FINESS lecture, Durham, Sept.14 -17, 2009

The hydrodynamics of ultracold superfluid gases at finite temperatures

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Outline

1. Review what is meant by the collision-dominated **hydrodynamic region**, using the ZNG theory of a dilute weakly interacting Bose gas as a concrete example.

2. Review the physics of the **Landau two-fluid hydrodynamic** equations which describe superfluid ^{4}He and second sound.

3. Discuss the hydrodynamic modes in a strongly interacting Fermi superfluid gas (at unitarity).

My collaborators

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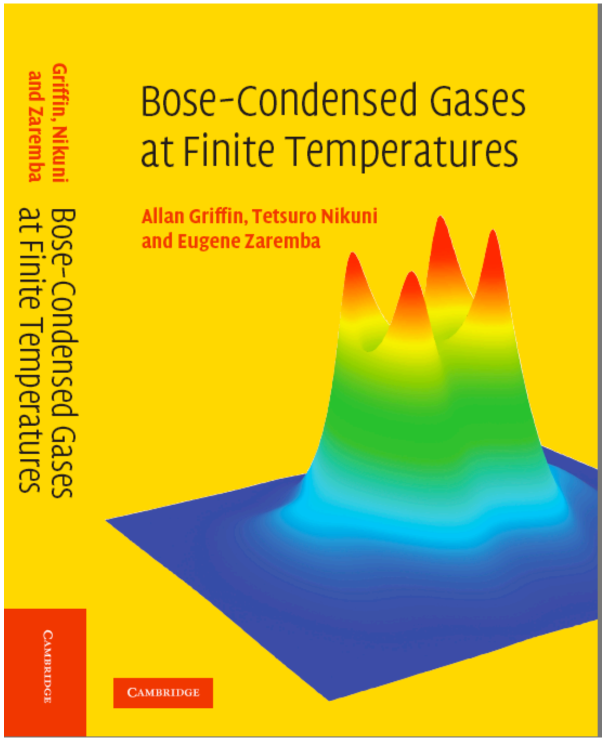
A. Griffin, T. Nikuni and E. Zaremba, ***Bose-Condensed Gases at Finite Temperatures*** (Cambridge, 2009)



Chs. 14-17 give a detailed treatment of the two-fluid hydrodynamics I discuss in this lecture.

Why is two-fluid hydrodynamics in atomic gases interesting?

It makes **connections** between superfluid **gases** and superfluid **liquids**.

It allows us understand superfluidity more deeply when the quantum **superfluid component is coupled to a normal fluid**.

The two-fluid collision dominated hydrodynamic region is the **most interesting** extension from simple GP theory at $T = 0$, because **both the condensate and the non-condensate** components are described by a few "hydrodynamic"variables, like density and velocity.

Finally, two-fluid hydrodynamics is by definition **correct in the limit of strong interactions**, which can be studied at **unitarity** in the BCS-BEC crossover region. This makes connection with all sorts of exotic quantum systems, like quark-gluon plasmas.

Dynamics of a pure condensate at T = 0

This is based on the familiar Gross-Pitaevskii time-dependent equation :

$$\boxed{i\hbar\frac{\partial\Phi(\mathbf{r},t)}{\partial t} = \left[-\frac{\hbar^2\nabla^2}{2m} + V_{\text{trap}}(\mathbf{r}) + g|\Phi(\mathbf{r},t)|^2\right]\Phi(\mathbf{r},t)}$$

All the atoms are in the **same** single-particle quantum state described by the macroscopic wave function with amplitude and phase

$$\Phi(r,t) = \sqrt{n_c(r,t)}\,e^{i\phi(r,t)} \qquad with\ \vec{v}_c(r,t) = \frac{\hbar\nabla\phi}{m}$$

The GP equation can be reduced to two **coupled equations for two local variables**, the density and velocity, which depend on **r** and *t*. This description recalls **ordinary fluid dynamics** which involves a few differential equations for local macroscopic variables such as the local density, velocity, pressure, temperature, etc. The GP theory written in terms of the condensate density n_c and velocity v_c is often called the **hydrodynamic theory of superfluids**. It really isn't!

