}[\tilde{\epsilon}_p + \mathbf{v}_s \cdot \mathbf{p}] \cdot \nabla_{\mathbf{p}} + \nabla_{\mathbf{p}}[\tilde{\epsilon}_p + \mathbf{v}_s \cdot \mathbf{p}] \cdot \nabla_{\mathbf{R}} \right] f_1(\mathbf{p}, \mathbf{R}, T) \\
 &= -ig [m(\mathbf{R}, T)f_2(-\mathbf{p}, \mathbf{R}, T) - m^*(\mathbf{R}, T)f_2(\mathbf{p}, \mathbf{R}, T)] \\
 &+ \frac{g}{2} [\nabla_{\mathbf{R}}m(\mathbf{R}, T) \cdot \nabla_{\mathbf{p}}f_2(-\mathbf{p}, \mathbf{R}, T) + \nabla_{\mathbf{R}}m^*(\mathbf{R}, T) \cdot \nabla_{\mathbf{p}}f_2(\mathbf{p}, \mathbf{R}, T)]. \quad (4.28)
 \end{aligned}$$

Here the “normal” single-particle energy is *defined* by

$$\tilde{\epsilon}_p(\mathbf{R}, T) \equiv \frac{p^2}{2m} + U_{eff}(\mathbf{R}, T) - \mu(\mathbf{R}, T). \quad (4.29)$$

We emphasize that $\tilde{\epsilon}_p$ is not the local HFB excitation energy. The corresponding kinetic equation for $f_2(\mathbf{p}, \mathbf{R}, T)$ is

$$\begin{aligned}
 & \left[\frac{\partial}{\partial T} + i2\tilde{\epsilon}_p(\mathbf{R}, T) - \nabla_{\mathbf{R}}[\mathbf{v}_s \cdot \mathbf{p}] \cdot \nabla_{\mathbf{p}} + \nabla_{\mathbf{p}}[\mathbf{v}_s \cdot \mathbf{p}] \cdot \nabla_{\mathbf{R}} \right] f_2(\mathbf{p}, \mathbf{R}, T) = \\
 & -igm(\mathbf{R}, T) [f_1(\mathbf{p}, \mathbf{R}, T) + f_1(-\mathbf{p}, \mathbf{R}, T) + 1] \\
 & + \frac{g}{2} \nabla_{\mathbf{R}}m(\mathbf{R}, T) \cdot \nabla_{\mathbf{p}} [f_1(\mathbf{p}, \mathbf{R}, T) - f_1(-\mathbf{p}, \mathbf{R}, T)]. \quad (4.30)
 \end{aligned}$$

Eqs. (4.28) and (4.30) may be viewed as coupled HFB collisionless kinetic equations (in a frame moving with the velocity \mathbf{v}_s), for the case of slowly varying disturbances.

The equation of motion (4.25) for the amplitude of the order parameter is exact within our HFB approximation, i.e. it is not limited to a slowly varying disturbances. Equating the real and imaginary parts of (4.25), we obtain two hydrodynamic equations of motion for the condensate

$$\frac{\partial n_c(\mathbf{R}, T)}{\partial T} = -\nabla_{\mathbf{R}} [n_c(\mathbf{R}, T)\mathbf{v}_s(\mathbf{R}, T)] + 2gn_c(\mathbf{R}, T)\text{Im} [\tilde{m}(\mathbf{R}, T)], \quad (4.31)$$

$$\frac{\partial \theta(\mathbf{R}, T)}{\partial T} + \frac{1}{2}m\mathbf{v}_s^2(\mathbf{R}, T) - \mu_0 = -\mu(\mathbf{R}, T) - \eta'(\mathbf{R}, T) \frac{1}{\sqrt{n_c(\mathbf{R}, T)}}, \quad (4.32)$$

where the condensate chemical potential $\mu(\mathbf{R}, T)$ is *defined* by (compare with (4.11))

$$\begin{aligned}
 \mu(\mathbf{R}, T) &\equiv -\frac{\nabla_{\mathbf{R}}^2 \sqrt{n_c(\mathbf{r}, T)}}{2m\sqrt{n_c(\mathbf{R}, T)}} + U_{ext}(\mathbf{R}) + U(\mathbf{R}, T) \\
 &+ g[2\tilde{n}(\mathbf{R}, T) + n_c(\mathbf{R}, T)] + g\text{Re} [\tilde{m}(\mathbf{R}, T)]. \quad (4.33)
 \end{aligned}$$

Taking the gradient of (4.32), we obtain the generalized Landau equation [37]

$$m \left[\frac{\partial \mathbf{v}_s(\mathbf{R}, T)}{\partial T} + \frac{1}{2} \nabla_{\mathbf{R}} \mathbf{v}_s^2(\mathbf{R}, T) \right] = -\nabla_{\mathbf{R}} \left[\mu(\mathbf{R}, T) + \eta'(\mathbf{R}, T) \frac{1}{\sqrt{n_c(\mathbf{R}, T)}} \right]. \quad (4.34)$$

In ZGN [37], the contribution of the external symmetry-breaking field η' was left implicit. In the often-used Thomas-Fermi (TF) approximation, the kinetic energy of the condensate is omitted and the HFB approximation for the condensate chemical potential in (4.33) then simplifies to

$$\mu^{TF}(\mathbf{R}, T) = U_{ext}(\mathbf{R}) + 2g\tilde{n}(\mathbf{R}, T) + gn_c(\mathbf{R}, T) + g\text{Re}[\tilde{m}(\mathbf{R}, T)]. \quad (4.35)$$

Next we consider (4.28) in the high temperature limit. In this case, n_c is small and therefore we can neglect the terms proportional to $gn_c f_2$ in (4.28) as small (recall that $m = n_c + \tilde{m}$). Also, we know that \tilde{m} must be at least of order g [6], and therefore the $g\tilde{m}$ contribution to the self-energy is $\mathcal{O}(g^2)$. Thus we can neglect the right side of (4.28), leaving

$$\left[\frac{\partial}{\partial T} - \nabla_{\mathbf{R}}[\tilde{\epsilon}_p + \mathbf{v}_s \cdot \mathbf{p}] \cdot \nabla_{\mathbf{p}} + \nabla_{\mathbf{p}}[\tilde{\epsilon}_p + \mathbf{v}_s \cdot \mathbf{p}] \cdot \nabla_{\mathbf{R}} \right] f_1(\mathbf{p}, \mathbf{R}, T) = 0. \quad (4.36)$$

This is precisely the expected collisionless Boltzmann equation for f_1 , valid in the HF-Popov approximation. This approximation is valid only at finite temperatures, in which case $\tilde{\epsilon}_p(\mathbf{R}, T)$ as defined in (4.29) gives the correct excitation energy. In this limit, the kinetic equation (4.36) becomes equivalent to that first derived (using a different formalism) by Kirkpatrick and Dorfman (KD) [28] for a uniform gas. It is the local rest frame equivalent of the kinetic equation used by Zaremba, Nikuni and Griffin (ZGN) [37] if one ignores the second-order collision terms. In this high temperature limit, we can neglect the $g\tilde{m}$ term on the right side of (4.25), and therefore (4.25) reduces to the equation for the condensate in the HFP approximation. Within this approximation, the off-diagonal distribution function f_2 can be neglected as a higher-order contribution.

4.4 Static HFB equilibrium solution in the semi-classical approximation

For a uniform system, the matrix Green's functions defined in (3.15) depend only on the relative coordinates, i.e. $\tilde{g}_{\alpha\beta}(1, 1') = \tilde{g}_{\alpha\beta}(1 - 1')$. One can then solve (4.1)-(4.2) by Fourier transformation to obtain expressions for $\tilde{g}_{11}(\mathbf{p}, \omega)$ and $\tilde{g}_{12}(\mathbf{p}, \omega)$. Using these, we obtain the well-known single-particle spectral densities [10]

$$\begin{aligned}
A_{11}(\mathbf{p}, \omega) &= -2\text{Im } \tilde{g}_{11}(\mathbf{p}, \omega + i0^+) = -2\text{Im} \left[\frac{u_p^2}{\omega - E_p + i0^+} - \frac{v_p^2}{\omega + E_p + i0^+} \right] \\
&= 2\pi \left[u_p^2 \delta(\omega - E_p) - v_p^2 \delta(\omega + E_p) \right] \\
A_{12}(\mathbf{p}, \omega) &= -2\text{Im } \tilde{g}_{12}(\mathbf{p}, \omega + i0^+) = 2\text{Im} \left[\frac{u_p v_p^*}{\omega - E_p + i0^+} - \frac{u_p v_p^*}{\omega + E_p + i0^+} \right] \\
&= -2\pi u_p v_p^* [\delta(\omega - E_p) - \delta(\omega + E_p)].
\end{aligned} \tag{4.37}$$

The HFB quasiparticle energy E_p is given by

$$E_p^2 = \tilde{\epsilon}_p^2 - (gm)^2, \tag{4.38}$$

with

$$\tilde{\epsilon}_p \equiv \frac{p^2}{2m} + 2gn - \mu, \tag{4.39}$$

and

$$u_p^2 = \frac{1}{2} \left[\frac{\tilde{\epsilon}_p}{E_p} + 1 \right], \quad v_p^2 = \frac{1}{2} \left[\frac{\tilde{\epsilon}_p}{E_p} - 1 \right], \quad u_p v_p^* = \frac{gm}{2E_p}. \tag{4.40}$$

The results in (4.37) have the same structure as in the simpler $T = 0$ Bogoliubov approximation (see Ch.14 of Ref. [10]). From (4.7), with $\Phi(1) = \text{const.}$, it follows that the chemical potential in the HFB approximation is given by [24, 11]

$$\mu = g(n + \tilde{n} + \tilde{m}). \tag{4.41}$$

Using this result in (4.38), we obtain

$$E_{p=0}^2 = g^2 [n_c - \tilde{m}]^2 - g^2 [n_c + \tilde{m}]^2 = -4g^2 \tilde{m} n_c. \tag{4.42}$$

Therefore, the long-wavelength HFB excitation spectrum in a uniform Bose gas has an unphysical finite energy gap.

The density of non-condensate atoms in a uniform system can be found using the single-particle spectral density given in (4.37)

$$\begin{aligned}
 \tilde{n} &= \int \frac{d\mathbf{p}}{(2\pi)^3} \int \frac{d\omega}{2\pi} f_0(\omega) A_{11}(\mathbf{p}, \omega) \\
 &= \int \frac{d\mathbf{p}}{(2\pi)^3} \left[v_p^2 + (u_p^2 + v_p^2) f_0(E_p) \right] \\
 &= \int \frac{d\mathbf{p}}{(2\pi)^3} \left[\frac{\tilde{\epsilon}_p}{2E_p} [2f_0(E_p) + 1] - \frac{1}{2} \right], \tag{4.43}
 \end{aligned}$$

where $f_0(E_p)$ is the quasiparticle Bose distribution function. Similarly, the anomalous density is given by

$$\begin{aligned}
 \tilde{m} &= \int \frac{d\mathbf{p}}{(2\pi)^3} \int \frac{d\omega}{2\pi} f_0(\omega) A_{12}(\mathbf{p}, \omega) \\
 &= - \int \frac{d\mathbf{p}}{(2\pi)^3} v_p u_p^* [2f_0(E_p) + 1] \\
 &= -gm \int \frac{d\mathbf{p}}{(2\pi)^3} \frac{1}{2E_p} [2f_0(E_p) + 1]. \tag{4.44}
 \end{aligned}$$

As is well-known in the Bose gas literature, the anomalous density given by (4.44) has a ultraviolet (high-energy) divergence. This divergence is result of using the simple contact interaction in (1.1). Namely, by including the pair correlation function \tilde{m} , we are including the effect of interactions twice - part of this correlations has already been included in the contact interaction which represents the effect of pair correlations in a vacuum. This divergence is removed by subtracting the high-momentum limit of the integrand in (4.44) ($1/2\epsilon_p$) [6, 39].

We can find an approximate solution for f_1 and f_2 given by (4.28) and (4.30) for a trapped Bose gas which will be the analogue of the uniform gas results given in (4.43) and (4.44). We start from the coupled *static* HFB equations, as derived in Ref. [24]:

$$\begin{aligned}
 \hat{\mathcal{L}}u_i(\mathbf{R}) - gm_0(\mathbf{R})v_i(\mathbf{R}) &= E_i u_i(\mathbf{R}) \\
 \hat{\mathcal{L}}v_i(\mathbf{R}) - gm_0(\mathbf{R})u_i(\mathbf{R}) &= -E_i v_i(\mathbf{R}), \tag{4.45}
 \end{aligned}$$

where the differential operator is

$$\hat{\mathcal{L}} \equiv -\frac{\nabla^2}{2m} + U_{ext}(\mathbf{R}) - \mu + 2gn(\mathbf{R}). \quad (4.46)$$

We can solve (4.45) using the well-known “semi-classical approximation” [40, 41]. This assumes that the normal and anomalous densities are smooth functions of \mathbf{R} on the scale of length of $a_{HO} \equiv \sqrt{\frac{\hbar}{m\omega_0}}$, which defines the size of the condensate in a harmonic potential with a trap frequency ω_0 (we reinsert \hbar in the present discussion for physical clarity). Hence the self-consistent Hartree-Fock mean field varies slowly (as a function of \mathbf{R}) on the length scale of order a_{HO} . Therefore we can assume that $u_i(\mathbf{R})$ and $v_i(\mathbf{R})$ have the form of plane waves with a slowly varying amplitude in that region [40], i.e.

$$\begin{aligned} u_i(\mathbf{R}) &\equiv u(\mathbf{p}, \mathbf{R})e^{i\varphi(\mathbf{R})} \\ v_i(\mathbf{R}) &\equiv v(\mathbf{p}, \mathbf{R})e^{i\varphi(\mathbf{R})}. \end{aligned} \quad (4.47)$$

We define the local momentum of elementary excitations by

$$\mathbf{p} = \hbar\nabla_{\mathbf{R}}\varphi(\mathbf{R}), \quad (4.48)$$

which satisfies the condition $\mathbf{p} \gg \hbar/a_{HO}$. This condition, when expressed in terms of wavelength, reduces to the small wavelength limit ($\lambda \ll a_{HO}$), or equivalently, to the semi-classical approximation limit expressed as $k_B T \gg \hbar\omega_0$, where $\hbar\omega_0$ is the harmonic well energy level spacing. If this condition is satisfied, we can neglect the spatial derivatives of u and v , as well as the second spatial derivative of the phase φ in (4.47). This is consistent with a so-called *quasi-classical condition*, which requires that a spatial change in wavelength of the particle must satisfy the condition $d\lambda/dx \ll 1$ [41]. The assumed form given by (4.47) is valid only in the regions of space where this condition is satisfied. To treat the condensate in the corresponding approximation, we use the Thomas-Fermi approximation which is valid in the large N limit. The only region where the TF approximation for the order parameter is inadequate is close to the classical turning points at the

condensate boundary [40, 2], which is consistent with inapplicability of the semi-classical approximation near these points.

Putting all this together, we can easily solve the coupled equations (4.45) for $u(\mathbf{p}, \mathbf{R})$ and $v(\mathbf{p}, \mathbf{R})$,

$$u^2(\mathbf{p}, \mathbf{R}) = \frac{\tilde{\epsilon}_p(\mathbf{R}) + E_p(\mathbf{R})}{2E_p(\mathbf{R})}, \quad v^2(\mathbf{p}, \mathbf{R}) = \frac{\tilde{\epsilon}_p(\mathbf{R}) - E_p(\mathbf{R})}{2E_p(\mathbf{R})}, \quad u(\mathbf{p}, \mathbf{R})v^*(\mathbf{p}, \mathbf{R}) = \frac{gm_0(\mathbf{R})}{2E_p(\mathbf{R})}. \quad (4.49)$$

The local HFB quasiparticle energy $E_p(\mathbf{R})$ is given by

$$E_p^2(\mathbf{R}) = \tilde{\epsilon}_p^2(\mathbf{R}) - (gm_0(\mathbf{R}))^2 = \left[\frac{p^2}{2m} - \mu_0 + U_{ext}(\mathbf{R}) + 2gn_0(\mathbf{R}) \right]^2 - (gm_0(\mathbf{R}))^2. \quad (4.50)$$

We use the expressions for \tilde{n} and \tilde{m} given in terms of u and v [24]

$$\begin{aligned} \tilde{n}(\mathbf{R}) &= \sum_{\mathbf{p}} \left(|u_p(\mathbf{R})|^2 + |v_p(\mathbf{R})|^2 \right) f_0(E_p) + |v_p(\mathbf{R})|^2, \\ \tilde{m}(\mathbf{R}) &= - \sum_{\mathbf{p}} u_p(\mathbf{R})v_p^*(\mathbf{R}) [2f_0(E_p) + 1], \end{aligned} \quad (4.51)$$

where $f_0(E)$ is the Bose distribution for the quasiparticle excitations. The sum over the quantum states is now replaced by the integral $\int d\mathbf{p}/(2\pi)^3$ and the semi-classical approximation for the diagonal static distribution function $f_{10}(\mathbf{p}, \mathbf{R})$ is found to be given by

$$\begin{aligned} f_{10}(\mathbf{p}, \mathbf{R}) &= v^2(\mathbf{p}, \mathbf{R}) + [u^2(\mathbf{p}, \mathbf{R}) + v^2(\mathbf{p}, \mathbf{R})] f_0(E_p(\mathbf{R})) \\ &= \frac{\tilde{\epsilon}_p(\mathbf{R})}{2E_p(\mathbf{R})} [2f_0(E_p(\mathbf{R})) + 1] - \frac{1}{2}, \end{aligned} \quad (4.52)$$

while for the off-diagonal distribution function $f_{20}(\mathbf{p}, \mathbf{R})$, we obtain

$$f_{20}(\mathbf{p}, \mathbf{R}) = - \frac{gm_0(\mathbf{R})}{2E_p(\mathbf{R})} [2f_0(E_p(\mathbf{R})) + 1]. \quad (4.53)$$

In summary, in the semi-classical approximation, the local static equilibrium non-condensate density is given by

$$\tilde{n}_0(\mathbf{R}) = \int \frac{d\mathbf{p}}{(2\pi)^3} \left[\frac{\tilde{\epsilon}_p(\mathbf{R})}{2E_p(\mathbf{R})} [2f_0(E_p(\mathbf{R})) + 1] - \frac{1}{2} \right], \quad (4.54)$$

and the local static equilibrium anomalous non-condensate density by

$$\tilde{m}_0(\mathbf{R}) = -gm_0(\mathbf{R}) \int \frac{d\mathbf{p}}{(2\pi)^3} \left[\frac{2f_0(E_p(\mathbf{R})) + 1}{2E_p(\mathbf{R})} \right]. \quad (4.55)$$

As expected, these semi-classical approximation results (or local density approximation) are the natural generalizations of the results obtained for a *uniform* gas given by (4.43) and (4.44). The same kind of semi-classical results have also been obtained in Ref. [40] for a trapped Bose gas using the Popov approximation (which corresponds to the HFB with $\tilde{m}=0$). Since the local quasiparticle energy $E_p(\mathbf{R})$ given by (4.50) depends on the normal and anomalous densities, the quantities in (4.50), (4.54) and (4.55) must be solved self-consistently, analogous to the discussion in Ref. [40].

We now show that these semi-classical static HFB results for f_{10} and f_{20} are solutions of our collisionless static HFB kinetic equations. First of all, in the static limit, we note that the kinetic equation in (4.30) reduces to

$$\begin{aligned} 2\tilde{\epsilon}_p(\mathbf{R})f_{20}(\mathbf{p}, \mathbf{R}) = & - gm_0(\mathbf{R}) [f_{10}(\mathbf{p}, \mathbf{R}) + f_{10}(-\mathbf{p}, \mathbf{R}) + 1] \\ & - i\frac{g}{2}\nabla_{\mathbf{R}}m(\mathbf{R}, T) \cdot \nabla_{\mathbf{p}} [f_{10}(\mathbf{p}, \mathbf{R}) - f_{10}(-\mathbf{p}, \mathbf{R})]. \end{aligned} \quad (4.56)$$

In static equilibrium without any mass current, we have $f_{10}(\mathbf{p}, \mathbf{R}) = f_{10}(-\mathbf{p}, \mathbf{R})$, and thus (4.56) can be further simplified and rewritten in the form

$$f_{20}(\mathbf{p}, \mathbf{R}) = -\frac{gm_0(\mathbf{R})}{2\tilde{\epsilon}_p(\mathbf{R})} [2f_{10}(\mathbf{p}, \mathbf{R}) + 1]. \quad (4.57)$$

We note that the HFB approximation, $\tilde{\epsilon}_p(\mathbf{R})$ in (4.29) is given by

$$\tilde{\epsilon}_p(\mathbf{R}) = \frac{p^2}{2m} + \frac{\nabla_{\mathbf{R}}^2 \sqrt{n_{c0}(\mathbf{R})}}{2m\sqrt{n_{c0}(\mathbf{R})}} + g[n_{c0}(\mathbf{R}) - \tilde{m}_0(\mathbf{R})]. \quad (4.58)$$

Similarly, in static thermal equilibrium, the equation of motion (4.28) for the diagonal distribution function $f_1(\mathbf{p}, \mathbf{R}, T)$ reduces to

$$[-\nabla_{\mathbf{R}}\tilde{\epsilon}_p(\mathbf{R}) \cdot \nabla_{\mathbf{p}} + \nabla_{\mathbf{p}}\tilde{\epsilon}_p(\mathbf{R}) \cdot \nabla_{\mathbf{R}}] f_{10}(\mathbf{p}, \mathbf{R}) = g\nabla_{\mathbf{R}}m_0(\mathbf{R}) \cdot \nabla_{\mathbf{p}}f_{20}(\mathbf{p}, \mathbf{R}). \quad (4.59)$$

In the last step, we again have used the relation $f_{20}(-\mathbf{p}, \mathbf{R}) = f_{20}(\mathbf{p}, \mathbf{R})$ and the fact that m_0 is real.

The local equilibrium functions $f_{10}(\mathbf{p}, \mathbf{R})$ and $f_{20}(\mathbf{p}, \mathbf{R})$ given by (4.52) and (4.53) can be shown to satisfy static HFB kinetic equations. Inserting (4.52) into (4.56) [or (4.57)], we obtain (4.53). This shows that the static equilibrium kinetic equation (4.56) for the off-diagonal distribution function has a solution consistent with the semi-classical approximation. Similarly, substituting the local equilibrium distribution functions f_{10} and f_{20} given by (4.52) and (4.53), a lengthy but straightforward calculation shows that they satisfy the static limit HFB kinetic equation given by (4.59). In a sense, these results may be surprising since the HFB kinetic equations ignore collisions which are needed to produce a static thermal equilibrium. However, this is just a generalization of the fact that in normal Bose systems, the equilibrium Bose distribution function is a solution of the collisionless kinetic equation (see p.71 of Ref. [14]).

We recall that the semi-classical local equilibrium expressions for f_1 and f_2 are only valid under the conditions $k_B T \gg \hbar\omega_0$ and $E_{\mathbf{p}}(\mathbf{R}) \gg \hbar\omega_0$. This means that these approximate forms cease to be valid at *very* low temperatures (for further discussion, see Ref. [40]).

4.5 Kohn mode

In this section, following the approach used in Ref. [17], we prove that our coupled HFB kinetic equations exhibit the rigid in-phase collective mode solution which has the trap frequency (the Kohn mode) [37]. This mode is an exact solution for Bosons in a parabolic trap. It is independent of the interaction and hence valid for all dynamical regimes. It is thus an important check on any approximate dynamical theory. This rigid in-phase center-of-mass oscillation of the condensate and the non-condensate corresponds to

$$n_c(\mathbf{R}, T) \equiv n_{c0}(\mathbf{R} - \boldsymbol{\eta}(T))$$

$$\begin{aligned}\tilde{n}(\mathbf{R}, T) &\equiv \tilde{n}_0(\mathbf{R} - \boldsymbol{\eta}(T)) \\ \tilde{m}(\mathbf{R}, T) &\equiv \tilde{m}_0(\mathbf{R} - \boldsymbol{\eta}(T)).\end{aligned}\tag{4.60}$$

Here the center-of-mass displacement $\boldsymbol{\eta}(T)$ (with $\mathbf{v}_s = \mathbf{v}_n = \dot{\boldsymbol{\eta}}$) is independent of position and will be shown to satisfy the harmonic oscillator equation of motion (independent of the interactions)

$$m \frac{\partial^2 \eta_\alpha}{\partial T^2} = -\omega_\alpha^2 \eta_\alpha,\tag{4.61}$$

where ω_α is the harmonic well trap frequency in the α^{th} direction.

We first recall the equation of motion for the superfluid velocity given by (4.34). For the case of a rigid in-phase oscillation described by (4.60), this equation can be written in the form

$$m \frac{\partial^2 \boldsymbol{\eta}}{\partial T^2} = -\nabla_{\mathbf{R}} [\mu_0(\mathbf{R} - \boldsymbol{\eta}(T)) + U_{ext}(\mathbf{R}) - U_{ext}(\mathbf{R} - \boldsymbol{\eta}(T))].\tag{4.62}$$

Since the equilibrium chemical potential μ_0 is position independent, it follows that the first term on the right hand side makes no contribution and we are left with

$$m \frac{\partial^2 \boldsymbol{\eta}}{\partial T^2} = -\nabla_{\mathbf{R}} [U_{ext}(\mathbf{R}) - U_{ext}(\mathbf{R} - \boldsymbol{\eta})].\tag{4.63}$$

For a harmonic trap potential described by

$$U_{ext}(\mathbf{R}) = \frac{1}{2} m (\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2),\tag{4.64}$$

a simple calculation shows that (4.63) reduces to (4.61), as claimed.

We next argue that the rigid in-phase oscillation described by (4.60) corresponds to the following distribution functions:

$$\begin{aligned}f_1(\mathbf{p}, \mathbf{R}, T) &\equiv f_{10}(\mathbf{p}, \mathbf{R} - \boldsymbol{\eta}(T)) \\ f_2(\mathbf{p}, \mathbf{R}, T) &\equiv f_{20}(\mathbf{p}, \mathbf{R} - \boldsymbol{\eta}(T)).\end{aligned}\tag{4.65}$$

Here, $f_{10}(\mathbf{p}, \mathbf{R})$ and $f_{20}(\mathbf{p}, \mathbf{R})$ satisfy the static equilibrium kinetic equations (4.59) and (4.57), respectively, and the static equilibrium densities satisfy (4.60). In this case, the

kinetic equation (4.28) for $f_1(\mathbf{p}, \mathbf{R}, T)$ reduces to

$$\begin{aligned} & \left[\frac{\partial}{\partial T} - \nabla_{\mathbf{R}} [\tilde{\epsilon}_p(\mathbf{R}, T) + \mathbf{v}_s \cdot \mathbf{p}] \cdot \nabla_{\mathbf{p}} + \nabla_{\mathbf{p}} [\tilde{\epsilon}_p(\mathbf{R}, T) + \mathbf{v}_s \cdot \mathbf{p}] \cdot \nabla_{\mathbf{R}} \right] f_{10}(\mathbf{p}, \mathbf{R} - \boldsymbol{\eta}) \\ &= -ig [m(\mathbf{R}, T) f_{20}(-\mathbf{p}, \mathbf{R} - \boldsymbol{\eta}) - m^*(\mathbf{R}, T) f_{20}(\mathbf{p}, \mathbf{R} - \boldsymbol{\eta})] + \\ &+ \frac{g}{2} [\nabla_{\mathbf{R}} m(\mathbf{R}, T) \cdot \nabla_{\mathbf{p}} f_2(-\mathbf{p}, \mathbf{R} - \boldsymbol{\eta}) + \nabla_{\mathbf{R}} m(\mathbf{R}, T) \cdot \nabla_{\mathbf{p}} f_2(\mathbf{p}, \mathbf{R} - \boldsymbol{\eta})]. \end{aligned} \quad (4.66)$$

From the definition of $f_{i0}(\mathbf{p}, \mathbf{R})$, we see that $f_{i0}(-\mathbf{p}, \mathbf{R} - \boldsymbol{\eta}(T)) = f_{i0}(\mathbf{p}, \mathbf{R} - \boldsymbol{\eta}(T))$. We also note that for the in-phase oscillation under consideration, $m(\mathbf{R}, T) = m_0(\mathbf{R} - \boldsymbol{\eta}(T))$, where m_0 is real. Therefore the first term on the right side of (4.66) vanishes, leaving us with

$$\begin{aligned} & \left[\frac{\partial}{\partial T} - \nabla_{\mathbf{R}} [\tilde{\epsilon}_p(\mathbf{R}, T) + \mathbf{v}_s \cdot \mathbf{p}] \cdot \nabla_{\mathbf{p}} + \nabla_{\mathbf{p}} [\tilde{\epsilon}_p(\mathbf{R}, T) + \mathbf{v}_s \cdot \mathbf{p}] \cdot \nabla_{\mathbf{R}} \right] f_{10}(\mathbf{p}, \mathbf{R} - \boldsymbol{\eta}) \\ &= g \nabla_{\mathbf{R}} m_0(\mathbf{R} - \boldsymbol{\eta}) \cdot \nabla_{\mathbf{p}} f_{20}(\mathbf{p}, \mathbf{R} - \boldsymbol{\eta}). \end{aligned} \quad (4.67)$$

Using (4.29), (4.33) and (4.60), it is straightforward to show that $\tilde{\epsilon}_p(\mathbf{R}, T) = \tilde{\epsilon}_p(\mathbf{R} - \boldsymbol{\eta})$.

Using this, (4.67) can be rewritten as (recall that $\mathbf{v}_s = \dot{\boldsymbol{\eta}}$ is spatially independent)

$$\begin{aligned} & \left[\frac{\partial}{\partial T} - \nabla_{\mathbf{R}} \tilde{\epsilon}_p(\mathbf{R} - \boldsymbol{\eta}) \cdot \nabla_{\mathbf{p}} + \nabla_{\mathbf{p}} [\tilde{\epsilon}_p(\mathbf{R} - \boldsymbol{\eta}) + \dot{\boldsymbol{\eta}} \cdot \mathbf{p}] \cdot \nabla_{\mathbf{R}} \right] f_{10}(\mathbf{p}, \mathbf{R} - \boldsymbol{\eta}) \\ &= g \nabla_{\mathbf{R}} m_0(\mathbf{R} - \boldsymbol{\eta}) \cdot \nabla_{\mathbf{p}} f_{20}(\mathbf{p}, \mathbf{R} - \boldsymbol{\eta}). \end{aligned} \quad (4.68)$$

If we introduce the new variable $\mathbf{R}'(T) \equiv \mathbf{R} - \boldsymbol{\eta}(T)$, and note that

$$\nabla_{\mathbf{R}} = \nabla_{\mathbf{R}'} \quad \frac{\partial}{\partial T} = -\dot{\boldsymbol{\eta}} \cdot \nabla_{\mathbf{R}'}, \quad (4.69)$$

then (4.68) can be rewritten as

$$\begin{aligned} & [-\dot{\boldsymbol{\eta}} \cdot \nabla_{\mathbf{R}'} - \nabla_{\mathbf{R}'} \tilde{\epsilon}_p(\mathbf{R}') \cdot \nabla_{\mathbf{p}} + \nabla_{\mathbf{p}} \tilde{\epsilon}_p(\mathbf{R}') \cdot \nabla_{\mathbf{R}'} + \dot{\boldsymbol{\eta}} \cdot \nabla_{\mathbf{R}'}] f_{10}(\mathbf{p}, \mathbf{R}') \\ &= g \nabla_{\mathbf{R}'} m_0(\mathbf{R}') \cdot \nabla_{\mathbf{p}} f_{20}(\mathbf{p}, \mathbf{R}'). \end{aligned} \quad (4.70)$$

Since the first and the last term on the left side of (4.70) cancel each other, we are left with an equation which is precisely the same as the static HFB equation (4.59) for the diagonal distribution function f_1 . This proves that $f_1(\mathbf{p}, \mathbf{R}, T)$ in (4.65) is a solution.

Similarly, for the in-phase mode, (4.30) for the off-diagonal distribution function f_2 reduces to

$$\left[\frac{\partial}{\partial T} + 2i\tilde{\epsilon}_p(\mathbf{R}, T) + \dot{\boldsymbol{\eta}} \cdot \nabla_{\mathbf{R}} \right] f_{20}(\mathbf{p}, \mathbf{R} - \boldsymbol{\eta}) = -igm_0(\mathbf{R} - \boldsymbol{\eta}) [2f_{10}(\mathbf{p}, \mathbf{R} - \boldsymbol{\eta}) + 1]. \quad (4.71)$$

Using the transformed variable in (4.69), (4.71) simplifies to

$$f_{20}(\mathbf{p}, \mathbf{R}') = -\frac{gm_0(\mathbf{R}')}{2\tilde{\epsilon}_p(\mathbf{R}')} [2f_{10}(\mathbf{p}, \mathbf{R}') + 1]. \quad (4.72)$$

Again, this result is seen to be equivalent to the static HFB equation result expressed in the form given by (4.57). Thus $f_2(\mathbf{p}, \mathbf{R}, T)$ in (4.65) is a solution.

In summary, we have explicitly verified that the ansatz given in (4.60) satisfies our coupled HFB kinetic equations for the two distribution function f_1 and f_2 . The oscillating center-of-mass displacement $\boldsymbol{\eta}(T)$ satisfies the SHO equation of motion in (4.61). Thus our HFB coupled kinetic equations exhibit a solution corresponding to a rigid SHO oscillation of the three static equilibrium density profiles (n_c , \tilde{n} and \tilde{m}) in the direction of η_α , with the associated trap frequency ω_α . This expected Kohn solution is an important check on the correctness of our HFB equations of motion. We might note that we have not made use of the approximate semi-classical local equilibrium forms for f_1 and f_2 (given in Section 4.4) to prove that this Kohn mode solution exists for (4.28) and (4.30).

4.6 Conclusions

To summarize this chapter, we have used the simple first order HFB self-energy approximation to derive the equation of motion for the condensate order parameter (given by (4.7)) and the kinetic equations for the diagonal and the off-diagonal distribution functions for the non-condensate atoms (given by (4.28) and (4.30)). We have only solved these coupled equations to exhibit the rigid in-phase Kohn mode (Section 4.5). However, we emphasize that, more generally, these kinetic equations can be used to derive collective modes which satisfy all conservation laws [11], even though they were generated

from the HFB single-particle self-energies (which gives a single-particle spectrum with certain problems, such as an unphysical energy gap in a uniform Bose gas). Another way of saying this is that for a uniform Bose gas, the coupled equations of motion in this chapter will lead to the same “conserving” dynamic density response function discussed by Cheung and Griffin [33]. The essential physics of such conserving approximations is discussed in Sections 6 and 7 of Griffin’s Varenna lectures [6].

Clearly the next step is to extend our present analysis by including the appropriate second-order self-energy contributions, and hence incorporate the effect of collisions into our kinetic equations. This generalization was originally studied in a *uniform* Bose-condensed gas by Kane and Kadanoff (KK) [15], work which will be extended to *trapped* Bose gases in Chs. 5 and 6 of this thesis [19, 29].

Chapter 5

Generalized Boltzmann equation at finite temperatures

5.1 Equations of motion for non-equilibrium Green's functions

In this chapter, starting from the general results of Ch.3, we extend the analysis of Ch.4 to include collisions between the atoms. We thus derive kinetic equations that include second order collision processes. However, we limit our discussion to high enough temperatures, where the important thermal excitations can be treated as free particles moving in a dynamic Hartree-Fock field. This will set the stage for the more general theory in Ch.6.

We start our analysis with the Dyson-Beliaev equations of motion for the real-time non-condensate propagators $\hat{g}(1, 1')$ written in 2×2 matrix form [see (3.16) and (3.17) in Ch. 3]

$$\begin{aligned} & \int d\bar{1} \left[\hat{g}_0^{-1}(1, \bar{1}) - \hat{\Sigma}^{HF}(1, \bar{1}) \right] \hat{g}^{\lessgtr}(1, 1') \\ &= \int_{-\infty}^{t_1} d\bar{1} \hat{\Gamma}(1, \bar{1}) \hat{g}^{\lessgtr}(1, 1') - \int_{-\infty}^{t_1'} d\bar{1} \hat{\Sigma}^{\lessgtr}(1, \bar{1}) \hat{a}(\bar{1}, 1'), \end{aligned} \quad (5.1)$$

and

$$\begin{aligned} & \int d\bar{1} \hat{g}^{\lessgtr}(1, \bar{1}) [\hat{g}_0^{-1}(\bar{1}, 1') - \hat{\Sigma}^{HF}(\bar{1}, 1')] \\ &= \int_{-\infty}^{t_1} d\bar{1} \hat{a}(1, \bar{1}) \hat{\Sigma}^{\lessgtr}(\bar{1}, 1') - \int_{-\infty}^{t_1'} d\bar{1} \hat{g}^{\lessgtr}(1, \bar{1}) \hat{\Gamma}(\bar{1}, 1'). \end{aligned} \quad (5.2)$$

where the 2×2 matrices a and Γ are defined in (3.18). The Hartree-Fock self-energy contribution is included into the left-hand side of (5.1) and (5.2) and will be shown to give the mean-field contribution to the “streaming” term. The second order self-energy describing binary collisions is included on the right-hand side of (5.1) and (5.2); it will eventually be shown to give rise to collision integrals of the kind shown in (1.6).

We write (3.8) for the order parameter in terms of a condensate self-energy function S , with the condensate propagator h as given by (3.6). In place of (3.8), we use (3.14)

$$\int d\bar{1} [\hat{g}_0^{-1}(1, \bar{1}) - \hat{S}^{HF}(1, \bar{1})] h(\bar{1}, 1') = \int_{-\infty}^{t_1} d\bar{1} (\hat{S}^>(1, \bar{1}) - \hat{S}^<(1, \bar{1})) \hat{h}(\bar{1}, 1'). \quad (5.3)$$

As before, the mean-field contributions are included in the Hartree-Fock part of the condensate self-energy S^{HF} given in (4.4). Similarly, one can find the expression for the second-order contribution to the condensate self-energy S^{\lessgtr} . If we recall the definition for the condensate propagator \hat{h} in (3.6) and use the explicit form for the condensate Hartree-Fock self energy \hat{S}^{HF} in (4.4), we obtain a generalized GP equation for the macroscopic wave function $\Phi(\mathbf{r}, t)$, namely

$$\begin{aligned} & \left[-i \frac{\partial}{\partial t_1} + \frac{\nabla_1^2}{2m} - U_{ext}(\mathbf{r}_1) + \mu_0 - g(n_c(1) + 2\tilde{n}(1)) \right] \Phi(1) = g\tilde{m}(1)\Phi^*(1) \\ & + \int_{-\infty}^{t_1} d\bar{1} (S_{11}^> - S_{11}^<)(1, \bar{1})\Phi(\bar{1}) + \int_{-\infty}^{t_1} d\bar{1} (S_{12}^> - S_{12}^<)(1, \bar{1})\Phi^*(\bar{1}). \end{aligned} \quad (5.4)$$

The specific form of Σ_c and \hat{S}^{\lessgtr} will depend on the approximation that we use. In this chapter, we work with the second-order self-energy given by the Beliaev (gapless) approximation [11, 24], plus the normal second-order contributions. The advantage of the Beliaev approximation is that the non-condensate Green’s function exhibits the correct spectrum (phonon-like in the long-wavelength, uniform gas limit).