Dynamics of a Bose condensate coupled to a thermal cloud

The first topic I want to review is the ZNG theory of the coupled dynamics of a Bose condensate and the thermal cloud, extending the $T = 0$ GP theory. This extension allows us to deal with a Bose-condensed gas at **finite temperatures in a nonequilibrium state.**

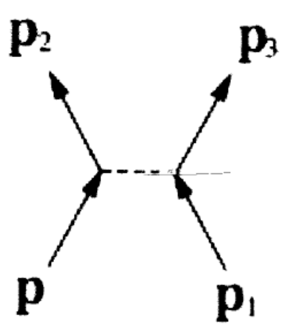
This coupled dynamics allows one, in the case of short relaxation times, to give an explicit derivation of collisional hydrodynamics, starting from a Bose condensate. This simplified theory is important since it gives us a **deeper understanding** of the superfluid two-fluid hydrodynamic description which Landau first developed for liquid ^{4}He.

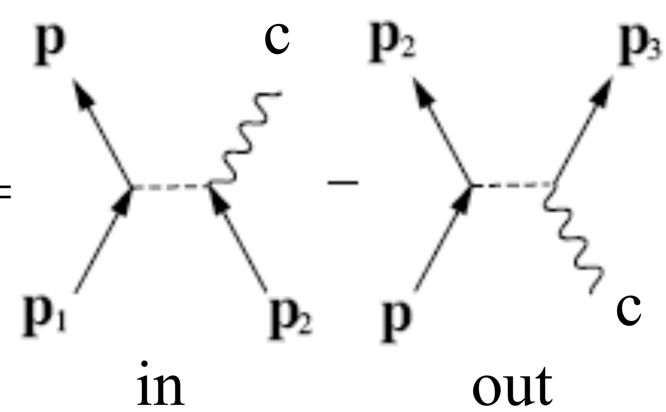
What can we expect this new theory to involve? Clearly we will now have two coupled components:

Condensate dynamics $\rightleftarrows$ Thermal cloud dynamics

Both systems will influence each other:

- By **self-consistent mean fields** produced by the condensate and by the thermal atoms act on one another and themselves.
- **Collisions** between atoms in each part,which can transfer atoms between the condensate and the thermal cloud.

$C_{22} =$ 

$C_{12} =$ 

C_{22} collisions between atoms in the thermal cloud- as in a normal Bose gas.

These C_{12} collisions change the number of atoms in the condensate. Atoms scatter into or out of condensate.

$$p_1 + p \Rightarrow p_2 + p_3$$

How do we describe the thermal cloud atoms?

Clearly their dynamics will be governed by some sort of **Boltzmann or kinetic equation -** just as in a classical gas.

In the ZNG theory, we use the simplest kind of Boltzmann equation for a semi-classical **single particle distribution function** $f(\boldsymbol{p}, \boldsymbol{r}, t)$ This distribution tells us, in a non-equilibrium state, how many atoms have **momentum** $\boldsymbol{p}$, at **position** $\boldsymbol{r}$ and at **time** t. Collisions with other atoms in the cloud and in the condensate relax this distribution function back to the thermal equilbrium value $f^0(\boldsymbol{p}, \boldsymbol{r})$, which is **time-independent** but still depends on the position.

This **semiclassical** approximation is okay at higher tempertures, where the thermal cloud can be treated as atoms moving in the trap potential and in **self-consistent HF fields.** However, the collisions described by C_{22} and C_{12} do take into account **Bose statistics**. This is very important.

Condensate dynamics at finite temperatures

At finite T, ZNG obtain a **generalized GP equation**

$$i\hbar\frac{\partial\Phi}{\partial t} = \left[-\frac{\hbar^2\nabla^2}{2m} + V_{\rm trap}(\mathbf{r}) + gn_{\rm c}(\mathbf{r},t) + 2g\tilde{n}(\mathbf{r},t) - iR(\mathbf{r},t)\right]\Phi$$

where the new imaginary term is given by

$$R(\mathbf{r},t) \equiv \frac{\hbar\Gamma_{12}[f,\Phi]}{2n_{\rm c}(\mathbf{r},t)}$$

in terms of the new **source function** associated with coupling to the thermal cloud atoms:

$$\boxed{\begin{aligned}\Gamma_{12}(\mathbf{r},t) = 2\ g^2\frac{n_{\rm c}(\mathbf{r},t)}{(2\pi)^5}\int d\mathbf{p}_1\int d\mathbf{p}_2\int d\mathbf{p}_3\ \delta(\mathbf{p}_{\rm c}+\mathbf{p}_1-\mathbf{p}_2-\mathbf{p}_3)\\ \times\,\delta\left(\varepsilon_{\rm c}+\tilde{\varepsilon}_1-\tilde{\varepsilon}_2-\tilde{\epsilon}_3\right)\\ \times\,[f_1(1+f_2)(1+f_3)-(1+f_1)f_2f_3]\,.\end{aligned}}$$