5.2 Generalized kinetic equations

To derive a generalized kinetic equation coupled to a condensate, we follow Kane and Kadanoff and gauge-transform (5.1) and (5.2) to a local rest frame to remove the phase of the macroscopic wavefunction [15]. It is important that we work in a frame of reference where the order parameter is defined without the phase because gradient expansions are a crucial step in the KB method. If we do not gauge transform into the local rest frame (where $\mathbf{v}_s = 0$), one obtains terms proportional to the gradient of the phase and these terms would not be small. Hence, the gradient expansion would not work. After transformation, (5.1) and (5.2) remain unchanged in form if g_0^{-1} is replaced by [compare with (3.10)]

$$g_0^{-1}(1, 1') = \left[i\tau_3 \frac{\partial}{\partial t_1} - \frac{\partial\theta(1)}{\partial t_1} + \frac{1}{2} [\nabla_1 + i\tau_3 \nabla_1 \theta(1)]^2 - U_{ext}(\mathbf{r}_1) + \mu_0 \right] \delta(1, 1'). \quad (5.5)$$

We recall that the superfluid velocity $\mathbf{v}_s(\mathbf{R}, T)$ and the local chemical potential $\mu(\mathbf{R}, T)$ are defined by (4.11) [15]

$$\begin{aligned} m\mathbf{v}_s(\mathbf{R}, T) &\equiv \nabla_{\mathbf{R}}\theta(\mathbf{R}, T) \\ \frac{\partial\theta(\mathbf{R}, T)}{\partial T} &\equiv - \left[\mu(\mathbf{R}, T) - \mu_0 + \frac{1}{2} m v_s^2(\mathbf{R}, T) \right]. \end{aligned} \quad (5.6)$$

In the lab frame, the condensate wavefunction is given by $\Phi(\mathbf{R}, T) = \sqrt{n_c(\mathbf{R}, T)} e^{i\theta(\mathbf{R}, T)}$, i.e., the phase is not removed.

To illustrate how this method works, in this chapter we neglect the off-diagonal anomalous single-particle Green's functions (\tilde{g}_{12} and \tilde{g}_{21}). In this approximation, the 2×2 matrix equations in (5.1) and (5.2) reduce to scalar equations for the $\tilde{g}_{11}^{\lessgtr}$ component

$$\begin{aligned} &\left[i \frac{\partial}{\partial t_1} - \frac{\partial\theta(1)}{\partial t_1} + \frac{1}{2} [\nabla_1 + i m \mathbf{v}_s(1)]^2 - U(1) + \mu_0 \right] \tilde{g}_{11}^{\lessgtr}(\bar{1}, 1') \\ &= \int_{-\infty}^{t_1} d\bar{1} \Gamma_{11}(1, \bar{1}) \tilde{g}_{11}^{\lessgtr}(\bar{1}, 1') - \int_{-\infty}^{t_1'} d\bar{1} \Sigma_{11}^{\lessgtr}(1, \bar{1}) a_{11}(\bar{1}, 1') \end{aligned} \quad (5.7)$$

and

$$\begin{aligned} & \left[-i \frac{\partial}{\partial t_{1'}} - \frac{\partial \theta(1')}{\partial t_{1'}} + \frac{1}{2} [\nabla_{1'} - im \mathbf{v}_s(1')]^2 - U(1') + \mu_0 \right] \tilde{g}_{11}^{\leq}(\bar{1}, 1') \\ & = \int_{-\infty}^{t_1} d\bar{1} a_{11}(1, \bar{1}) \Sigma_{11}^{\leq}(\bar{1}, 1') - \int_{-\infty}^{t_{1'}} d\bar{1} \tilde{g}_{11}^{\leq}(1, \bar{1}) \Gamma_{11}(\bar{1}, 1'). \end{aligned} \quad (5.8)$$

The effective self-consistent Hartree-Fock dynamic mean field $U(1)$ is given by (4.15).

As in previous chapters, in the KB approach, we express correlation functions in terms of relative and center-of-mass coordinates, defined in (4.8) and (4.9). Correlation functions (like \tilde{g} , Σ , etc.) are dominated by the small values of relative coordinates (\mathbf{r}, t) (or high momenta and frequencies in the Fourier transforms), but vary slowly as functions of the center-of-mass coordinates (\mathbf{R}, T) . Using these key properties of correlation functions to simplify the equations, we can write (5.7) and (5.8) in terms of the center-of-mass and relative coordinates to obtain

$$\begin{aligned} & \left[i \frac{\partial}{\partial T} + \left[\mathbf{r} \cdot \nabla_{\mathbf{R}} + t \frac{\partial}{\partial T} \right] (\mu(\mathbf{R}, T) - U(\mathbf{R}, T)) + i \nabla_{\mathbf{R}} \cdot \mathbf{v}_s(\mathbf{R}, T) \right. \\ & + \frac{1}{m} \nabla_{\mathbf{R}} \cdot \nabla_{\mathbf{r}} + i \left[\left(\mathbf{r} \cdot \nabla_{\mathbf{R}} + t \frac{\partial}{\partial T} \right) \mathbf{v}_s(\mathbf{R}, T) \right] \cdot \nabla_{\mathbf{r}} + i \mathbf{v}_s(\mathbf{R}, T) \cdot \nabla_{\mathbf{R}} \left. \right] \tilde{g}_{11}^{\leq}(\mathbf{r}, t) \\ & = \int d\bar{\mathbf{r}} d\bar{t} [\tilde{g}_{11}^{\leq}(\bar{\mathbf{r}}, \bar{t}) \Sigma_{11}^{\leq}(\mathbf{r} - \bar{\mathbf{r}}, t - \bar{t}; \mathbf{R}, T) - \tilde{g}_{11}^{\leq}(\bar{\mathbf{r}}, \bar{t}) \Sigma_{11}^{\leq}(\mathbf{r} - \bar{\mathbf{r}}, t - \bar{t}; \mathbf{R}, T)]. \end{aligned} \quad (5.9)$$

The (\mathbf{R}, T) dependence of $\tilde{g}_{11}(\mathbf{r}, t; \mathbf{R}, T)$ is left implicit. The double Fourier transform of (5.9) is given by

$$\begin{aligned} & \left[\frac{\partial}{\partial T} + \nabla_{\mathbf{p}} [\tilde{\epsilon}_p + \mathbf{p} \cdot \mathbf{v}_s] \cdot \nabla_{\mathbf{R}} - \nabla_{\mathbf{R}} [\tilde{\epsilon}_p + \mathbf{p} \cdot \mathbf{v}_s] \cdot \nabla_{\mathbf{p}} \right. \\ & + \frac{\partial}{\partial T} [\tilde{\epsilon}_p + \mathbf{p} \cdot \mathbf{v}_s] \frac{\partial}{\partial \omega} \left. \right] \tilde{g}_{11}^{\leq}(\mathbf{p}, \omega; \mathbf{R}, T) \\ & = \tilde{g}_{11}^{\leq}(\mathbf{p}, \omega; \mathbf{R}, T) \Sigma_{11}^{\leq}(\mathbf{p}, \omega; \mathbf{R}, T) - \tilde{g}_{11}^{\leq}(\mathbf{p}, \omega; \mathbf{R}, T) \Sigma_{11}^{\leq}(\mathbf{p}, \omega; \mathbf{R}, T). \end{aligned} \quad (5.10)$$

As before in (4.29), we define the “normal” HF single-particle energy by

$$\tilde{\epsilon}_p(\mathbf{R}, T) \equiv \frac{p^2}{2m} + U(\mathbf{R}, T) - \mu(\mathbf{R}, T). \quad (5.11)$$

In the Thomas-Fermi (TF) approximation [2], we note that this single-particle energy reduces to

$$\tilde{\epsilon}_p(\mathbf{R}, T) \equiv \frac{p^2}{2m} + g n_c(\mathbf{R}, T). \quad (5.12)$$

We recall that the TF approximation, which neglects the kinetic energy associated with the condensate wavefunction, is typically valid when the number of atoms is 10^4 or larger [2]. Using (5.10) to calculate the single-particle spectral density defined in (3.18), we find that the self-energies Σ^{\lessgtr} cancel out, leaving

$$\begin{aligned} & \left[\frac{\partial}{\partial T} + \nabla_{\mathbf{p}} [\tilde{\epsilon}_p + \mathbf{p} \cdot \mathbf{v}_s] \cdot \nabla_{\mathbf{R}} - \nabla_{\mathbf{R}} [\tilde{\epsilon}_p + \mathbf{p} \cdot \mathbf{v}_s] \cdot \nabla_{\mathbf{p}} \right. \\ & \left. + \frac{\partial}{\partial T} [\tilde{\epsilon}_p + \mathbf{p} \cdot \mathbf{v}_s] \frac{\partial}{\partial \omega} \right] a_{11}(\mathbf{p}, \omega; \mathbf{R}, T) = 0. \end{aligned} \quad (5.13)$$

This equation is a key step in deriving a kinetic equation for the distribution function for the non-condensate atoms. One may explicitly verify that the HF quasiparticle approximation for the spectral density, given by

$$a_{11}(\mathbf{p}, \omega; \mathbf{R}, T) = (2\pi)\delta(\omega - \mathbf{p} \cdot \mathbf{v}_s - \tilde{\epsilon}_p(\mathbf{R}, T)) \quad (5.14)$$

does indeed satisfy (5.13). We note that the assumption that the correlation functions $\tilde{g}^{\lessgtr}(\mathbf{r}, t; \mathbf{R}, T)$ are peaked at small values of \mathbf{r} and t implies that large values of \mathbf{p} and ω determine $\tilde{g}^{\lessgtr}(\mathbf{p}, \omega; \mathbf{R}, T)$.

Following Ref.[14], we now define a *quasiparticle* distribution function f in the following way

$$\begin{aligned} \tilde{g}_{11}^{\lessgtr}(\mathbf{p}, \omega; \mathbf{R}, T) & \equiv a_{11}(\mathbf{p}, \omega; \mathbf{R}, T)f(\mathbf{p}, \mathbf{R}, T) \\ \tilde{g}_{11}^{\lessgtr}(\mathbf{p}, \omega; \mathbf{R}, T) & \equiv a_{11}(\mathbf{p}, \omega; \mathbf{R}, T)[1 + f(\mathbf{p}, \mathbf{R}, T)], \end{aligned} \quad (5.15)$$

where the spectral density a_{11} is given by (5.14). We recall that the usual Wigner distribution function is related to the KB diagonal single-particle Green's function by

$$\begin{aligned} f_W(\mathbf{p}, \mathbf{R}, T) & \equiv \int d\mathbf{r} e^{-i\mathbf{p}\mathbf{r}} i\tilde{g}_{11}^{\lessgtr}(\mathbf{r}, t = 0; \mathbf{R}, T) \\ & = \int_{-\infty}^{\infty} d\omega \tilde{g}_{11}^{\lessgtr}(\mathbf{p}, \omega; \mathbf{R}, T). \end{aligned} \quad (5.16)$$

Using (5.15) in conjunction with the explicit result for spectral density a_{11} in (5.14), we see that the quasiparticle distribution function defined in (5.15) does indeed reduce to

the Wigner distribution function (5.16). This is correct in the high-temperature limit only, where the HF quasiparticle spectrum reduces to the Hartree-Fock energy of atoms; this quasiparticle distribution function is the same as the distribution function for atoms. This is not the case at very low temperatures, where (5.14) is not valid.

Using (5.13), (5.14), and (5.15) and after integration over ω , (5.10) reduces to a kinetic equation for the distribution function f , namely

$$\begin{aligned} & \left[\frac{\partial}{\partial T} + \nabla_{\mathbf{p}} [\tilde{\epsilon}_p + \mathbf{p} \cdot \mathbf{v}_s] \cdot \nabla_{\mathbf{R}} - \nabla_{\mathbf{R}} [\tilde{\epsilon}_p + \mathbf{p} \cdot \mathbf{v}_s] \cdot \nabla_{\mathbf{p}} \right] f(\mathbf{p}, \mathbf{R}, T) \\ & = (1 + f) \Sigma_{11}^{\leq}(\mathbf{p}, \omega = \tilde{\epsilon}_p + \mathbf{p} \cdot \mathbf{v}_s; \mathbf{R}, T) - f \Sigma_{11}^{\geq}(\mathbf{p}, \omega = \tilde{\epsilon}_p + \mathbf{p} \cdot \mathbf{v}_s; \mathbf{R}, T), \end{aligned} \quad (5.17)$$

where $f \equiv f(\mathbf{p}, \mathbf{R}, T)$. A generalized quantum kinetic equation of this kind has been derived by Stoof [42] using the related Keldysh formalism and path integrals.

5.3 Collision integrals in the second-order Beliaev approximation

Neglecting the off-diagonal Green's functions, the Fourier transform of the collisional part of the self-energy Σ^{\lessgtr} , given by (3.38), is given by the following expression

$$\begin{aligned} \Sigma_{11}^{\geq}(\mathbf{p}, \omega) & = 2g^2 \int \frac{d\mathbf{p}_i d\omega_i}{(2\pi)^{12}} (2\pi)^4 \delta(\omega + \omega_1 - \omega_2 - \omega_3) \delta(\mathbf{p} + \mathbf{p}_1 - \mathbf{p}_2 - \mathbf{p}_3) \\ & \times \left[\tilde{g}_{11}^{\leq}(\mathbf{p}_1, \omega_1) \tilde{g}_{11}^{\geq}(\mathbf{p}_2, \omega_2) h_{11}(\mathbf{p}_3, \omega_3) + \tilde{g}_{11}^{\leq}(\mathbf{p}_1, \omega_1) h_{11}(\mathbf{p}_2, \omega_2) \tilde{g}_{11}^{\geq}(\mathbf{p}_3, \omega_3) \right. \\ & \left. + h_{11}(\mathbf{p}_1, \omega_1) \tilde{g}_{11}^{\leq}(\mathbf{p}_2, \omega_2) \tilde{g}_{11}^{\geq}(\mathbf{p}_3, \omega_3) + \tilde{g}_{11}^{\leq}(\mathbf{p}_1, \omega_1) \tilde{g}_{11}^{\geq}(\mathbf{p}_2, \omega_2) \tilde{g}_{11}^{\geq}(\mathbf{p}_3, \omega_3) \right]. \end{aligned} \quad (5.18)$$

Here and elsewhere, the (\mathbf{R}, T) dependence of the self-energies and the Green's functions is left implicit. Using (5.14), (5.15) and (5.18), one can evaluate the collision integral C_{22} which involves collisions between excited atoms (i.e., the parts of the self-energy in (5.18) which involve three propagators for non-condensate atoms). We obtain

$$C_{22}[f] = 2g^2 \int \frac{d\mathbf{p}_1 d\mathbf{p}_2 d\mathbf{p}_3}{(2\pi)^3} \delta(\tilde{\epsilon}_p + \tilde{\epsilon}_{p_1} - \tilde{\epsilon}_{p_2} - \tilde{\epsilon}_{p_3}) \delta(\mathbf{p} + \mathbf{p}_1 - \mathbf{p}_2 - \mathbf{p}_3)$$

$$\times [(1+f)(1+f_1)f_2f_3 - ff_1(1+f_2)(1+f_3)], \quad (5.19)$$

where $f \equiv f(\mathbf{p}, \mathbf{R}, T)$ and $f_i \equiv f(\mathbf{p}_i, \mathbf{R}, T)$. C_{22} describes collisions where 2 excited atoms are scattered into 2 excited atoms, hence the subscript “22”.

The collision integral $C_{12}[f]$ involving collisions between thermally-excited atoms and one condensate atom comes from the terms involving h_{11} in (5.18). One can show that, for slowly varying external disturbances [15], one has

$$h_{11}(\mathbf{p}, \omega; \mathbf{R}, T) = n_c(\mathbf{R}, T)\delta(\mathbf{p})\delta(\omega)(2\pi)^4. \quad (5.20)$$

Using (5.20), the contribution $C_{12}[f]$ is found to be given by

$$\begin{aligned} C_{12}[f] &= 4\pi g^2 n_c(\mathbf{R}, T) \int \frac{d\mathbf{p}_1 d\mathbf{p}_2 d\mathbf{p}_3}{(2\pi)^3} [\delta(\mathbf{p} - \mathbf{p}_1) - \delta(\mathbf{p} - \mathbf{p}_2) - \delta(\mathbf{p} - \mathbf{p}_3)] \\ &\delta(\tilde{\epsilon}_{p_1} - \tilde{\epsilon}_{p_2} - \tilde{\epsilon}_{p_3}) \delta(\mathbf{p}_1 - \mathbf{p}_2 - \mathbf{p}_3) [(1+f_1)f_3f_2 - f_1(1+f_3)(1+f_2)]. \end{aligned} \quad (5.21)$$

C_{12} describes collisions where we go from 1 excited and one condensate atom to 2 excited atoms, hence the subscript “12”. Inserting (5.19) and (5.21) into (5.17), we arrive at a kinetic equation which is equivalent to that derived by ZNG [17], namely

$$\begin{aligned} &\left[\frac{\partial}{\partial T} + \nabla_{\mathbf{p}} [\tilde{\epsilon}_p + \mathbf{p} \cdot \mathbf{v}_s] \cdot \nabla_{\mathbf{R}} - \nabla_{\mathbf{R}} [\tilde{\epsilon}_p + \mathbf{p} \cdot \mathbf{v}_s] \cdot \nabla_{\mathbf{p}} \right] f(\mathbf{p}, \mathbf{R}, T) \\ &= C_{22}[f(\mathbf{p}, \mathbf{R}, T)] + C_{12}[f(\mathbf{p}, \mathbf{R}, T)]. \end{aligned} \quad (5.22)$$

The left-hand side of (5.22) is the expected streaming term in the frame of reference in which the superfluid velocity is zero (ZGN have derived the equivalent of (5.22) in the lab frame). For a uniform gas at elevated temperatures, (5.22) was first obtained by Kirkpatrick and Dorfman [28] using a completely different approach.

5.4 Generalized Gross-Pitaevskii equation

We finally turn to the derivation of an equation of motion for the order parameter. We first write equation (5.4) for $\Phi(\mathbf{r}, t)$ in the local rest frame, namely

$$\left[i \frac{\partial}{\partial t} - \frac{\partial \theta(1)}{\partial t} + \frac{1}{2m} [\nabla_{\mathbf{r}} + im\mathbf{v}_s(1)]^2 + \mu_0 - U_{ext}(\mathbf{r}) - g(2\tilde{n}(1) + n_c(1)) \right] \Phi(1)$$

$$= \int_{-\infty}^t d\bar{t} [S_{11}^> - S_{11}^<] (\mathbf{r} - \bar{\mathbf{r}}, t - \bar{t}; (\mathbf{r} + \bar{\mathbf{r}})/2, (t + \bar{t})/2) \Phi(\bar{\mathbf{r}}, \bar{t}). \quad (5.23)$$

We assume, as usual in the KB approach, that the S_{11} correlation function related to (3.9) is dominated by small values of the relative coordinates $(\mathbf{r} - \bar{\mathbf{r}}, t - \bar{t})$, and therefore we can approximate $S_{11}^<$ in (5.23) by $S^>(\mathbf{r} - \bar{\mathbf{r}}, t - \bar{t}; \mathbf{r}, t)$. For the same reason, we approximate the macroscopic wavefunction in (5.23) by $\Phi(\bar{1}) \simeq \Phi(1) \equiv \sqrt{n_c(\mathbf{r}, t)}$. Hence we can further approximate (5.23) by

$$\begin{aligned} & \left[i \frac{\partial}{\partial t} - \frac{\partial \theta(1)}{\partial t} + \frac{1}{2m} [\nabla_{\mathbf{r}} + im\mathbf{v}_s(1)]^2 + \mu_0 - U_{ext}(\mathbf{r}) - g(2\tilde{n}(1) + n_c(1)) \right] \Phi(1) \\ &= \int_{-\infty}^t d\bar{\mathbf{r}} d\bar{t} [S_{11}^> - S_{11}^<] (\mathbf{r} - \bar{\mathbf{r}}, t - \bar{t}; \mathbf{r}, t) \Phi(\mathbf{r}, t). \end{aligned} \quad (5.24)$$

We can rewrite (5.24) as (renaming $(\mathbf{r}, t) \rightarrow (\mathbf{R}, T)$)

$$\begin{aligned} & \left[i \frac{\partial}{\partial T} - \frac{\partial \theta(\mathbf{R}, T)}{\partial T} + \frac{1}{2m} [\nabla_{\mathbf{R}} + im\mathbf{v}_s(\mathbf{R}, T)]^2 + \mu_0 - U_{ext}(\mathbf{R}) \right. \\ & \left. - g(2\tilde{n}(\mathbf{R}, T) + n_c(\mathbf{R}, T)) \right] \Phi(\mathbf{R}, T) \\ &= \Phi(\mathbf{R}, T) \int \frac{d\mathbf{p} d\omega}{(2\pi)^4} [S_{11}^> - S_{11}^<] (\mathbf{p}, \omega; \mathbf{R}, T) \int_{-\infty}^T d\bar{\mathbf{r}} d\bar{t} e^{i\mathbf{p}(\mathbf{R}-\bar{\mathbf{r}}) - i\omega(T-\bar{t})}. \end{aligned} \quad (5.25)$$

In the Beliaev approximation, $S_{11}(\mathbf{p}, \omega; \mathbf{R}, T)$ is found to be given by

$$\begin{aligned} S_{11}^<(\mathbf{p}, \omega; \mathbf{R}, T) &= 2g^2 \int \frac{d\mathbf{p}_i d\omega_i}{(2\pi)^{12}} (2\pi)^4 \delta(\omega + \omega_1 - \omega_2 - \omega_3) \delta(\mathbf{p} + \mathbf{p}_1 - \mathbf{p}_2 - \mathbf{p}_3) \\ &\times \tilde{g}_{11}^>(\mathbf{p}_1, \omega_1; \mathbf{R}, T) \tilde{g}_{11}^<(\mathbf{p}_2, \omega_2; \mathbf{R}, T) \tilde{g}_{11}^<(\mathbf{p}_3, \omega_3; \mathbf{R}, T). \end{aligned} \quad (5.26)$$

In evaluating the right-hand side of (5.25), we have used the following identity

$$\lim_{\delta \rightarrow 0^+} \int_{-\infty}^T d\bar{t} e^{-i(\omega + i\delta)(T-\bar{t})} \simeq \pi \delta(\omega) + iP \left(\frac{1}{\omega} \right), \quad (5.27)$$

in conjunction with (5.14) and (5.15).

We thus finally obtain a generalized Gross-Pitaevskii equation of the following form

$$\begin{aligned} & i \frac{\partial \sqrt{n_c(\mathbf{R}, T)}}{\partial T} = \left[\frac{\partial \theta(\mathbf{R}, T)}{\partial T} - \frac{1}{2m} [\nabla_{\mathbf{R}} + im\mathbf{v}_s(\mathbf{R}, T)]^2 - \mu_0 \right. \\ & \left. + U_{ext}(\mathbf{R}) + g[2\tilde{n}(\mathbf{R}, T) + n_c(\mathbf{R}, T)] - iR(\mathbf{R}, T) \right] \sqrt{n_c(\mathbf{R}, T)}. \end{aligned} \quad (5.28)$$

The new dissipative term R in the GP equation is seen to be related to the C_{12} collision term in the non-condensate kinetic equation in (5.22), namely [17]

$$R(\mathbf{R}, T) \equiv \int \frac{d\mathbf{p}}{(2\pi)^3} \frac{C_{12}[f(\mathbf{p}, \mathbf{R}, T)]}{2n_c(\mathbf{R}, T)}. \quad (5.29)$$

This term describes the damping of condensate amplitude fluctuations due to collisions with the atoms in the thermal cloud. The appearance of this dissipative term in (5.28) is expected, since the C_{12} collisions can change the number of atoms in the condensate and hence can modify the magnitude of the condensate macroscopic wavefunction $\sqrt{n_c(\mathbf{r}, t)}$. The real part of the right-hand side of (5.25) is omitted since (as mentioned in Ch. 1), we only work to first order in the interaction as far as renormalized energies are concerned. If we transform back into the lab frame (recall that in the lab frame, one has $\Phi = \sqrt{n_c}e^{i\theta}$), (5.28) reduces to the time-dependent generalized Gross-Pitaevskii equation for $\Phi(\mathbf{R}, T)$ discussed by ZNG [17].

5.5 Conclusions

In this chapter, we have shown how the elegant and powerful Kadanoff-Baym formalism can be used in a trapped Bose gas to derive a generalized Gross-Pitaevskii equation for the condensate wavefunction as well as a quantum kinetic equation for distribution function for the non-condensate atoms.

We have limited ourselves to the case of finite temperatures, where the single-particle spectral density can be approximated by (5.14). In the next chapter, we show how this formalism can be extended to also deal with the case of low temperatures, where both the diagonal and the off-diagonal components of the single-particle spectral density $a_{\alpha\beta}$ must be kept. This means that the quasiparticles are now described by a Bogoliubov-Popov spectrum, in place of the particle-like Hartree-Fock spectrum in (5.14) valid at high temperatures.

Chapter 6

Quasiparticle kinetic equation at low temperatures

6.1 Introduction

In Ch.5, we have shown how one can generalize the Gross-Pitaevskii equation coupled to the kinetic equation for the distribution function for thermally excited atoms to study a trapped Bose-condensed system at temperatures $T > 0.5T_{BEC}$. However, the kinetic equation (5.22) is valid in the semiclassical limit only: it assumes that the thermal energy is much greater than the spacing between the trap SHO energy levels ($k_B T \gg \hbar\omega_0$, where ω_0 is the harmonic trap frequency) as well as the average interaction energy ($k_B T \gg gn$). ZNG have first have given a detailed derivation [17] of (5.22) at finite temperatures for a trapped Bose gas using the approach of Kirkpatrick and Dorfman [28], who considered a uniform Bose gas. From various studies [2, 40], one expects that the ZNG type of model based on particle-like excitations will be adequate for temperatures higher than $T \geq 0.5T_{BEC}$ and perhaps much lower. However, the ZNG model ultimately breaks down at very low temperatures because the thermal excitations on which it is based do not include the collective (or phonon) part of the Bogoliubov spectrum $E_p = \sqrt{(p^2/2m)^2 + gn_cp^2/m}$.

To deal with this generalization, one has to formulate a kinetic theory in terms of such Bogoliubov quasiparticle excitations. In this chapter, we do this by using the Kadanoff-Baym (KB) non-equilibrium Green's functions method to derive a generalized kinetic equation for the thermally excited Bogoliubov quasiparticles. We work within the second-order Beliaev-Popov approximation [43], rather than the full Beliaev approximation. We expect that our equations of motion will be useful in describing the very low temperature dynamics of trapped Bose gases.

It is important to emphasize that, although our analysis involves the non-equilibrium generalization of the Beliaev second-order self-energy used for systems in thermal equilibrium, our work is quite different from the recent papers that calculate the poles of equilibrium Green's functions within the second-order Beliaev approximation [43, 44, 45]. In a very elegant formulation, Giorgini [44] has calculated the quasiparticle energy and damping at finite temperatures in a uniform Bose gas in the collisionless regime by linearizing the equations of motion for fluctuations using the first-order dynamic Hartree-Fock-Bogoliubov approximation. This leads, as expected, to the same excitation spectrum found by Hua Shi and Griffin [43], who calculated directly the poles of the single-particle equilibrium Green's functions using the second-order Beliaev self-energy diagram contributions.

In this chapter, we use the second-order Beliaev approximation to discuss the *non-equilibrium dynamics* of a trapped Bose-condensed gas at finite temperatures. We use the second-order Beliaev self-energies with the lower order Bogoliubov excitation spectrum, including off-diagonal single-particle propagators, but we still omit the pair anomalous correlation functions \tilde{m} . This last assumption defines what we call the Bogoliubov-Popov approximation [24]. In this chapter, we are primarily interested in the damping arising from the collisions between atoms. We do not explicitly calculate the second-order corrections in g to the quasiparticle energy spectrum or to the condensate chemical potential. These corrections are associated with the real parts of the second-order Beliaev

self-energies. One effect of including the real part of the second-order self-energies in an improved kinetic equation is the appearance of a renormalized quasiparticle energy [see Eq. (6.3.77) in Ref. [20]]. In this context, the original work of Kane and Kadanoff [15] is superior to our present study since they use the KB method to derive equations of motion for the order parameter and non-condensate Green's functions keeping the real parts of their (simplified) second-order self energies.

In this chapter, we derive a kinetic equation for the distribution function for the thermally excited quasiparticles, as well as a generalized equation for the Bose condensate order parameter. The kinetic equation we obtain is the same as the one given for a uniform Bose gas by Eckern [27] in 1984, and discussed in great detail by Kirkpatrick and Dorfman (KD) in 1985 [28]. The KD derivation was based on a direct extension of the traditional method used to derive kinetic equations for classical gases, which we do not believe is as physically transparent as the KB approach [14]. An additional major difference from our work is that is that KD did not explicitly derive a GP-like equation of motion for the condensate degree of freedom. In a non-Bose condensed uniform gas, similar kinetic equations are derived in Ref. [20] using the related Keldysh formalism.

The present chapter is a natural generalization of work presented in two earlier chapters. In Ch. 4, we have derived kinetic equations within the full Hartree-Fock-Bogoliubov approximation but ignored collisions. In Ch. 5, we derived a kinetic equation including collisions, based on a simple Hartree-Fock particle-like spectrum and hence not valid at very low temperatures.

In Section 6.2, we review the general equations of motion for the non-equilibrium Green's functions describing the non-condensate atoms as well as the equation of motion for the macroscopic order parameter. In Section 6.3, we transform these equations to a local rest frame of reference (see also Ch. 5) where the order parameter $\Phi(\mathbf{r}, t)$ is real, i.e., to a frame where the local superfluid velocity is zero. This naturally introduces the superfluid velocity and the local chemical potential as the spatial and time derivatives,

respectively, of the phase of the order parameter [15]. We then specialize our equations of motion for the non-equilibrium Green's functions to the case of slowly varying external perturbations. We make the key assumption that all correlation functions vary slowly as a function of center-of-mass space-time coordinates but are dominated by small values of the relative coordinates. Following the Kadanoff-Baym approach [14, 15, 16], we derive a generalized quantum Boltzmann equation for the quasiparticle distribution function $f(\mathbf{p}, \omega; \mathbf{R}, T)$ which is now frequency-dependent.

In Section 6.4, we use this generalized KB quantum Boltzmann equation to derive the kinetic equation for quasiparticles at low temperatures, with the collision integral describing collisions between quasiparticles in the Bogoliubov-Popov approximation. Finally, in Section 6.6, we derive a generalized Gross-Pitaevskii equation.

6.2 Equations of motion for non-equilibrium Green's functions

In this chapter, we work with the second-order self-energy Σ_c as given by the Beliaev (gapless) approximation (see Section 3.1 and Refs. [43, 24, 11]). The advantage of the Beliaev approximation is that the non-condensate single-particle Green's function exhibits a quasiparticle spectrum with the correct form [11]. In the second-order Beliaev approximation, the second-order self-energy Σ_c^{\gtrless} is given by (3.38), namely

$$\begin{aligned} \hat{\Sigma}_c^{\gtrless}(1, 1') = & - \frac{1}{2}v(13)v(21')\hat{g}^{\gtrless}(11') \left[\hat{g}^{\gtrless}(23)\hat{g}^{\gtrless}(32) + \hat{g}^{\gtrless}(23)\hat{h}(32) + \hat{h}(23)\hat{g}^{\gtrless}(32) \right] \\ & - v(13)v(21')\hat{g}^{\gtrless}(12) \left[\hat{g}^{\gtrless}(23)\hat{h}(31') + \hat{h}(23)\hat{g}^{\gtrless}(31') + \hat{g}^{\gtrless}(23)\hat{g}^{\gtrless}(31') \right] \\ & - \frac{1}{2}v(13)v(21') \left[\hat{h}(11')\hat{g}^{\gtrless}(23)\hat{g}^{\gtrless}(32) + 2\hat{h}(12)\hat{g}^{\gtrless}(23)\hat{g}^{\gtrless}(31') \right]. \end{aligned} \quad (6.1)$$

For convenience, we write again the equations of motion for the non-condensate 2×2 matrix propagators $\tilde{g}_{\alpha\beta}$ as given in (3.16) and (3.17),

$$\begin{aligned} & \int d\bar{1} \left[\hat{g}_0^{-1}(1, \bar{1}) - \hat{\Sigma}^{HF}(1, \bar{1}) \right] \hat{g}^{\lessgtr}(1, \bar{1}, 1') \\ &= \int_{-\infty}^{t_1} d\bar{1} \hat{\Gamma}(1, \bar{1}) \hat{g}^{\lessgtr}(1, \bar{1}, 1') - \int_{-\infty}^{t_1'} d\bar{1} \hat{\Sigma}_c^{\lessgtr}(1, \bar{1}) \hat{a}(1, \bar{1}, 1'), \end{aligned} \quad (6.2)$$

and

$$\begin{aligned} & \int d\bar{1} \hat{g}^{\lessgtr}(1, \bar{1}) \left[\hat{g}_0^{-1}(\bar{1}, 1') - \hat{\Sigma}^{HF}(\bar{1}, 1') \right] \\ &= \int_{-\infty}^{t_1} d\bar{1} \hat{a}(1, \bar{1}) \hat{\Sigma}_c^{\lessgtr}(\bar{1}, 1') - \int_{-\infty}^{t_1'} d\bar{1} \hat{g}^{\lessgtr}(1, \bar{1}) \hat{\Gamma}(\bar{1}, 1'). \end{aligned} \quad (6.3)$$

As before, the equation for the condensate in terms of the 2-component order parameter is given by

$$\int d\bar{1} \hat{g}_0^{-1}(1, \bar{1}) \hat{G}_{1/2}(\bar{1}) = \sqrt{-i} \hat{\eta}(1) + \sqrt{-i} \hat{\eta}_{ext}(1), \quad (6.4)$$

where the condensate source function η is defined by the three-field correlation function in (3.9). The exact coupled equations of motion (6.2), (6.3) and (6.4) are the starting point of our analysis in this chapter. The external generating fields U and η_{ext} will be left implicit in the rest of this chapter. As before, we gauge transform (6.2), (6.3) and (6.4) in the local rest frame. We recall that, after the gauge transformation defined in (4.12), equations (6.2), (6.3) and (6.4) remain unchanged in form if the g_0^{-1} is replaced by (5.5). This gauge transformation changes the momentum $\mathbf{p} \rightarrow \mathbf{p} - m\mathbf{v}_s$, as expected for the momentum in the local rest frame (when we Fourier transform, the gradient operator in (4.13) becomes the momentum).

6.3 Generalized kinetic equation

In Section 6.2, we have written down the equations of motion (6.2) and (6.3) for the non-equilibrium real-time Green's functions. We now want to use these to derive a kinetic equation for an appropriately defined quasiparticle distribution function. We recall that

these are 2×2 matrix equations. In addition, (6.2) involves differential operators with the respect to the coordinates (\mathbf{r}, t) and (6.3) involves derivatives with respect to coordinates (\mathbf{r}', t') . Since our single-particle Green's functions are functions of both coordinates $(1, 1')$, one has to find a way to combine both equations to derive a single kinetic equation for a quasiparticle distribution function. We will discuss later how these Green's functions can be related to the quasiparticle distribution function. However, we emphasize that we need both (6.2) and (6.3) to derive a kinetic equation for quasiparticles.

Using the same physics as in earlier chapters, we now write (6.2) and (6.3) in terms of the center-of-mass and relative coordinates [19, 15, 16, 26], and take the trace of the resulting matrix equation to obtain:

$$\begin{aligned} & \hat{\mathcal{L}}_{11} \tilde{g}_{11}^<(\mathbf{r}, t; \mathbf{R}, T) + \hat{\mathcal{L}}_{22} \tilde{g}_{22}^<(\mathbf{r}, t; \mathbf{R}, T) \\ &= g \left[\left((\mathbf{r} \cdot \nabla_{\mathbf{R}} + t \frac{\partial}{\partial T}) m \right) \tilde{g}_{21}^< + \left(\left(\mathbf{r} \cdot \nabla_{\mathbf{R}} + t \frac{\partial}{\partial T} \right) m^* \right) \tilde{g}_{12}^< \right] \\ &+ \int_{-\infty}^{\infty} d\bar{\mathbf{r}} d\bar{t} \text{Tr} \left[\Sigma_c^>(\mathbf{r} - \bar{\mathbf{r}}, t - \bar{t}) \hat{g}^<(\bar{\mathbf{r}}, \bar{t}) - \Sigma_c^<(\mathbf{r} - \bar{\mathbf{r}}, t - \bar{t}) \hat{g}^>(\bar{\mathbf{r}}, \bar{t}) \right]. \end{aligned} \quad (6.5)$$

For simplicity, the (\mathbf{R}, T) dependence of the \tilde{g} , Σ and m is left implicit. The new operators $\hat{\mathcal{L}}_{11}$ and $\hat{\mathcal{L}}_{22}$ on the left-hand side of (6.5) are defined as

$$\begin{aligned} \hat{\mathcal{L}}_{11} &\equiv i \frac{\partial}{\partial T} + (\mathbf{r} \cdot \nabla_{\mathbf{R}} + t \frac{\partial}{\partial T}) (\mu(\mathbf{R}, T) - U_{eff}(\mathbf{R}, T)) + \frac{1}{m} \nabla_{\mathbf{R}} \cdot \nabla_{\mathbf{r}} \\ &+ i \left((\mathbf{r} \cdot \nabla_{\mathbf{R}} + t \frac{\partial}{\partial T}) \mathbf{v}_s(\mathbf{R}, T) \right) \cdot \nabla_{\mathbf{r}} + i \mathbf{v}_s(\mathbf{R}, T) \cdot \nabla_{\mathbf{R}} + i \nabla_{\mathbf{R}} \cdot \mathbf{v}_s(\mathbf{R}, T) \\ \hat{\mathcal{L}}_{22} &\equiv -i \frac{\partial}{\partial T} + (\mathbf{r} \cdot \nabla_{\mathbf{R}} + t \frac{\partial}{\partial T}) (\mu(\mathbf{R}, T) - U_{eff}(\mathbf{R}, T)) + \frac{1}{m} \nabla_{\mathbf{R}} \cdot \nabla_{\mathbf{r}} \\ &- i \left((\mathbf{r} \cdot \nabla_{\mathbf{R}} + t \frac{\partial}{\partial T}) \mathbf{v}_s(\mathbf{R}, T) \right) \cdot \nabla_{\mathbf{r}} - i \mathbf{v}_s(\mathbf{R}, T) \cdot \nabla_{\mathbf{R}} - i \nabla_{\mathbf{R}} \cdot \mathbf{v}_s(\mathbf{R}, T), \end{aligned} \quad (6.6)$$

where the effective dynamic HF field $U_{eff}(\mathbf{R}, T)$ is given by

$$U_{eff}(\mathbf{R}, T) = U_{ext}(\mathbf{R}) + 2g(n_c(\mathbf{R}, T) + \tilde{n}(\mathbf{R}, T)). \quad (6.7)$$

We emphasize that, in the expansion for the small values of relative coordinates (\mathbf{r}, t) , we did not keep all terms of order $\partial/\partial T$ and $\nabla_{\mathbf{R}}$ in (6.5). These additional terms that we have

neglected contribute to the many-body renormalization effects, i.e. how the two-body interactions change the dispersion relation of the quasiparticles. Such corrections, second order in g , involve the real part of the second-order Beliaev self-energies. The Bogoliubov-Popov quasiparticle approximation that we use for the Beliaev spectral densities $a_{\alpha\beta}$ defined in (3.18) leaves out such second-order renormalization effects. In the present chapter, we concentrate on the damping associated with the collisional self-energies Σ_c^{\lessgtr} on the right-hand side of (6.5). For further discussion of the KB formalism related to going past the simple Bogoliubov quasiparticle approximation, see Ch. 9 of Ref. [14] and Ch. 6 in Ref. [20]. We also note that Kane and Kadanoff [15, 16] have, albeit working within a truncated second-order self-energy approximation, derived equations of motion which keep the extra terms left out of (6.5) and (6.6). Thus they include many-body renormalization effects to second order in g . Their work is thus a more sophisticated implementation of the Kadanoff-Baym formalism than our present discussion.