Kinetic equation for the dynamics of the thermal atoms

f (p, r, t) describes the single-particle distribution for atoms of momentum ***p***, at position ***r*** *and* time *t*. A kinetic equation describes how it change in time when perturbed from equilbrium:

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \vec{\nabla}_r f \cdot \frac{\partial \vec{r}}{\partial t} + \vec{\nabla}_p f \cdot \frac{\partial \vec{p}}{\partial t} = \frac{\partial f}{\partial t} + \frac{\vec{p}}{m} \cdot \vec{\nabla}_r f + \vec{F} \cdot \vec{\nabla}_p f$$

$$= C_{12}[f, \Phi] + C_{22}[f, \Phi]$$

Atoms move in a **self-consistent time-dependent Hartree-Fock potential**

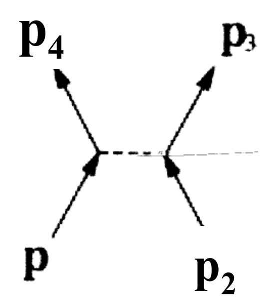
$$\tilde{\varepsilon}_p(r,t) = \frac{p^2}{2m} + U(r,t)$$

$$U(r,t) \equiv V_{trap}(r) + 2g[n_c(r,t) + \tilde{n}(r,t)]$$

$$\vec{F} \equiv \frac{\partial \vec{p}}{\partial t} = -\vec{\nabla}_r U(r,t)$$

Two kinds of collision terms in the kinetic equation

Collisions between thermal atoms are described by



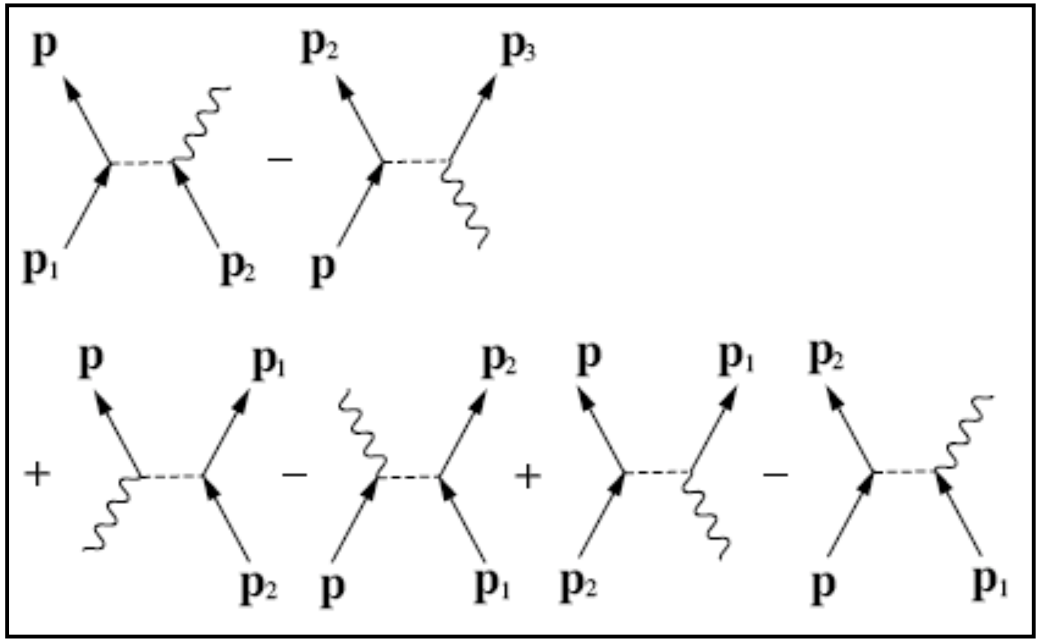
$$C_{22}[f] = \frac{2g^2}{(2\pi)^5\hbar^7}\int d\mathrm{p}_2 \int d\mathrm{p}_3 \int d\mathrm{p}_4 \delta(\mathrm{p}+\mathrm{p}_2-\mathrm{p}_3-\mathrm{p}_4)$$
$$\times\delta(\tilde{\varepsilon}_p+\tilde{\varepsilon}_{p_2}-\tilde{\varepsilon}_{p_3}-\tilde{\varepsilon}_{p_4})\left[(1+f)(1+f_2)f_3f_4 - ff_2(1+f_3)(1+f_4)\right]$$

Because of **Bose statistics**, in collisions, we have the statistical factors $(1+f_i)$ for the creation or f_i for the destruction of an atom in state i . For Bose atoms, f_i can be **large** >>1.