A double Fourier transform of (6.5) gives

$$\begin{aligned} & \hat{\mathcal{L}}_{11}\tilde{g}_{11}^{\lessgtr} + \hat{\mathcal{L}}_{22}\tilde{g}_{22}^{\lessgtr} - g\nabla_{\mathbf{R}}m^* \cdot \nabla_{\mathbf{p}}\tilde{g}_{12}^{\lessgtr} - g\nabla_{\mathbf{R}}m \cdot \nabla_{\mathbf{p}}\tilde{g}_{21}^{\lessgtr} + g\frac{\partial m^*}{\partial T}\frac{\partial\tilde{g}_{12}^{\lessgtr}}{\partial\omega} + g\frac{\partial m}{\partial T}\frac{\partial\tilde{g}_{21}^{\lessgtr}}{\partial\omega} \\ & = \text{Tr} \left(\hat{\Sigma}_c^{\gtr}(\mathbf{p}, \omega; \mathbf{R}, T) \hat{g}^{\lessgtr}(\mathbf{p}, \omega; \mathbf{R}, T) - \hat{\Sigma}_c^{\lessgtr}(\mathbf{p}, \omega; \mathbf{R}, T) \hat{g}^{\gtr}(\mathbf{p}, \omega; \mathbf{R}, T) \right), \end{aligned} \quad (6.8)$$

with $\tilde{g}_{\alpha\beta}^{\lessgtr} \equiv \tilde{g}_{\alpha\beta}^{\lessgtr}(\mathbf{p}, \omega; \mathbf{R}, T)$ and

$$\begin{aligned} \hat{\mathcal{L}}_{11} &= \frac{\partial}{\partial T} + \nabla_{\mathbf{p}}[\tilde{\epsilon}_p + \mathbf{p} \cdot \mathbf{v}_s] \cdot \nabla_{\mathbf{R}} - \nabla_{\mathbf{R}}[\tilde{\epsilon}_p + \mathbf{p} \cdot \mathbf{v}_s] \cdot \nabla_{\mathbf{p}} + \frac{\partial}{\partial T}[\tilde{\epsilon}_p + \mathbf{p} \cdot \mathbf{v}_s] \frac{\partial}{\partial\omega} \\ \hat{\mathcal{L}}_{22} &= -\frac{\partial}{\partial T} + \nabla_{\mathbf{p}}[\tilde{\epsilon}_p - \mathbf{p} \cdot \mathbf{v}_s] \cdot \nabla_{\mathbf{R}} - \nabla_{\mathbf{R}}[\tilde{\epsilon}_p - \mathbf{p} \cdot \mathbf{v}_s] \cdot \nabla_{\mathbf{p}} + \frac{\partial}{\partial T}[\tilde{\epsilon}_p - \mathbf{p} \cdot \mathbf{v}_s] \frac{\partial}{\partial\omega}. \end{aligned} \quad (6.9)$$

Here $\tilde{\epsilon}_p$ is defined by

$$\tilde{\epsilon}_p(\mathbf{R}, T) = \frac{p^2}{2m} + U_{ext}(\mathbf{R}) + 2gn(\mathbf{R}, T) - \mu_c(\mathbf{R}, T). \quad (6.10)$$

In Section 6.6, we shall find that the condensate chemical potential $\mu_c(\mathbf{R}, T)$ is given by

$$\mu_c = -\frac{\nabla_{\mathbf{R}}^2 \sqrt{n_c(\mathbf{R}, T)}}{2m\sqrt{n_c(\mathbf{R}, T)}} + U_{ext}(\mathbf{R}) + g[2\tilde{n}(\mathbf{R}, T) + n_c(\mathbf{R}, T)]. \quad (6.11)$$

The result in (6.8) gives an equation closely related to the quasiparticle kinetic equation we are trying to derive.

A kinetic equation for thermally excited atoms in a trapped Bose gas can be written in terms of distribution functions for the atoms directly or for quasiparticle excitations. At high temperatures, we were able to transform the equations of motion for non-equilibrium Green's functions into a kinetic equation for a distribution function $f(\mathbf{p}, \mathbf{R}, T)$ directly describing non-condensate atoms with a Hartree-Fock spectrum. If one wants to use a more realistic spectrum valid at low temperatures, it is much more convenient to work within a quasiparticle picture. In the theory of Bose-condensed trapped gases, one introduces quasiparticles by expressing the quantum field operators for the non-condensate atoms as a coherent superposition of creation and annihilation operators for Bose quasiparticles, with the weights given by the usual Bogoliubov amplitudes u and v

$$\tilde{\psi}(\mathbf{R}, T) \equiv \sum_i \left[u_i(\mathbf{R}) \hat{\alpha}_i e^{-iE_i T/\hbar} + v_i^*(\mathbf{R}) \hat{\alpha}_i^\dagger e^{iE_i T/\hbar} \right]. \quad (6.12)$$

In the semiclassical approximation, (6.12) becomes

$$\tilde{\psi}(\mathbf{R}, T) \equiv \int \frac{d\mathbf{p}}{(2\pi)^3} \left[u_p(\mathbf{R}) \hat{\alpha}_p e^{-iE_p T/\hbar} + v_p^*(\mathbf{R}) \hat{\alpha}_p^\dagger e^{iE_p T/\hbar} \right]. \quad (6.13)$$

Here $\hat{\alpha}_p^\dagger$ and $\hat{\alpha}_p$ are the Bogoliubov quasiparticle creation and annihilation operators, respectively, which obey the usual Bose commutation relations. One can see that creating an atom with momentum \mathbf{p} is equivalent to creating a quasiparticle with momentum \mathbf{p} with amplitude u_p and at the same time, destroying a quasiparticle with momentum $-\mathbf{p}$ and amplitude v_p . The quasiparticle distribution function is given by the statistical average of the quasiparticle operators, i.e., $f(\mathbf{p}) \equiv \langle \hat{\alpha}_p^\dagger \hat{\alpha}_p \rangle$. We recall [19, 26] that distribution function for atoms $f_{at}(\mathbf{p}, \omega; \mathbf{R}, T)$ is directly related to the diagonal Green's function, namely

$$f_{at}(\mathbf{p}, \mathbf{R}, T) = \int \frac{d\omega}{2\pi} (-i) \tilde{g}_{11}^<(\mathbf{p}, \omega; \mathbf{R}, T). \quad (6.14)$$

In terms of quantum field operators, $\tilde{g}_{11}^<$ is given by [see Ch. 9 of Ref.[14]]

$$-i \tilde{g}_{11}^<(\mathbf{p}, \omega; \mathbf{R}, T) = \int d\mathbf{r} dt e^{-i\mathbf{p} \cdot \mathbf{r} + i\omega t} \langle \tilde{\psi}^\dagger(\mathbf{R} - \frac{\mathbf{r}}{2}, T - \frac{t}{2}) \tilde{\psi}(\mathbf{R} + \frac{\mathbf{r}}{2}, T + \frac{t}{2}) \rangle. \quad (6.15)$$

The usual Boltzmann equation is expressed in terms of the Wigner distribution function $f_W(\mathbf{p}, \mathbf{R}, T)$ [14, 19]. This limits the description to the semiclassical approximation, since it is assumed that the position and momentum of the atoms can be defined simultaneously. In order to use this kind of distribution function for quantum systems, it is necessary to perform some type of averaging in order to remove effects due to the uncertainty principle. In our present work, we want to derive a kinetic equation for the quasiparticles which is valid at all temperatures and therefore the semiclassical approximation will no longer be valid. To include the quantum effects, we introduce the *quasiparticle* distribution function $f(\mathbf{p}, \omega; \mathbf{R}, T)$ with an additional variable ω in the following way [15, 20, 21, 31]

$$\begin{aligned} -i\tilde{g}_{\alpha\beta}^<(\mathbf{p}, \omega; \mathbf{R}, T) &\equiv a_{\alpha\beta}(\mathbf{p}, \omega; \mathbf{R}, T)f(\mathbf{p}, \omega; \mathbf{R}, T) \\ -i\tilde{g}_{\alpha\beta}^>(\mathbf{p}, \omega; \mathbf{R}, T) &\equiv a_{\alpha\beta}(\mathbf{p}, \omega; \mathbf{R}, T)[1 + f(\mathbf{p}, \omega; \mathbf{R}, T)], \end{aligned} \quad (6.16)$$

where the spectral densities $a_{\alpha\beta}$ are defined in (3.18). Using the Bogoliubov-Popov approximation for the spectral density [see (6.22)], one can use (6.16) to obtain the well-known relation [40] between the quasiparticle distribution function $f(\mathbf{p}, \omega; \mathbf{R}, T)$ and the atom distribution function $f_{at}(\mathbf{p}, \mathbf{R}, T)$ in (6.14), namely

$$f_{at}(\mathbf{p}, \mathbf{R}, T) = \left(u_p^2(\mathbf{R}, T) + v_p^2(\mathbf{R}, T) \right) f(\mathbf{p}, \omega = E_p + \mathbf{v}_s \cdot \mathbf{p}; \mathbf{R}, T) + v_p^2(\mathbf{R}, T). \quad (6.17)$$

One can show, using (6.2) and (6.3), that the spectral function $a_{\alpha\beta}$ in (3.18) satisfies the equation

$$\hat{\mathcal{L}}_{11}a_{11} + \hat{\mathcal{L}}_{22}a_{22} - g\nabla_{\mathbf{R}}m^* \cdot \nabla_{\mathbf{p}}a_{12} - g\nabla_{\mathbf{R}}m \cdot \nabla_{\mathbf{p}}a_{21} - g\frac{\partial m^*}{\partial T}\frac{\partial a_{12}}{\partial \omega} - g\frac{\partial m}{\partial T}\frac{\partial a_{21}}{\partial \omega} = 0. \quad (6.18)$$

Using (6.16) and (6.18), one can rewrite the kinetic equation (6.8) for $\tilde{g}_{\alpha\beta}^>$ to obtain finally a new kinetic equation specifically for the quasiparticle distribution function $f(\mathbf{p}, \omega; \mathbf{R}, T)$ in the following form:

$$a_{11}\hat{\mathcal{L}}_{11}f + a_{22}\hat{\mathcal{L}}_{22}f - a_{12}g\nabla_{\mathbf{R}}m^* \cdot \nabla_{\mathbf{p}}f - a_{21}g\nabla_{\mathbf{R}}m \cdot \nabla_{\mathbf{p}}f$$

$$+ a_{12} \frac{\partial m^*}{\partial T} \frac{\partial f}{\partial \omega} + a_{21} \frac{\partial m}{\partial T} \frac{\partial f}{\partial \omega} = f \text{Tr}(\Sigma^>\hat{a}) - (1 + f) \text{Tr}(\Sigma^<\hat{a}). \quad (6.19)$$

We note that (6.19) includes terms involving $\partial/\partial\omega$. The additional variable ω in $f(\mathbf{p}, \omega; \mathbf{R}, T)$ results in new streaming terms on the left side of (6.19). These terms are not present in the semiclassical kinetic equation (which is obtained from (6.19) by integrating over ω), which remind us that the terms involving $\partial/\partial\omega$ are of quantum origin [20, 21, 31].

We emphasize that (6.19) is the most general form for a kinetic equation for the quasiparticle distribution function f within the KB formalism. To derive (6.19), we have assumed only that the external disturbances vary slowly in space and time, and therefore all relevant physical quantities vary slowly as function of center-of-mass coordinates (\mathbf{R}, T) defined in (4.8). Our other assumption is that one can introduce a quasiparticle distribution function f through the definition in (6.16). Of course, at this stage, one could say that we are only replacing one unknown function with another. Such a generalized kinetic equation for a Bose-condensed system (6.19) was first discussed by Kane in his unpublished Ph.D. thesis [16].

From (6.19), we see that the general structure of the collision integral I has the following form [15, 16, 42]

$$I[f(\mathbf{p}, \mathbf{R}, T)] \equiv \int \frac{d\omega}{2\pi} [f \text{Tr}(\hat{\Sigma}^>\hat{a}) - (1 + f) \text{Tr}(\hat{\Sigma}^<\hat{a})]. \quad (6.20)$$

Using (6.16) in the general expression for the non-equilibrium Beliaev self-energies Σ^{\lessgtr} given in (6.1), one can prove that the collision integral given by (6.20) conserves momentum (see Appendix of Ref. [29] for details),

$$\int d\mathbf{p} \mathbf{p} I[f(\mathbf{p}, \mathbf{R}, T)] = 0. \quad (6.21)$$

One can also prove that (6.20) conserves energy as well. To prove this, however, we need to work within a specific approximation for the single-particle spectral density $a_{\alpha\beta}(\mathbf{p}, \omega; \mathbf{R}, T)$.

6.4 Kinetic equation in the Bogoliubov-Popov approximation

In this section, we approximate the general KB kinetic equation in (6.19) to derive a quasiparticle kinetic equation, working within the Bogoliubov-Popov approximation (with $\tilde{m} = 0$). More precisely, this means that we use the spectral densities with the Bogoliubov-Popov quasiparticle excitation energies [14, 16, 10]

$$\begin{aligned}
a_{11}(\mathbf{p}, \omega; \mathbf{R}, T) &= 2\pi[u_p^2 \delta(\omega - \mathbf{v}_s \cdot \mathbf{p} - E_p) - v_p^2 \delta(\omega - \mathbf{v}_s \cdot \mathbf{p} + E_p)] \\
a_{12}(\mathbf{p}, \omega; \mathbf{R}, T) &= -2\pi u_p v_p [\delta(\omega - \mathbf{v}_s \cdot \mathbf{p} - E_p) - \delta(\omega - \mathbf{v}_s \cdot \mathbf{p} + E_p)] \\
a_{21}(\mathbf{p}, \omega; \mathbf{R}, T) &= a_{12}(\mathbf{p}, \omega; \mathbf{R}, T) \\
a_{22}(\mathbf{p}, \omega; \mathbf{R}, T) &= -a_{11}(-\mathbf{p}, -\omega; \mathbf{R}, T).
\end{aligned} \tag{6.22}$$

Here the Bose-coherence factors $u(\mathbf{R}, T)$ and $v(\mathbf{R}, T)$ are given by [10]

$$u_p^2(\mathbf{R}, T) = \frac{\tilde{\epsilon}_p(\mathbf{R}, T) + E_p(\mathbf{R}, T)}{2E_p(\mathbf{R}, T)}, \quad u_p^2 - v_p^2 = 1, \quad u_p v_p = \frac{gn_c(\mathbf{R}, T)}{2E_p(\mathbf{R}, T)} \tag{6.23}$$

and the quasiparticle energy E_p is given by

$$E_p(\mathbf{R}, T) = \sqrt{\tilde{\epsilon}_p^2(\mathbf{R}, T) - (gn_c(\mathbf{R}, T))^2}. \tag{6.24}$$

It is important to note that the spectral densities in (6.22) are consistent with the general equation of motion given in (6.18), when we set $\tilde{m} = 0$. This can be verified explicitly by direct substitution. We emphasize that the spectral densities in (6.22) could be derived in the quasiparticle approximation from the general equations of motion for the Green's functions, as it has been shown in Kane's thesis [16]. We simply start with them here as input into our general KB formalism. We might note that in the Bose-gas literature, v_p is sometimes defined with the opposite sign, such that $u_p v_p$ in (6.23) is negative.

In the Thomas-Fermi approximation [2], where one neglects the ‘‘quantum pressure’’ term in (6.11), the quasiparticle energy E_p reduces to the usual Bogoliubov excitation

energy (generalized to finite temperatures)

$$E_p(\mathbf{R}, T) = \sqrt{\epsilon_p^2 + 2gn_c(\mathbf{R}, T)\epsilon_p}, \quad (6.25)$$

where $\epsilon_p = p^2/2m$. We note that spectral densities $a_{\alpha\beta}(\mathbf{p}, \omega; \mathbf{R}, T)$ in (6.22) exhibit both positive and negative energy poles. In the Hartree-Fock approximation used in Ch. 5 [19], one has $u_p^2 = 1$ and $v_p^2 = 0$. Physically, (6.22) corresponds to the assumption that the thermal cloud can be considered as a gas of weakly interacting single-particle excitations with the excitation energy given by (6.25).

Substituting the spectral densities (6.22) into (6.19), we can now proceed to derive a kinetic equation for the quasiparticles. After lengthy algebra, we obtain

$$\begin{aligned} & \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \delta(\omega - \mathbf{v}_s \cdot \mathbf{p} - E_p) \left[\frac{\partial f}{\partial T} + \nabla_{\mathbf{p}} (E_p + \mathbf{v}_s \cdot \mathbf{p}) \cdot \nabla_{\mathbf{R}} f - \nabla_{\mathbf{R}} (E_p + \mathbf{v}_s \cdot \mathbf{p}) \cdot \nabla_{\mathbf{p}} f \right. \\ & + \frac{\partial}{\partial T} (E_p + \mathbf{v}_s \cdot \mathbf{p}) \frac{\partial f}{\partial \omega} + u_p^2 ((1+f)\Sigma_{11}^< - f\Sigma_{11}^>) + v_p^2 ((1+f)\Sigma_{22}^< - f\Sigma_{22}^>) \\ & - u_p v_p ((1+f)(\Sigma_{12}^< + \Sigma_{21}^<) - f(\Sigma_{12}^> + \Sigma_{21}^>))] \\ & + \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \delta(\omega - \mathbf{v}_s \cdot \mathbf{p} + E_p) \left[\frac{\partial f}{\partial T} + \nabla_{\mathbf{p}} (-E_p + \mathbf{v}_s \cdot \mathbf{p}) \cdot \nabla_{\mathbf{R}} f - \nabla_{\mathbf{R}} (-E_p + \mathbf{v}_s \cdot \mathbf{p}) \cdot \nabla_{\mathbf{p}} f \right. \\ & + \frac{\partial}{\partial T} (-E_p + \mathbf{v}_s \cdot \mathbf{p}) \frac{\partial f}{\partial \omega} + v_p^2 [(1+f)\Sigma_{11}^< - f\Sigma_{11}^>) + u_p^2 [(1+f)\Sigma_{22}^< - f\Sigma_{22}^>] \\ & - u_p v_p [(1+f)(\Sigma_{12}^< + \Sigma_{21}^<) - f(\Sigma_{12}^> + \Sigma_{21}^>)] = 0, \end{aligned} \quad (6.26)$$

where $\Sigma_{\alpha\beta} = \Sigma_{\alpha\beta}(\mathbf{p}, \omega; \mathbf{R}, T)$. This is a kinetic equation for the frequency-dependent quasiparticle distribution function $f(\mathbf{p}, \omega; \mathbf{R}, T)$ expressed in terms of an integral over both positive and negative energy poles. If we recall the expression for Bose coherence factors u and v given by (6.23), we note that the second term in (6.26) is the same as the first term in (6.26) if we replace $-E_p$ with E_p . Therefore, it is sufficient only to consider the first term to obtain the kinetic equation for the *quasiparticle* distribution function defined by

$$f_{qp}(\mathbf{p}, \mathbf{R}, T) \equiv f(\mathbf{p}, \omega = E_p + \mathbf{v}_s \cdot \mathbf{p}; \mathbf{R}, T). \quad (6.27)$$

We obtain finally

$$\left[\frac{\partial f_{qp}}{\partial T} + \nabla_{\mathbf{p}} (E_p + \mathbf{v}_s \cdot \mathbf{p}) \cdot \nabla_{\mathbf{R}} f_{qp} - \nabla_{\mathbf{R}} (E_p + \mathbf{v}_s \cdot \mathbf{p}) \cdot \nabla_{\mathbf{p}} f_{qp} \right] = I[f_{qp}]. \quad (6.28)$$

Here the collision integral $I[f_{qp}(p, \mathbf{R}, T)]$, defined in (6.20), becomes

$$\begin{aligned} I[f_{qp}(p, \mathbf{R}, T)] &\equiv \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \left[u_p^2 [(1 + f_{qp})\Sigma_{11}^< - f_{qp}\Sigma_{11}^>] + v_p^2 [(1 + f_{qp})\Sigma_{22}^< - f_{qp}\Sigma_{22}^>] \right. \\ &\quad \left. + u_p v_p [(1 + f_{qp})(\Sigma_{12}^< + \Sigma_{21}^<) - f_{qp}(\Sigma_{12}^> + \Sigma_{21}^>)] \right]. \end{aligned} \quad (6.29)$$

Such a kinetic equation is also derived by Kane in his thesis [16], but this was not included in Ref. [15]. We remark that in our Popov quasiparticle approximation, the quantum terms in (6.19) involving derivatives with respect to ω make no contribution. A similar quasiparticle kinetic equation has also been derived by Stoof [42] using the related Schwinger-Keldysh formalism for non-equilibrium processes. However Stoof mainly studied the high-temperature limit of the collision integral, as in Ch. 5 of this thesis.

What remains to be done is to choose a specific approximation for the second-order self-energy $\Sigma_{\alpha\beta}$, so that we can evaluate the collision integral in (6.29). In this chapter, we use the second order Beliaev approximation given by (6.1). The Fourier transform is

$$\begin{aligned} \hat{\Sigma}^<>(\mathbf{p}, \omega; \mathbf{R}, T) &= -\frac{1}{2}g^2 \int \frac{d\mathbf{p}_i d\omega_i}{(2\pi)^8} \delta(\omega + \omega_1 - \omega_2 - \omega_3) \delta(\mathbf{p} + \mathbf{p}_1 - \mathbf{p}_2 - \mathbf{p}_3) \\ &\times \left[\hat{g}^<>(\mathbf{p}_2, \omega_2) Tr \left[\hat{g}^<>(\mathbf{p}_1, \omega_1) \hat{g}^<>(\mathbf{p}_3, \omega_3) \right] + 2\hat{g}^<>(\mathbf{p}_2, \omega_2) \hat{g}^<>(\mathbf{p}_1, \omega_1) \hat{g}^<>(\mathbf{p}_3, \omega_3) \right. \\ &+ \hat{g}^<>(\mathbf{p}_2, \omega_2) Tr \left[\hat{h}(\mathbf{p}_1, \omega_1) \hat{g}^<>(\mathbf{p}_3, \omega_3) + \hat{g}^<>(\mathbf{p}_1, \omega_1) \hat{h}(\mathbf{p}_3, \omega_3) \right] \\ &+ \hat{h}(\mathbf{p}_2, \omega_2) Tr \left[\hat{g}^<>(\mathbf{p}_1, \omega_1) \hat{g}^<>(\mathbf{p}_3, \omega_3) \right] + 2\hat{h}(\mathbf{p}_2, \omega_2) \hat{g}^<>(\mathbf{p}_1, \omega_1) \hat{g}^<>(\mathbf{p}_3, \omega_3) \\ &\left. + 2\hat{g}^<>(\mathbf{p}_2, \omega_2) \left[\hat{h}(\mathbf{p}_1, \omega_1) \hat{g}^<>(\mathbf{p}_3, \omega_3) + \hat{g}^<>(\mathbf{p}_1, \omega_1) \hat{h}(\mathbf{p}_3, \omega_3) \right] \right]. \end{aligned} \quad (6.30)$$

As usual, the (\mathbf{R}, T) dependence of the functions Σ, \tilde{g} and h on the right-hand side has been suppressed for simplicity of notation. The quasiparticle energy $E_p(\mathbf{R}, T)$ in (6.28) is the energy of the quasiparticles in the local rest frame ($\mathbf{v}_s = 0$). The Beliaev second-order expression (6.30) consists of two kinds of contributions: (1) Terms that include both the condensate propagator h and the non-condensate propagators \tilde{g} ; (2) Terms that include the non-condensate propagators \tilde{g} only. The first type of contribution will give rise to the collision integral that describe collisions that include one condensate atom interacting

with the thermally excited quasiparticles. As in Ch.5 [17, 19, 28], we denote this part of the collision integral as C_{12} , indicating that we go from 1 thermally excited quasiparticle (and one condensate atom) to 2 thermally excited quasiparticles. The second type of contribution in (6.30) only includes non-condensate propagators. We denote this part of the collision integral as C_{22} , indicating that it describes collisions where 2 thermally excited quasiparticles are scattered into 2 excited quasiparticles. At low temperatures, when the number of thermally-excited quasiparticles is very small, we can neglect the C_{22} collision integral relative to C_{12} .

From (6.21), it follows that both C_{12} and C_{22} conserve momentum, i.e.,

$$\begin{aligned}\int d\mathbf{p}\mathbf{p}C_{12} &= 0 \\ \int d\mathbf{p}\mathbf{p}C_{22} &= 0.\end{aligned}\tag{6.31}$$

In addition, using (6.22) in (6.30) one can show that the collision integral in (6.20) conserves quasiparticle energy E_p and therefore both C_{12} and C_{22} will satisfy the conditions

$$\begin{aligned}\int d\mathbf{p}E_pC_{12} &= 0 \\ \int d\mathbf{p}E_pC_{22} &= 0.\end{aligned}\tag{6.32}$$

However, we note that the number of condensate atoms is not conserved by C_{12} , i.e., $\int d\mathbf{p}C_{12} \neq 0$.

One can show that for slowly varying external disturbances, the condensate propagator in (3.6) can be approximated by

$$\hat{h}(\mathbf{p}, \omega; \mathbf{R}, T) = n_c(\mathbf{R}, T)(2\pi)^4 \delta(p) \delta(\omega) \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.\tag{6.33}$$

To evaluate the collision integral C_{12} in terms of the Bose coherence factors u and v , we need to substitute (6.22) and (6.33) into (6.30). One can simplify (6.30) greatly using the following exact symmetry relations:

$$\tilde{g}_{22}^>(\mathbf{p}, \omega; \mathbf{R}, T) = \tilde{g}_{11}^<(-\mathbf{p}, -\omega; \mathbf{R}, T),$$

$$\begin{aligned}
\tilde{g}_{12}^{\lessgtr}(\mathbf{p}, \omega; \mathbf{R}, T) &= \tilde{g}_{12}^{\lessgtr}(-\mathbf{p}, -\omega; \mathbf{R}, T), \\
\tilde{g}_{21}^{\lessgtr}(\mathbf{p}, \omega; \mathbf{R}, T) &= \tilde{g}_{21}^{\lessgtr}(-\mathbf{p}, -\omega; \mathbf{R}, T).
\end{aligned} \tag{6.34}$$

After considerable algebra, one finds the following expressions for the non-equilibrium self-energy (limiting ourselves to only the C_{12} collision terms, which include one condensate propagator):

$$\begin{aligned}
\Sigma_{11}^{\lessgtr}(\mathbf{p}, \omega; \mathbf{R}, T) &= -g^2 \int \frac{d\mathbf{p}_2 d\omega_2}{(2\pi)^4} n_c(\mathbf{R}, T) \left[2\tilde{g}_{11}^{\lessgtr}(\mathbf{p}_2, \omega_2) \tilde{g}_{11}^{\lessgtr}(\mathbf{p} - \mathbf{p}_2, \omega - \omega_2) \right. \\
&+ 4\tilde{g}_{11}^{\lessgtr}(\mathbf{p}_2, \omega_2) \tilde{g}_{11}^{\lessgtr}(\mathbf{p}_2 - \mathbf{p}, \omega_2 - \omega) + 8\tilde{g}_{12}^{\lessgtr}(\mathbf{p}_2, \omega_2) \tilde{g}_{11}^{\lessgtr}(\mathbf{p} - \mathbf{p}_2, \omega - \omega_2) \\
&\left. + 4\tilde{g}_{12}^{\lessgtr}(\mathbf{p}_2, \omega_2) \tilde{g}_{12}^{\lessgtr}(\mathbf{p} - \mathbf{p}_2, \omega - \omega_2) \right],
\end{aligned} \tag{6.35}$$

$$\begin{aligned}
\Sigma_{22}^{\lessgtr}(\mathbf{p}, \omega; \mathbf{R}, T) &= -g^2 \int \frac{d\mathbf{p}_2 d\omega_2}{(2\pi)^4} n_c(\mathbf{R}, T) \left[4\tilde{g}_{12}^{\lessgtr}(\mathbf{p}_2, \omega_2) \tilde{g}_{12}^{\lessgtr}(\mathbf{p} - \mathbf{p}_2, \omega - \omega_2) \right. \\
&+ 4\tilde{g}_{11}^{\lessgtr}(\mathbf{p}_2, \omega_2) \tilde{g}_{11}^{\lessgtr}(\mathbf{p}_2 - \mathbf{p}, \omega_2 - \omega) + 8\tilde{g}_{12}^{\lessgtr}(\mathbf{p}_2, \omega_2) \tilde{g}_{11}^{\lessgtr}(\mathbf{p}_2 - \mathbf{p}, \omega_2 - \omega) \\
&\left. + 2\tilde{g}_{11}^{\lessgtr}(-\mathbf{p}_2, -\omega_2) \tilde{g}_{11}^{\lessgtr}(\mathbf{p}_2 - \mathbf{p}, \omega_2 - \omega) \right],
\end{aligned} \tag{6.36}$$

$$\begin{aligned}
\Sigma_{12}^{\lessgtr}(\mathbf{p}, \omega; \mathbf{R}, T) &= -g^2 \int \frac{d\mathbf{p}_2 d\omega_2}{(2\pi)^4} n_c(\mathbf{R}, T) \left[6\tilde{g}_{12}^{\lessgtr}(\mathbf{p}_2, \omega_2) \tilde{g}_{12}^{\lessgtr}(\mathbf{p} - \mathbf{p}_2, \omega - \omega_2) \right. \\
&+ 4\tilde{g}_{12}^{\lessgtr}(\mathbf{p}_2, \omega_2) \tilde{g}_{11}^{\lessgtr}(\mathbf{p} - \mathbf{p}_2, \omega - \omega_2) + 4\tilde{g}_{11}^{\lessgtr}(\mathbf{p}_2, \omega_2) \tilde{g}_{11}^{\lessgtr}(\mathbf{p}_2 - \mathbf{p}, \omega_2 - \omega) \\
&\left. + 4\tilde{g}_{12}^{\lessgtr}(\mathbf{p}_2, \omega_2) \tilde{g}_{11}^{\lessgtr}(\mathbf{p}_2 - \mathbf{p}, \omega_2 - \omega) \right],
\end{aligned} \tag{6.37}$$

$$\Sigma_{21}^{\lessgtr}(\mathbf{p}, \omega; \mathbf{R}, T) = \Sigma_{12}^{\lessgtr}(\mathbf{p}, \omega; \mathbf{R}, T). \tag{6.38}$$

Again, the (\mathbf{R}, T) dependence of the \tilde{g} 's is suppressed on the right-hand side of these equations. One notices that these expressions have the same structure as the thermal equilibrium ones obtained by Hua Shi and Griffin as well as others [43, 44, 45]. This is expected since the whole structure of our theory is only valid for a systems slightly perturbed from thermal equilibrium, with all physical quantities assumed to vary slowly as functions of center-of-mass coordinates (\mathbf{R}, T) . The entire KB formalism reduces to

the usual equilibrium self-energies in the appropriate limit, which is one of its strengths.

One can show, using the general properties of the non-equilibrium Green's functions in (6.34) in conjunction with (6.16), that the quasiparticle distribution function $f(\mathbf{p}, \omega; \mathbf{R}, T)$ satisfies the exact relation

$$f(-\mathbf{p}, -\omega; \mathbf{R}, T) = -(1 + f(\mathbf{p}, \omega; \mathbf{R}, T)). \quad (6.39)$$

If we introduce the following standard abbreviations for the Bose-coherence factors

$$A_p \equiv u_p^2, \quad B_p \equiv v_p^2, \quad C_p \equiv -u_p v_p, \quad (6.40)$$

the self-energies in (6.35)-(6.37) can be finally written as [using the key identity (6.39)]

$$\begin{aligned} \Sigma_{11}^{\leq}(\mathbf{p}, \omega; \mathbf{R}, T) &= g^2 \int \frac{d\mathbf{p}_2 d\omega_2}{(2\pi)^2} n_c(\mathbf{R}, T) \begin{pmatrix} (1 + f_1)(1 + f_2) \\ f_1 f_2 \end{pmatrix} \\ &[(2A_1 A_2 + 8A_1 C_2 + 4C_1 C_2 + 4B_1 A_2) \delta(\omega_2 - E_2) \delta(\omega_1 - E_1) \\ &- (2B_1 A_2 + 8B_1 C_2 + 4C_1 C_2 + 4A_1 A_2) \delta(\omega_2 - E_2) \delta(\omega_1 + E_1) \\ &- (2A_1 B_2 + 8A_1 C_2 + 4C_1 C_2 + 4B_1 B_2) \delta(\omega_2 + E_2) \delta(\omega_1 - E_1) \\ &+ (2B_1 B_2 + 8B_1 C_2 + 4C_1 C_2 + 4A_1 B_2) \delta(\omega_2 + E_2) \delta(\omega_1 + E_1)], \end{aligned} \quad (6.41)$$

$$\begin{aligned} \Sigma_{12}^{\leq}(\mathbf{p}, \omega; \mathbf{R}, T) &= g^2 \int \frac{d\mathbf{p}_2 d\omega_2}{(2\pi)^2} n_c(\mathbf{R}, T) \begin{pmatrix} (1 + f_1)(1 + f_2) \\ f_1 f_2 \end{pmatrix} \\ &[(6C_1 C_2 + 4A_1 C_2 + 4B_1 A_2 + 4B_1 C_2) \delta(\omega_2 - E_2) \delta(\omega_1 - E_1) \\ &- (4B_1 C_2 + 4A_1 C_2 + 6C_1 C_2 + 4A_1 A_2) \delta(\omega_2 - E_2) \delta(\omega_1 + E_1) \\ &- (4A_1 C_2 + 4B_1 C_2 + 6C_1 C_2 + 4B_1 B_2) \delta(\omega_2 + E_2) \delta(\omega_1 - E_1) \\ &+ (4A_1 B_2 + 4B_1 C_2 + 6C_1 C_2 + 4A_1 C_2) \delta(\omega_2 + E_2) \delta(\omega_1 + E_1)], \end{aligned} \quad (6.42)$$

$$\Sigma_{22}^{\leq}(\mathbf{p}, \omega; \mathbf{R}, T) = g^2 \int \frac{d\mathbf{p}_2 d\omega_2}{(2\pi)^2} n_c(\mathbf{R}, T) \begin{pmatrix} (1 + f_1)(1 + f_2) \\ f_1 f_2 \end{pmatrix}$$

$$\begin{aligned}
& [(2B_1B_2 + 8B_1C_2 + 4C_1C_2 + 4B_1A_2) \delta(\omega_2 - E_2)\delta(\omega_1 - E_1) \\
& - (2A_1B_2 + 8A_1C_2 + 4C_1C_2 + 4A_1A_2) \delta(\omega_2 - E_2)\delta(\omega_1 + E_1) \\
& - (2B_1A_2 + 8B_1C_2 + 4C_1C_2 + 4B_1B_2) \delta(\omega_2 + E_2)\delta(\omega_1 - E_1) \\
& + (2A_1A_2 + 8A_1C_2 + 4C_1C_2 + 4A_1B_2) \delta(\omega_2 + E_2)\delta(\omega_1 + E_1)],
\end{aligned} \tag{6.43}$$

where $\mathbf{p} - \mathbf{p}_2 \equiv \mathbf{p}_1$ and $\omega - \omega_2 \equiv \omega_1$.

Using these results, we can finally evaluate the C_{12} collision integral contribution to given in (6.29)

$$\begin{aligned}
C_{12}[f] &= 2g^2 n_c(\mathbf{R}, T) \int \frac{d\mathbf{p}_1 d\mathbf{p}_2}{(2\pi)^2} [(1+f)f_1 f_2 - f(1+f_1)(1+f_2)] \\
& \delta(\mathbf{p} - \mathbf{p}_1 - \mathbf{p}_2) [((u_1 - v_1)(u_p u_2 + v_p v_2) + (u_2 - v_2)(u_p u_1 + v_p v_1) \\
& - (u_p - v_p)(u_1 v_2 + v_1 u_2))^2 \delta(E_p - E_1 - E_2) \\
& + 2((u_1 - v_1)(u_p u_2 + v_p v_2) + (u_p - v_p)(u_1 u_2 + v_1 v_2) \\
& - (u_2 - v_2)(u_p v_1 + u_1 v_p))^2 \delta(E_p + E_1 - E_2) \\
& + (v_p u_1 u_2 + u_p u_2 v_1 + u_1 v_2 u_p - v_p v_2 u_1 - v_1 u_2 v_p - u_p v_2 v_1)^2 \delta(E_p + E_1 + E_2)].
\end{aligned} \tag{6.44}$$

The last term in (6.44) clearly vanishes because of the energy delta function. We recall that all u 's, v 's and the quasiparticle energy E_p in (6.44) have an implicit (\mathbf{R}, T) dependence. If we change $\mathbf{p}_1 \rightarrow -\mathbf{p}_1$ in the second term in (6.44), and use (6.39), we can simplify (6.44) slightly to obtain

$$\begin{aligned}
C_{12}[f] &= 2g^2 n_c(\mathbf{R}, T) \int \frac{d\mathbf{p}_1 d\mathbf{p}_2}{(2\pi)^2} [(1+f)f_1 f_2 - f(1+f_1)(1+f_2)] \\
& [((u_1 - v_1)(u_p u_2 + v_p v_2) + (u_2 - v_2)(u_p u_1 + v_p v_1) - (u_p - v_p)(u_1 v_2 + v_1 u_2))^2 \\
& \times \delta(\mathbf{p} - \mathbf{p}_1 - \mathbf{p}_2) \delta(E_p - E_1 - E_2) \\
& - 2((u_1 - v_1)(u_p u_2 + v_p v_2) + (u_p - v_p)(u_1 u_2 + v_1 v_2) - (u_2 - v_2)(u_p v_1 + u_1 v_p))^2 \\
& \times \delta(\mathbf{p} + \mathbf{p}_1 - \mathbf{p}_2) \delta(E_p + E_1 - E_2)].
\end{aligned} \tag{6.45}$$

The first term in (6.45) describes the decay of an excitation with momentum \mathbf{p} into two excitations with momenta \mathbf{p}_1 and \mathbf{p}_2 . At $T = 0$, this is the only scattering process possible since there are no excitations thermally excited. The second term describes an excitation of momentum \mathbf{p} absorbing a thermal excitation of momentum \mathbf{p}_1 , leaving an excitation with momentum $\mathbf{p}_2 = \mathbf{p} + \mathbf{p}_1$.

The expression for the collision integral in (6.45) was first written down by Eckern [27], and shortly after by Kirkpatrick and Dorfman [28] who gave a more detailed derivation. Here, we have used the Kadanoff-Baym approach to give what we believe is a cleaner derivation of this expression for C_{12} , in a form which is also valid for a trapped Bose-condensed gas.