We note that the collision integrals are **second order** in the interaction strength g (binary collisions since the gas is dilute).

A **new** kind of collision term C_{12} arises when there is a Bose condensate, involving **scattering between atoms in the condensate and thermal cloud**. These collisions play an important role in the whole theory.

C_{12} collisions



wiggly lines are condensate atoms

$$C_{12}[f] = \frac{2g^2 n_c}{(2\pi)^2\hbar^4}\int d\mathbf{p}_1 \int d\mathbf{p}_2 \int d\mathbf{p}_3 \delta(m\mathbf{v}_c + \mathbf{p}_1 - \mathbf{p}_2 - \mathbf{p}_3)$$
$$\times\delta(\varepsilon_c + \tilde{\varepsilon}_{p_1} - \tilde{\varepsilon}_{p_2} - \tilde{\varepsilon}_{p_3})[\delta(\mathbf{p}-\mathbf{p}_1) - \delta(\mathbf{p}-\mathbf{p}_2) - \delta(\mathbf{p}-\mathbf{p}_3)]$$
$$\times[(1+f_1)f_2 f_3 - f_1(1+f_2)(1+f_3)],$$

Summary: ZNG coupled equations

Generalized GP equation for the condensate dynamics

$$i\hbar\frac{\partial\Phi}{\partial t} = \left(-\frac{\hbar^2\nabla^2}{2m} + V + g[n_c + 2\tilde{n}] - iR\right)\Phi$$

is coupled to the kinetic equation for the thermal cloud

Collisional coupling
(atom transfer)

mean field coupling

$$\frac{\partial f}{\partial t} + \frac{\mathbf{p}}{m}\cdot\nabla f - \nabla U\cdot\nabla_{\mathbf{p}} f = C_{22}[f] + C_{12}[f]$$

C_{12} collisions $\Rightarrow$ $$R = \frac{\hbar}{2n_c}\int\frac{d\mathbf{p}}{(2\pi\hbar)^3}C_{12}$$

The coupled generalized GP equation for the condensate atoms and the Boltzmann kinetic equation for the thermal cloud atom distribution function allow us to work out the complete **non-equilibrium behavior of a trapped Bose-condensed gas.**

There are two traditional regions of interest at finite T:

Collisionless region - where mean fields mainly determine the solution of the kinetic equation.

Collisional hydrodynamic region - where atomic collisions determine the **form** of the solution of the kinetic equation.

The hydrodynamic **collision-dominated** region describes **collective oscillations** with a period T only if the appropriate atomic collision time τ satisfies the condition

$$\tau << T \Rightarrow \omega\tau << 1$$

This is equivalent to saying that the mean free path of the atoms must be **much smaller** than the wavelength of the collective mode.

In a **normal fluid** , hydrodynamics allows one to describe the dynamics in terms of a few macroscopic local variables, like **local** density $n(\mathbf{r},t)$,pressure $P(\mathbf{r}, t)$, temperature $T(\mathbf{r}, t)$, etc. This requires **local equilibrium**.

In the two-fluid region for Bose superfluids, both the condensate and thermal cloud have a hydrodynamic description in terms of a few variables like density, velocity, etc.

Bose distribution function in local equilibrium

$$C_{22}[f] = \frac{2g^2}{(2\pi)^5\hbar^7}\int d\mathbf{p}_2\int d\mathbf{p}_3\int d\mathbf{p}_4\delta(\mathbf{p}+\mathbf{p}_2-\mathbf{p}_3-\mathbf{p}_4)$$
$$\times\delta(\tilde{\varepsilon}_p+\tilde{\varepsilon}_{p_2}-\tilde{\varepsilon}_{p_3}-\tilde{\varepsilon}_{p_4})\left[(1+f)(1+f_2)f_3f_4-ff_2(1+f_3)(1+f_4)\right]$$

For $\tilde{f}(\mathbf{p},\mathbf{r},t) = \frac{1}{e^{\beta[\frac{1}{2m}(\mathbf{p}-m\mathbf{v}_n)^2+U-\tilde{\mu}]}-1}$;

This local equilibrium distribution denoted by a tilde ~

$C_{22}[\tilde{f},\Phi]=0$ for **any** value of $\Phi(\mathbf{r}, \mathrm{t})$

$$\boxed{(1+f_1)(1+f_2)f_3f_4 - f_1f_2(1+f_3)(1+f_4) = 0}$$

All the quantities temperature, local velocity, chemical potential and the effective HF potential now depend on the position *r* in the trap and the time *t*. This is what local hydrodynamic equilibrium means.