After some algebra, one can also rewrite (6.45) in the following more compact form [28]

$$C_{12}[f] = 2g^2 n_c(\mathbf{R}, T) \int \frac{d\mathbf{p}_1 d\mathbf{p}_2 d\mathbf{p}_3}{(2\pi)^2} |A(2, 3; 1)|^2 \delta(\mathbf{p}_1 - \mathbf{p}_2 - \mathbf{p}_3) \delta(E_1 - E_2 - E_3) \\ [\delta(\mathbf{p} - \mathbf{p}_1) - \delta(\mathbf{p} - \mathbf{p}_2) - \delta(\mathbf{p} - \mathbf{p}_3)] [(1 + f_1)f_2f_3 - f_1(1 + f_2)(1 + f_3)]. \quad (6.46)$$

Here the scattering amplitude in $|A|^2$ is given in term of the Bose coherence factors u and v

$$A(2, 3; 1) \equiv (u_3 - v_3)(u_1 u_2 + v_1 v_2) + (u_2 - v_2)(u_1 u_3 + v_1 v_3) - (u_1 - v_1)(u_2 v_3 + v_2 u_3). \quad (6.47)$$

In conclusion, the thermally-excited quasiparticle distribution function defined by (6.27) satisfies the kinetic equation

$$\left[\frac{\partial}{\partial T} + \nabla_{\mathbf{p}} (E_p + \mathbf{v}_s \cdot \mathbf{p}) \cdot \nabla_{\mathbf{R}} - \nabla_{\mathbf{R}} (E_p + \mathbf{v}_s \cdot \mathbf{p}) \cdot \nabla_{\mathbf{p}} \right] f_{qp}(\mathbf{p}, \mathbf{R}, T) = C_{12}[f_{qp}]. \quad (6.48)$$

Here, C_{12} given explicitly by (6.45), or equivalently, (6.46). The derivation of this equation using the KB formalism is the main result of this chapter. As we have noted, a kinetic equation of this kind has been derived by other authors [16, 27, 28, 42] using a variety of techniques and approximations. The earliest microscopic derivation was given

in Kane's thesis [16], but he did not evaluate explicitly the collision integral to obtain results such as (6.46) and (6.45).

To remove the rapidly varying phase of the order parameter, we have gauge-transformed to the local rest frame where the condensate is at rest. Hence the energy of the thermally excited quasiparticles is measured relative to this local frame. Since the thermal excitations are moving with the superfluid velocity \mathbf{v}_s relative to the condensate, the energy of quasiparticles measured relative to the condensate is $E_p + \mathbf{v}_s \cdot \mathbf{p}$. Therefore, the expression $E_p + \mathbf{v}_s \cdot \mathbf{p}$ in the streaming term on the left-hand side of (6.48) is expected. Similarly, if we denote the quasiparticle distribution function in the coordinate system where the macroscopic velocity of the quasiparticles is zero by $f(\mathbf{p}, \omega; \mathbf{R}, T)$, the quasiparticles moving with the velocity \mathbf{v}_s relative to the condensate will be described by the distribution function $f(\mathbf{p}, \omega = E_p + \mathbf{v}_s \cdot \mathbf{p}; \mathbf{R}, T) \equiv f_{qp}(\mathbf{p}, \mathbf{R}, T)$, which occurs in (6.48) [30, 46]. If we use the frame of reference where the macroscopic velocity of the quasiparticles is zero, the streaming term will include the energy of the quasiparticles only (i.e., the $\mathbf{v}_s \cdot \mathbf{p}$ term in (6.48) will not be present).

To understand the C_{12} collision integral in (6.45) better and the corresponding scattering processes that it describes, it is useful to consider a few limiting cases for a uniform gas. We define $p_0^2 \equiv 2mgn_c$ as the characteristic momentum for the crossover between the linear and the quadratic part of the quasiparticle spectrum ($p_0 = \hbar k_0 \equiv \hbar\xi^{-1}$, where ξ is the so-called healing length). We then consider the following special cases:

1) If all momenta $p_i \gg p_0$, then the quasiparticle spectrum E_p defined by (6.25) is equal to a single-particle spectrum $\tilde{\epsilon}_p$ defined in (6.10). Moreover, in this limit, it follows from (6.23) that Bogoliubov amplitudes $u \rightarrow 1$ and $v \rightarrow 0$. Hence, the scattering amplitude A in (6.47) becomes unity and the collision integral in (6.46) then reduces to the result in ((5.21) of Ch.5 and derived by Zaremba, Nikuni and Griffin [19, 17] using a different approach (see also Ref. [42]). Clearly, this approximation is only valid at finite temperatures, where the dominant excitation spectrum is described by the Hartree-Fock

single-particle spectrum described by (6.10).

2) In the opposite limit, when all three momenta p_i are small, one can expand the Bose coherence factors u and v in the following way [47, 48]

$$\begin{aligned} u_p &\simeq \left(\frac{gn_c}{2E_p}\right)^{1/2} + \frac{1}{2} \left(\frac{E_p}{2gn_c}\right)^{1/2} \\ v_p &\simeq \left(\frac{gn_c}{2E_p}\right)^{1/2} - \frac{1}{2} \left(\frac{E_p}{2gn_c}\right)^{1/2}, \end{aligned} \quad (6.49)$$

where $E_p \simeq cp$ and $c = \sqrt{gn_c/m}$ is the speed of Bogoliubov sound. In this limit, one obtains for the scattering amplitude the following expression [27]

$$A(1; 2, 3) \simeq \frac{3}{2^{7/4}} \sqrt{\frac{p_1 p_2 p_3}{p_0^3}}. \quad (6.50)$$

This approximation is valid at low temperatures, where only low-momentum excitations are relevant. We remark that the sign of v_p in (6.49) is opposite from the one given in [47, 48] because we have defined $u_p v_p$ in (6.23) to be positive.

3) Finally, one can consider the scattering of phonons (low-momentum excitations) with momentum $p_3 \ll p_0$ with atoms (high-momentum excitations) with momenta $p_1, p_2 \gg p_0$. The corresponding amplitude for this process is given by [27]

$$A(1; 2, 3) \simeq 2^{3/4} (p_3/p_0)^{1/2}. \quad (6.51)$$

Therefore, in the case of a sound wave scattering with particle-like excitations, the scattering amplitude depends only on the wavevector of the sound wave: it is independent of the momenta of the scattering particles.

6.5 Local equilibrium solution

In this thesis, we do not discuss solutions of the quasiparticle kinetic equation (6.48) in any detail. However, it is useful to make some remarks about the local equilibrium solution. To describe the thermalization of quasiparticles, it is sufficient to consider the

C_{22} collision integral. From (6.16), (6.20) and (6.30), one obtains

$$\begin{aligned}
C_{22}[f] &= -\frac{1}{2}g^2 \int \frac{d\mathbf{p}_i d\omega_i}{(2\pi)^8} \delta(E_p + \omega_1 - \omega_2 - \omega_3) \delta(\mathbf{p} + \mathbf{p}_1 - \mathbf{p}_2 - \mathbf{p}_3) \\
&\times [ff_1(1+f_2)(1+f_3) - (1+f)(1+f_1)f_2f_3] \\
&\quad [\text{Tr}(\hat{a}(\mathbf{p}_2, \omega_2)\hat{a}(\mathbf{p}, E_p)) \text{Tr}(\hat{a}(\mathbf{p}_1, \omega_1)\hat{a}(\mathbf{p}_3, \omega_3)) \\
&\quad + \text{Tr}(2\hat{a}(\mathbf{p}_2, \omega_2)\hat{a}(\mathbf{p}_1, \omega_1)\hat{a}(\mathbf{p}_3, \omega_3)\hat{a}(\mathbf{p}, E_p))] .
\end{aligned} \tag{6.52}$$

The ‘‘local equilibrium’’ distribution function for quasiparticles $f_0(\mathbf{p}, \omega; \mathbf{R}, T)$ is determined by the requirement that $C_{22}[f_0] = 0$. One can see from (6.52) that one doesn’t have to choose a specific approximation for the single-particle spectral densities. We only need a solution for f such that the expression in (6.52) containing the f ’s vanishes. One can verify that $C_{22}[f_0] = 0$ as long as $f_0(\mathbf{p}, \omega; \mathbf{R}, T)$ has the following form

$$f_0(\mathbf{p}, \omega; \mathbf{R}, T) = \frac{1}{e^{\beta(\omega - \mathbf{p} \cdot \mathbf{v}_n - \mu_{qp}(\mathbf{R}, T))} - 1} . \tag{6.53}$$

In the local rest frame, the entire current is carried by thermally excited quasiparticles. For small velocities, we have for the quasiparticle current [7, 28]

$$\begin{aligned}
\mathbf{j} &\equiv \rho_n(\mathbf{v}_n - \mathbf{v}_s) = \int d\mathbf{p} \mathbf{p} f_0(\mathbf{p}, \omega = E_p + \mathbf{p} \cdot \mathbf{v}_s; \mathbf{R}, T) \\
&\simeq \int \frac{d\mathbf{p}}{(2\pi)^3} \mathbf{p} f_0(E_p) - \int \frac{d\mathbf{p}}{(2\pi)^3} \mathbf{p} \mathbf{p} \cdot (\mathbf{v}_n - \mathbf{v}_s) \frac{\partial f_0(E_p)}{\partial E_p} \\
&= (\mathbf{v}_n - \mathbf{v}_s) \int \frac{d\mathbf{p}}{(2\pi)^3} \left(-\frac{p^2}{3} \right) \frac{\partial f_0(E_p)}{\partial E_p} .
\end{aligned} \tag{6.54}$$

This *defines* the normal density ρ_n in the Landau quasiparticle picture [7]. The distribution function given in (6.53) differs from the usual equilibrium quasiparticle distribution function discussed in the standard literature for phonons and rotons in superfluid ^4He [7]. Since the number of quasiparticles is not conserved, the usual form for the equilibrium quasiparticle distribution function has no chemical potential (more precisely, the chemical potential is zero). Here we have introduced a quasiparticle chemical potential in (6.53) to allow for the possibility that the condensate atoms and the thermally excited quasiparticles are *not* in the diffusive equilibrium with each other.

To understand the physics of the quasiparticle chemical potential in (6.53) better, let us consider the high temperature case. At high temperatures, the atom and the quasiparticle excitation spectrum are equivalent (see Ch. 5) and the local equilibrium distribution function in the lab frame is given by [17]

$$f_0(\mathbf{p}, \mathbf{R}, T) = \frac{1}{e^{\beta\left(\frac{(\mathbf{p}-m\mathbf{v}_n)^2}{2m} + U(\mathbf{R}, T) - \tilde{\mu}(\mathbf{R}, T)\right)} - 1}, \quad (6.55)$$

with $U(\mathbf{R}, T) = U_{ext}(\mathbf{R}) + 2gn(\mathbf{R}, T)$ and $\tilde{\mu}(\mathbf{R}, T)$ is the local chemical potential of the non-condensate atoms. If we transform (6.55) to the local rest frame (where $\mathbf{v}_s = 0$), then $\mathbf{p}' \equiv \mathbf{p} - m\mathbf{v}_s$ is the momentum in the local rest frame and (6.55) becomes

$$f_0(\mathbf{p}', \mathbf{R}, T) = \frac{1}{e^{\beta[E_{p'} - \mathbf{p}' \cdot (\mathbf{v}_n - \mathbf{v}_s) - (\tilde{\mu} - \mu_c) + \frac{1}{2}m(\mathbf{v}_n - \mathbf{v}_s)^2]} - 1}. \quad (6.56)$$

Here $E_{p'} \equiv \frac{p'^2}{2m} + gn_c(\mathbf{R}, T)$ is the excitation energy in the local *rest* frame. Therefore, if we define [17]

$$\mu_{diff} \equiv \tilde{\mu} - \mu_c - \frac{1}{2}m(\mathbf{v}_n - \mathbf{v}_s)^2, \quad (6.57)$$

we see that the quasiparticle chemical potential μ_{qp} introduced in (6.53) can be identified precisely with μ_{diff} as discussed at length by Zaremba, Nikuni and Griffin [17]. Hence we see that, in the quasiparticle description, and in a local frame in which the condensate is stationary, the difference (μ_{diff}) between the chemical potentials of the condensate and non-condensate atoms that was introduced by ZNG to describe the non-diffusive equilibrium of these two components appears very naturally as the quasiparticle chemical potential μ_{qp} . The standard case discussed in the superfluid helium ^4He literature [7] corresponds to $\mu_{qp} = 0$ (see, however, the discussion [7] of the second viscosity coefficients in superfluid ^4He).

To summarize, we can distinguish two distribution functions f which satisfy $C_{22}[f_0] = 0$:

(1) The condensate atoms and the quasiparticle excitations are in diffusive thermal equilibrium, i.e. $\mu_{qp} = 0$ and hence $C_{12}[f] = 0$.

(2) The condensate atoms and the quasiparticle excitations are not in the diffusive thermal equilibrium, i.e. $\mu_{qp} \neq 0$. In this case, one finds that $C_{12}[f]$ is proportional to $[1 - e^{\beta\mu_{qp}}]$, as discussed in ZNG [17].

6.6 Generalized Gross-Pitaevskii equation

It is clear that the second-order collisions which we have included in deriving the quasiparticle kinetic equation in Section 6.4 must also be included in a generalized GP equation for the condensate wavefunction $\Phi(\mathbf{R}, T)$. Technically, this arises from the three-field correlation function given in (3.9). In Ch.5, we have derived [19] the same generalized GP equation as ZNG using the KB method for finite temperatures. In this section, we now extend this kind of calculation so that it is valid at low temperatures. The new GP equation of motion for the order parameter will be shown to be identical to that obtained in Refs. [19, 17], apart from the fact that C_{12} is now given by the expression in (6.45). That is, it now involves the Bogoliubov-Popov quasiparticles and the collision cross-section is renormalized by various Bose-coherence factors involving the u 's and v 's.

To derive an equation of motion for the condensate order parameter, we first write equation (5.4) for $\Phi(\mathbf{r}, t)$ in the local rest frame. As before, under the gauge transformation (4.12), the only change is that the non-interacting propagator is now given by (4.13). The equation of motion in the new local frame is

$$\begin{aligned} & \left[i \frac{\partial}{\partial t} - \frac{\partial \theta(1)}{\partial t} + \frac{1}{2m} [\nabla_{\mathbf{r}} + im\mathbf{v}_s(1)]^2 + \mu_0 - U_{ext}(\mathbf{r}) - g(2\tilde{n}(1) + n_c(1)) \right] \Phi(1) \\ &= \int_{-\infty}^t d\bar{t} [S_{11}^> - S_{11}^<](\mathbf{r} - \bar{\mathbf{r}}, t - \bar{t}; (\mathbf{r} + \bar{\mathbf{r}})/2, (t + \bar{t})/2) \Phi(\bar{\mathbf{r}}, \bar{t}) \\ &+ \int_{-\infty}^t d\bar{t} [S_{12}^> - S_{12}^<](\mathbf{r} - \bar{\mathbf{r}}, t - \bar{t}; (\mathbf{r} + \bar{\mathbf{r}})/2, (t + \bar{t})/2) \Phi^*(\bar{\mathbf{r}}, \bar{t}). \end{aligned} \quad (6.58)$$

Here we have rewritten the condensate self-energy in the center-of-mass and relative coordinates and, as usual, set $\tilde{m} = 0$ (the Popov approximation). We recall that in the local rest frame, the order parameter phase is removed and hence $\Phi(\mathbf{r}, t) = \sqrt{n_c(\mathbf{r}, t)}$.

We assume, as in Ch. 5, that the S correlation function defined in (3.13) is dominated by small values of the relative space-time coordinates $(\mathbf{r} - \bar{\mathbf{r}}, t - \bar{t})$. Hence we can approximate S_{11}^{\lessgtr} in (6.58) by $S^{\lessgtr}(\mathbf{r} - \bar{\mathbf{r}}, t - \bar{t}; \mathbf{r}, t)$. For the same reason, we can also approximate the macroscopic wavefunction $\Phi(\bar{\mathbf{r}}, \bar{t})$ in the integrand of (6.58) by $\Phi(\mathbf{r}, t) \equiv \sqrt{n_c(\mathbf{r}, t)}$. Hence (6.58) simplifies to

$$\begin{aligned} & \left[i \frac{\partial}{\partial t} - \frac{\partial \theta(1)}{\partial t} + \frac{1}{2m} [\nabla_{\mathbf{r}} + im\mathbf{v}_s(1)]^2 + \mu_0 - U_{ext}(\mathbf{r}) - g(2\tilde{n}(1) + n_c(1)) \right] \Phi(1) \\ &= \int_{-\infty}^t d\bar{\mathbf{r}} d\bar{t} [(S_{11}^> - S_{11}^<)(\mathbf{r} - \bar{\mathbf{r}}, t - \bar{t}; \mathbf{r}, t) \Phi(\mathbf{r}, t) \\ &+ (S_{12}^> - S_{12}^<)(\mathbf{r} - \bar{\mathbf{r}}, t - \bar{t}; \mathbf{r}, t) \Phi^*(\mathbf{r}, t)]. \end{aligned} \quad (6.59)$$

We can rewrite (6.59) [labeling $(\mathbf{r}, t) \rightarrow (\mathbf{R}, T)$] as follows

$$\begin{aligned} & \left[i \frac{\partial}{\partial T} - \frac{\partial \theta(\mathbf{R}, T)}{\partial T} + \frac{1}{2m} [\nabla_{\mathbf{R}} + im\mathbf{v}_s(\mathbf{R}, T)]^2 + \mu_0 - U_{ext}(\mathbf{R}) \right. \\ & - \left. g(2\tilde{n}(\mathbf{R}, T) + n_c(\mathbf{R}, T)) \right] \Phi(\mathbf{R}, T) \\ &= \Phi(\mathbf{R}, T) \int \frac{d\mathbf{p} d\omega}{(2\pi)^4} [S_{11}^> - S_{11}^< + S_{12}^> - S_{12}^<](\mathbf{p}, \omega; \mathbf{R}, T) \int_{-\infty}^T d\bar{\mathbf{r}} d\bar{t} e^{i\mathbf{p}(\mathbf{R}-\bar{\mathbf{r}}) - i\omega(T-\bar{t})}. \end{aligned} \quad (6.60)$$

In our second-order Beliaev approximation, the condensate self-energy is given by (3.36),

$$\begin{aligned} \hat{S}^{\lessgtr}(1, 1') &= - \frac{1}{2} v(13)v(21') \hat{g}^{\lessgtr}(11') \left[\hat{g}^{\lessgtr}(23) \hat{g}^{\lessgtr}(32) \right] \\ & - v(13)v(21') \hat{g}^{\lessgtr}(12) \left[\hat{g}^{\lessgtr}(23) \hat{g}^{\lessgtr}(31') \right]. \end{aligned} \quad (6.61)$$

The Fourier transform of this is

$$\begin{aligned} \hat{S}^{\lessgtr}(\mathbf{p}, \omega; \mathbf{R}, T) &= -\frac{1}{2} g^2 \int \frac{d\mathbf{p}_i d\omega_i}{(2\pi)^8} \delta(\omega + \omega_1 - \omega_2 - \omega_3) \delta(\mathbf{p} + \mathbf{p}_1 - \mathbf{p}_2 - \mathbf{p}_3) \\ & \times \left[\hat{g}^{\lessgtr}(\mathbf{p}_2, \omega_2; \mathbf{R}, T) \text{Tr} \left[\hat{g}^{\lessgtr}(\mathbf{p}_1, \omega_1; \mathbf{R}, T) \hat{g}^{\lessgtr}(\mathbf{p}_3, \omega_3; \mathbf{R}, T) \right] \right. \\ & \left. + 2 \hat{g}^{\lessgtr}(\mathbf{p}_2, \omega_2; \mathbf{R}, T) \hat{g}^{\lessgtr}(\mathbf{p}_1, \omega_1; \mathbf{R}, T) \hat{g}^{\lessgtr}(\mathbf{p}_3, \omega_3; \mathbf{R}, T) \right]. \end{aligned} \quad (6.62)$$

In evaluating the right-hand side of (6.60), we use the identity

$$\lim_{\delta \rightarrow 0^+} \int_{-\infty}^T d\bar{t} e^{-i(\omega + i\delta)(T-\bar{t})} \simeq \pi \delta(\omega) + iP \left(\frac{1}{\omega} \right), \quad (6.63)$$

and only keep the delta function part, to obtain

$$\begin{aligned}
& \left[i \frac{\partial}{\partial T} - \frac{\partial \theta(\mathbf{R}, T)}{\partial T} + \frac{1}{2m} [\nabla_{\mathbf{R}} + im\mathbf{v}_s(\mathbf{R}, T)]^2 \right. \\
& + \mu_0 - U_{ext}(\mathbf{R}) - g(2\tilde{n}(\mathbf{R}, T) + n_c(\mathbf{R}, T)) \\
& \left. - (S_{11}^> - S_{11}^< + S_{12}^> - S_{12}^<) (\mathbf{p} = 0, \omega = 0; \mathbf{R}, T) \right] \Phi(\mathbf{R}, T) = 0. \quad (6.64)
\end{aligned}$$