Two-fluid collisional hydrodynamic region

In this **limit,** the interactions are **strong** enough to produce **local equilibrium** in the thermal gas, as described by the local Bose distribution function $\tilde{f}$. This means the thermal gas can be described by a few **local** variables, just as in **ordinary fluid dynamics.** Inserting this local distribution into the Boltzmann equation and taking momentum moments, we can derive hydrodynamic equations for these coarse-grained local variables. However, the new aspect in a Bose-condensed gas is that these normal fluid variables are coupled to the variables describing the atoms Bose-condensed into a single-particle state.

This leads to the kind of **two-fluid hydrodynamics** first discussed for **superfluid Helium** by Landau (1941).

Time-scales for non-equilibrium behaviour

As with classical gases studied by Boltzmann, our work shows that the non-equilibrium time development of a **Bose superfluid gas** can be naturally divided into **three regions:**

1. A **very short time** scale, where behavior is very complex.
2. An **intermediate time** scale comparable to a collision time, where a **kinetic** equation describes the time evolution.
3. A **very long time** scale, where local equilibrium has developed and the time evolution is described by **hydrodynamic** equations.

Remark: It would be interesting to try and develop the analogous picture using c-field concepts

Summary of what we have accomplished.

Assuming that the thermal cloud is described by the **local equilibrium Bose distribution**, ZNG have derived a set of coupled hydrodynamic equations for both the condensate and the thermal cloud. These equations are a closed set for the **macroscopic variables** for both components. All reference to the microscopic thermal atom **distribution function** has disappeared. We are left with local densities, velocities, etc. that **only** depend on ***r*** and ***t***.

The set of coupled equations derived by ZNG can be shown to be **precisely equivalent** to the Landau two-fluid hydrodynamic equations for a Bose gas. To do this, we need to rewrite the ZNG microscopic results which are given in terms of the condensate and thermal cloud **density and velocity fluctuations** in terms of the **thermodynamic** variables(pressure, entropy, etc) that Landau used.

In our **trapped Bose gas**, of course, the elementary excitations that must be used to calculate the thermodynamic properties in the Landau two-fluid equations are not the phonon-roton excitations used for liquid ^{4}He. In the ZNG theory, the atoms are described **a Hartree-Fock self-consistent energy spectrum**. In this **dilute** gas, moreover, the superfluid and normal fluid correspond to the condensate and the thermal cloud, to a very good approximation. In liquid ^{4}He, things are very different in that the strong interactions deplete the condensate to 10% at $T = 0$, and the phonon-roton excitations are quite **different** from bare ^{4}He atoms.

The Landau two fluid equations are now understood to be always correct for **superfluid hydrodynamics**, in liquid ^{4}He, Bose gases and Fermi gases. This statement is analogous to the statement that the equations of **classical fluid dynamics** are generic, ie, valid for **all** normal fluids, if we restrict ourselves to the **low frequency** region.

The ZNG **derivation** of Landau two-fluid hydrodynamics is important conceptually since we started with a **Bose condensate**. It is the **basis of whole ZNG discussion**. This is important since Landau in his derivation in 1941 did not explicitly use a Bose condensate, giving the **incorrect impression** that it was not essential. **It is !!**

Our derivation for a dilute Bose gas is also important since it gives one a feeling for the microscopic basis of the two-fluid hydrodynamic equations. As a result , one has more confidence in using them even in **strongly interacting systems** where it is **difficult** to carry out an “explicit” proof. We will use these equations in a Fermi gas superfluid at unitarity , when the *s*-wave interaction strength becomes **infinitely strong** .

Tisza-Landau two-fluid hydrodynamics 1938-1941

Tisza

Landau

Superfluid*: *component of liquid which is associated with macroscopic occupation (BEC) of* **one** ***single-particle *state .Carries zero entropy, flows without dissipation with an irrotational velocity.***

Normal fluid: *comprised of* ***incoherent*** *thermal excitations , behaves like any fluid at finite temperatures in* ***local thermodynamic*** *equilibrium. This requires strong collisions.*

Laszlo Tisza died on April 15, 2009. He was 101.