Using (6.16) in the condensate self-energy S given in (6.62), the second-order terms appearing in (6.64) reduce to

$$\begin{aligned}
& (S_{11}^> - S_{11}^< + S_{12}^> - S_{12}^<) (\mathbf{p} = 0, \omega = 0; \mathbf{R}, T) = i \frac{1}{2} g^2 \int \frac{d\mathbf{p}_i d\omega_i}{(2\pi)^8} \\
& \times \delta(\omega_1 - \omega_2 - \omega_3) \delta(\mathbf{p}_1 - \mathbf{p}_2 - \mathbf{p}_3) [f_1(1 + f_2)(1 + f_3) - (1 + f_1)f_2f_3] \\
& \times [(a_{11} + a_{12})(\mathbf{p}_2, \omega_2) \text{Tr}(\hat{a}(\mathbf{p}_1, \omega_1) \hat{a}(\mathbf{p}_3, \omega_3)) + 2(\hat{a}(\mathbf{p}_2, \omega_2) \hat{a}(\mathbf{p}_1, \omega_1) \hat{a}(\mathbf{p}_3, \omega_3))_{11} \\
& + 2(\hat{a}(\mathbf{p}_2, \omega_2) \hat{a}(\mathbf{p}_1, \omega_1) \hat{a}(\mathbf{p}_3, \omega_3))_{12}]. \quad (6.65)
\end{aligned}$$

Recalling that in the local rest frame we have $\Phi(\mathbf{R}, T) = \sqrt{n_c(\mathbf{R}, T)}$, we finally obtain a generalized Gross-Pitaevskii equation in the following form

$$\begin{aligned}
& i \frac{\partial \sqrt{n_c(\mathbf{R}, T)}}{\partial T} = \left[\frac{\partial \theta(\mathbf{R}, T)}{\partial T} - \frac{1}{2m} [\nabla_{\mathbf{R}} + im\mathbf{v}_s(\mathbf{R}, T)]^2 - \mu_0 \right. \\
& \left. + U_{ext}(\mathbf{R}) + g[2\tilde{n}(\mathbf{R}, T) + n_c(\mathbf{R}, T)] - iR(\mathbf{R}, T) \right] \sqrt{n_c(\mathbf{R}, T)}. \quad (6.66)
\end{aligned}$$

The new dissipative term R in the GP equation is precisely related to the C_{12} collision term in the kinetic equation (6.48), namely [17, 19]

$$R(\mathbf{R}, T) \equiv \int \frac{d\mathbf{p}}{(2\pi)^3} \frac{C_{12}[f(\mathbf{p}, \mathbf{R}, T)]}{2n_c(\mathbf{R}, T)}. \quad (6.67)$$

This term describes the damping of condensate amplitude fluctuations due to collisions with the thermal excitations. As in Ch. 5, the appearance of the dissipative term in (6.66) is expected since the C_{12} collisions change the number of atoms in the condensate and hence can modify the magnitude of the condensate macroscopic wavefunction. We note that since we ignore the real part of the second-order self-energies, the first order condensate chemical potential in (6.11) is not modified.

For consistency, since we have used the Thomas-Fermi approximation in defining our Bogoliubov quasiparticle energies in (6.25), we could neglect the $\nabla_{\mathbf{R}}^2$ term in the chemical potential in (6.11), as well as in our equation of motion for the condensate, as given by (6.58) and the final result (6.66). However, the structure is better shown by (6.66).

If we transform back into the lab frame (where we have $\Phi = \sqrt{n_c}e^{i\theta}$), (6.66) reduces to the time-dependent generalized Gross-Pitaevskii equation for $\Phi(\mathbf{R}, T)$ such as discussed by ZNG [17]. However, C_{12} now involves the Bogoliubov-Popov quasiparticle spectrum in place of the HF particle-like spectrum used in Ref. [17], and in addition, the collision integral matrix elements involve the characteristic Bose coherence factors u and v .

6.7 Kohn mode

In this section, we use our kinetic equation for quasiparticles to prove that it still exhibits a solution corresponding to the Kohn mode. This mode is also discussed in detail in Section 4.5 of this thesis. The present section shows this solution still exists in the presence of collisions between the quasiparticles. The analysis in Ref. [17] is easily generalized to the more general equations we are discussing. The center-of-mass oscillation of the non-condensate and condensate density profiles corresponding to the Kohn mode is given by

$$\begin{aligned} n_c(\mathbf{R}, T) &\equiv n_{c0}(\mathbf{R} - \boldsymbol{\eta}(T)) \\ \tilde{n}(\mathbf{R}, T) &\equiv \tilde{n}_0(\mathbf{R} - \boldsymbol{\eta}(T)). \end{aligned} \quad (6.68)$$

Here the center-of-mass displacement $\boldsymbol{\eta}(T)$ (with $\mathbf{v}_s = \dot{\boldsymbol{\eta}}$) satisfies the harmonic oscillator equation of motion

$$m \frac{\partial^2 \eta_\alpha}{\partial T^2} = -\omega_\alpha^2 \eta_\alpha, \quad (6.69)$$

where ω_α is the trap frequency in the α^{th} direction. The quasiparticle distribution function $f(\mathbf{p}, \mathbf{R}, T)$ corresponds to the equilibrium density profile oscillating around its center

of mass with the trap frequency, i.e.,

$$f(\mathbf{p}, \mathbf{R}, T) \equiv f_0(\mathbf{p}, \mathbf{R} - \boldsymbol{\eta}(T)). \quad (6.70)$$

To prove (6.70), we note that with (6.68), the expression for the Bogoliubov excitation energy in (6.25) reduces to

$$E_p(\mathbf{R}, T) = \sqrt{\epsilon_p^2 + 2gn_{c0}(\mathbf{R} - \boldsymbol{\eta})\epsilon_p} \equiv E_{p0}(\mathbf{R} - \boldsymbol{\eta}). \quad (6.71)$$

Therefore the kinetic equation for the quasiparticle distribution function given in (6.48) is

$$\begin{aligned} \left[\frac{\partial}{\partial T} + \nabla_{\mathbf{p}} (E_{p0}(\mathbf{R} - \boldsymbol{\eta}(T)) + \dot{\boldsymbol{\eta}} \cdot \mathbf{p}) \cdot \nabla_{\mathbf{R}} - \nabla_{\mathbf{R}} E_{p0}(\mathbf{R} - \boldsymbol{\eta}(T)) \cdot \nabla_{\mathbf{p}} \right] f_0(\mathbf{p}, \mathbf{R} - \boldsymbol{\eta}(T)) \\ = C_{12}[f_0(\mathbf{p}, \mathbf{R} - \boldsymbol{\eta}(T))]. \end{aligned} \quad (6.72)$$

If we expand $f_0(\mathbf{p}, \mathbf{R} - \boldsymbol{\eta}(T))$ around $\boldsymbol{\eta} = 0$,

$$f_0(\mathbf{p}, \mathbf{R} - \boldsymbol{\eta}(T)) = f_0(\mathbf{p}, \mathbf{R}) - \boldsymbol{\eta} \cdot \nabla_{\mathbf{R}} f_0(\mathbf{p}, \mathbf{R}), \quad (6.73)$$

and neglect the quadratic terms in $\boldsymbol{\eta}$, then (6.72) simplifies to

$$\begin{aligned} \nabla_{\mathbf{p}} E_{p0}(\mathbf{R} - \boldsymbol{\eta}(T)) \cdot \nabla_{\mathbf{R}} f_0(\mathbf{p}, \mathbf{R} - \boldsymbol{\eta}(T)) - \nabla_{\mathbf{R}} E_{p0}(\mathbf{R} - \boldsymbol{\eta}(T)) \cdot \nabla_{\mathbf{p}} f_0(\mathbf{p}, \mathbf{R} - \boldsymbol{\eta}(T)) \\ = C_{12}[f_0(\mathbf{p}, \mathbf{R} - \boldsymbol{\eta}(T))]. \end{aligned} \quad (6.74)$$

The left-hand side of (6.74) is seen to be the kinetic equation for the equilibrium distribution function. To prove that the Kohn mode (6.68) is a solution, one only has to show $C_{12}[f_0(\mathbf{p}, \mathbf{R} - \boldsymbol{\eta}(T))] = 0$. Assuming the equilibrium quasiparticle distribution function f_0 is given by (6.53) with $\mu_{qp} = 0$, and using the identity for the Bose distribution function

$$1 + f(x) = e^x f(x), \quad (6.75)$$

one obtains the following expression for $C_{12}[f_0]$,

$$\begin{aligned} C_{12}[f_0(\mathbf{p}, \mathbf{R} - \boldsymbol{\eta}(T))] &= 2g^2 n_c(\mathbf{R} - \boldsymbol{\eta}(T)) \int \frac{d\mathbf{p}_1 d\mathbf{p}_2 d\mathbf{p}_3}{(2\pi)^2} |A_0(2, 3; 1)|^2 \\ &\times \delta(\mathbf{p}_1 - \mathbf{p}_2 - \mathbf{p}_3) \delta[E_{10}(\mathbf{R} - \boldsymbol{\eta}(T)) - E_{20}(\mathbf{R} - \boldsymbol{\eta}(T)) - E_{30}(\mathbf{R} - \boldsymbol{\eta}(T))] \\ &\times [\delta(\mathbf{p} - \mathbf{p}_1) - \delta(\mathbf{p} - \mathbf{p}_2) - \delta(\mathbf{p} - \mathbf{p}_3)] (1 + f_{10}) f_{20} f_{30} \\ &\times \left[1 - e^{-\beta[E_{10} - E_{20} - E_{30} - (\mathbf{p}_1 - \mathbf{p}_2 - \mathbf{p}_3)(\mathbf{v}_n - \mathbf{v}_s)]} \right]. \end{aligned} \quad (6.76)$$

Using the delta functions in (6.76) corresponding to the conservation of energy and momentum, it immediately follows that

$$C_{12}[f_0(\mathbf{p}, \mathbf{R} - \boldsymbol{\eta}(T))] = 0. \quad (6.77)$$

This proves that the non-condensate exhibits a rigid simple harmonic displacement with the trap frequency, with C_{12} collisions having no effect.

Since we have proven that the collision integral vanishes for the Kohn mode type of oscillations, the dissipative term R in the generalized GP equation in (6.66) will vanish as well. Therefore it follows that the equilibrium condensate profiles oscillate with the trap frequency.

Chapter 7

Final remarks

In this thesis, we have given a detailed discussion of a microscopic theory for the dynamics of a trapped Bose gas at finite temperatures using the powerful Kadanoff-Baym formalism. Starting from several approximations to the Beliaev single-particle self-energies, we derived a generalized Gross-Pitaevskii equation of motion for the condensate order parameter that is coupled to a Boltzmann equation for the single-particle distribution function describing the thermally excited Bose excitations. The resulting equations can be used to study a variety of problems associated with the non-equilibrium dynamics in a trapped Bose gas, but this aspect is not addressed in this thesis.

In Ch. 4, we derived the equations of motion for the thermally-excited atoms that are coupled to the equation of motion for the atoms in the condensate in the collisionless regime by using the full first-order Hartree-Fock-Bogoliubov self-energies. The Hartree-Fock-Bogoliubov approximation includes both the dynamics of the condensate atoms as well as the dynamics of the non-condensate atoms self-consistently. Furthermore, the HFB theory also includes the pair correlation function \tilde{m} that describes the broken symmetry correlations between atoms in the presence of the condensate and excited atoms. However, as well known in the theory of Bose-condensed systems [11], this theory has some problems. For a uniform Bose gas, the HFB approximation leads to unphysical

energy gap in the long wavelength ($k \rightarrow 0$) excitation spectrum. However, the HFB is a conserving approximation (see Ch. 3) and thus it can give a good approximation for thermodynamic quantities, even though the HFB quasiparticle spectrum is not very realistic.

In Ch. 5, we used an improved approximation by including the second-order contributions to the self-energy functions and derived a kinetic equation that now includes the effect of binary collisions between thermally-excited atoms, as well as between atoms in the condensate and thermally-excited atoms. The latter collisions modify the Gross-Pitaevskii equation of motion for the Bose condensate order parameter as well, which now includes a damping term resulting from collisions between the condensate and the non-condensate atoms. However, the results of Ch. 5 are limited to finite temperatures (estimated to be $T \geq 0.5T_{BEC}$), since we used the normal Hartree-Fock excitation spectrum for the non-condensate atoms. This is much simpler than the HFB spectrum we used in Ch.4. Our emphasis in Ch. 5 was to incorporate the new effects arising from collisions. This simplified HF spectrum must become invalid at very low temperatures [40].

Finally, in Ch. 6 we derived a kinetic equation for the quasiparticle distribution function and a generalized Gross-Pitaevskii equation which, while still approximate, should be valid even at low temperatures. We have included both the diagonal and off-diagonal Beliaev propagators, but we again neglect the pair correlation described by \tilde{m} . The resulting kinetic equation describes the dynamics of Bogoliubov-Popov quasiparticles. As expected, Bose coherence factors involving the Bogoliubov u and v functions enter into the integrand of the collision integrals. Of course, these results reduce at higher temperatures to those obtained in Ch. 5.

We note that Kane and Kadanoff [15] based their pioneering work four decades ago on a simpler second-order approximation for the self-energies than the one we use in Ch. 6. It had the advantage of being a “conserving” approximation (see Section 3.2).

The quasiparticle approximation that we have used in Ch. 6 allowed us to derive the kinetic equation in a Boltzmann-like form as given in (6.48). In deriving these results, however we neglected the real part of the second-order self-energies. These will give rise to many-body corrections to the single-particle spectrum, first considered by Beliaev at $T = 0$ in 1957 [22]. Our present KB analysis (in Ch. 6) could be generalized to include such second-order renormalization effects in a more self-consistent manner, but this improved theory would be very complex. A first step might be to include some of these renormalization effects by using a Lorentzian quasiparticle approximation for single-particle Beliaev spectral densities. However, this simple procedure cannot be guaranteed to lead to a conserving approximation [15, 20].

An obvious question that arises in connection with the discussion in Ch.6 is to estimate at what temperatures will we expect that the use of a kinetic theory based on Bogoliubov quasiparticles will be necessary (vs. the simpler Hartree-Fock spectrum of Ch.5). We recall that Giorgini et al. have shown [40] that the thermodynamic properties of a trapped Bose-condensed gas at low temperatures are still well approximated using the Hartree-Fock single-particle spectrum. One thus could expect that the Hartree-Fock spectrum will also be good for the transport properties in a trapped Bose-condensed gas. The difficulty is that the corrections to the simple HF spectrum will probably always be small until we get into a region of temperature where the contribution of thermal excitations is very small to begin with.

We could use our results in Ch. 6 to derive the Landau-Khalatnikov two-fluid hydrodynamic equations [7], which are valid in the collision-dominated local equilibrium region. Indeed, this derivation was the original goal of the classic work of Kane and Kadanoff [15]. We note that the approach developed in Ref. [7] is based on a quasiparticle kinetic equation which is precisely of the kind we have derived in Ch. 6. Thus one sees how our kinetic equation in (6.48) could be used to derive the Landau-Khalatnikov two-fluid equations, including hydrodynamic damping from various transport processes.

Kane and Kadanoff [15] did not include these latter processes in their work. This would generalize recent work [18] in which the Landau-Khalatnikov two-fluid equations with hydrodynamic damping are derived starting from the kinetic equation discussed in Ch. 5.

The kinetic equation for the quasiparticle excitations which we derived in Ch.6, coupled to a generalized GP equation, provides the platform for studying different non-equilibrium aspects of a dynamics of a trapped, Bose-condensed gas at all temperatures in the collisionless regime. Some applications of these equations are:

(1) The coupled equations in (5.22) and (5.28) derived in Ch. 5 have been recently used to study theoretically the rate of condensate formation and growth in a trapped Bose gas [49]. The most detailed experimental study of condensate creation was done at MIT by Miesner et al. [50] using ^{23}Na atoms confined within a highly anisotropic trap. Using evaporative cooling, sodium atoms were cooled to a temperature just above the transition temperature and subsequently a large fraction of the atoms above a certain energy was removed. The gas then relaxes to a new equilibrium state below the BEC transition temperature and a condensate forms.

To describe these experiments, Bijlsma et al. [49] start from (5.22) and (5.28) to describe the non-equilibrium dynamics of the both condensate and non-condensate atoms. The collision integral in (5.21) allows the transfer of the atoms between the two components. To describe the condensate, they use the Thomas-Fermi approximation for the condensate density profile and neglect the dynamics of the condensate. To describe the non-condensate atoms, an ergodic approximation was used for simplicity. That is, all points in phase space having the same energy are assumed to be equally probable and therefore the distribution function describing the non-condensate atoms only depends on the energy and time variables, i.e. $f(\mathbf{r}, \mathbf{p}, t) \equiv g[E_p(\mathbf{r}, t), t]$. This ergodic approximation leads to a simplified kinetic equation which is solved numerically. The results reproduce the experimental data [50] quite well.

(2) Jackson and Zaremba [51, 52] have recently developed a powerful Monte-Carlo algorithm to solve the kinetic equation (5.22) numerically. They used this method, together with a numerical solution of the generalized GP equation (5.28), to determine the temperature-dependent frequency and damping of the so-called “scissors mode” in an anisotropic trap. This scissors mode is particularly interesting because it gives a direct probe of superfluidity [53] in a trapped atomic Bose gas and has been measured experimentally [54] as a function of temperature.

The scissors mode is excited in the experiment [54] by suddenly rotating the confining trap with respect to the equilibrium configuration resulting in the oscillations of the condensate in the horizontal plane. At finite temperatures, both the condensate and thermal cloud excite complex coupled damped oscillations. Starting from the (5.22) and (5.28), Jackson and Zaremba [51, 52] numerically simulate the Oxford experiment. They numerically solve the generalized Gross-Pitaevskii equation to describe the dynamics of the condensate, while the semiclassical Boltzmann equation is simulated using a Hamiltonian description of the particle dynamics. The trajectory for a given atom between collisions is calculated by using Newton’s equations of motion. To simulate the collision integrals, they calculate the probability of two atoms to undergo a collision. If this probability is greater than the randomly generated uniform number between 0 and 1, the collision is accepted. This procedure is then repeated for each pair of atoms. When the equilibrium state of the system is obtained, they rotate the particle coordinates and condensate wavefunction relative to the trap and then let the system evolve. The results they obtain are in very good agreement with the Oxford experimental results.

These results of Jackson and Zaremba show in a convincing manner that the ZNG coupled equations [17] as derived in Ch. 5 contain all the correct physics to describe the dynamics of trapped Bose gases at finite temperatures.

Stimulated by our work as presented in Ref. [29], results similar to those in Ch. 6 have been also recently derived by a group working at JILA [55, 56] starting from a

very different formalism. The JILA kinetic theory is based on a quantum kinetic master equation approach [57]. Wachter et al. [55, 56] have recently shown that this JILA kinetic theory can be reformulated in terms of Bogoliubov quasiparticles and shown to reduce to the results we obtained in Ch. 6, based on the Kadanoff-Baym formalism within the second-order Beliaev self-energy approximation. However, the KB formalism would seem to be more general than the JILA kinetic theory approach, since it allows one to derive the kinetic equations for any self-energy approximation. More precisely, given any specific approximation for an equilibrium self-energy functions, one can generalize them using the KB method to find a related theory of the non-equilibrium dynamics of a trapped Bose gas. An important area of future work on the non-equilibrium behaviour of a trapped Bose gas will be to understand the precise connection between the different formalisms used in the modern BEC literature.

Throughout the thesis, we have noted several extensions of our formalism which would be very worthwhile. One specific calculation would be to include collisions in a kinetic theory based on the full Hartree-Fock-Bogoliubov (HFB) spectrum discussed in Ch. 4. We also note that the kinetic equation given in (6.19) is still quite general and presumably would be a good starting point for further theoretical studies.

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