Landau two-fluid hydrodynamic equations (1941)

$$\frac{\partial n}{\partial t} + \nabla \cdot \mathbf{j} = 0$$

Conservation equation

$$m\frac{\partial \mathbf{j}}{\partial t} = -\nabla P - n\nabla V_{\mathrm{trap}},$$

Euler equation

$$m\frac{\partial \mathbf{v}_{\mathrm{s}}}{\partial t} = -\nabla \mu,$$

This says superfluid is irrotational.

$$\frac{\partial s}{\partial t} + \nabla \cdot (s\mathbf{v}_{\mathrm{n}}) = 0$$

This says the superfluid carries no entropy.

The mass density and current are the sum of the superfluid and normal fluid components:

$$mn \equiv \rho = \rho_{\mathrm{s}} + \rho_{\mathrm{n}},$$

$$m\mathbf{j} \equiv \rho_{\mathrm{s}}\mathbf{v}_{\mathrm{s}} + \rho_{\mathrm{n}}\mathbf{v}_{\mathrm{n}}$$

These equations can be written in different forms.

Landau two-fluid equations of motion (non-dissipative limit) in linearized form:

$$\frac{\partial \delta n}{\partial t} + \nabla \cdot \delta \mathbf{j} = 0$$

$$m\frac{\partial \delta \mathbf{j}}{\partial t} = -\nabla \delta P - \delta n \nabla U_{ext}$$

$$m\frac{\partial \delta \mathbf{v}_s}{\partial t} = -\nabla \delta \mu$$

All local variables are linearized around their equilibrium values.

$$\frac{\partial \delta s}{\partial t} + \nabla \cdot (s_0 \delta \mathbf{v}_n) = 0\,,$$

The linearized current and density fluctuations are

$$m\delta \mathbf{j}(\mathbf{r},t) = \rho_{s0}(\mathbf{r})\delta \mathbf{v}_s(\mathbf{r},t) + \rho_{n0}(\mathbf{r})\delta \mathbf{v}_n(\mathbf{r},t),$$
$$m\delta n(\mathbf{r},t) = \delta \rho_s(\mathbf{r},t) + \delta \rho_n(\mathbf{r},t).$$

What are conditions for the Landau equations?

We need the **normal fluid** dynamics to be described in terms of a few macroscopic local variables, like **local** pressure $P(\mathbf{r}, t)$, **local** temperature $T(\mathbf{r}, t)$, etc. This requires **local equilibrium**. Just as in normal fluids, this in turn requires strong collisions between the atoms.

The two-fluid equations will describe **collective oscillations** with a period T only if the appropriate **atomic collision time τ** satisfies the condition

$$\tau << T \Rightarrow \omega\tau << 1$$

Note that the Bose condensate involves only one single-particle quantum state and hence is **always** described by the density and velocity of the atoms in this quantum state.

First and second sound in uniform superfluids

Using the thermodynamic identity,

$$n_0 \delta\mu = -s_0 \delta T + \delta P$$

one can reduce the two-fluid equations to **two coupled wave equations**

$$\frac{\partial^2 \delta\rho}{\partial t^2} = -m\nabla \cdot \frac{\partial \delta \mathbf{j}}{\partial t} = \nabla^2 \delta P$$

$$\frac{\partial^2 \delta \bar{s}}{\partial t^2} = \bar{s}_0^2 \left(\frac{\rho_{s0}}{\rho_{n0}} \right) \nabla^2 \delta T$$

where the entropy per unit mass is defined as $\bar{s} \equiv s/\rho$

We can **eliminate** the fluctuations in pressure and temperature using the relations:

$$\delta P = \left(\frac{\partial P}{\partial \rho} \right)_{\bar{s}} \delta\rho + \left(\frac{\partial P}{\partial \bar{s}} \right)_{\rho} \delta\bar{s},$$

$$\delta T = \left(\frac{\partial T}{\partial \rho} \right)_{\bar{s}} \delta\rho + \left(\frac{\partial T}{\partial \bar{s}} \right)_{\rho} \delta\bar{s}.$$

In **uniform** Bose superfluids, we know that the solutions of the coupled wave equations are plane waves

$$\delta\rho, \delta\bar{s} \propto e^{i(\mathbf{q}\cdot\mathbf{r}-\omega t)}$$

Substituting these into the wave equations, one finds that they reduce to two coupled **algebraic equations** with phonon or **sound wave** solutions

$$\omega^2 = u^2 q^2$$

The two values of the sound velocity are the solutions of the quadratic equation for u^2 (Landau,1941)

$$\boxed{u^4 - u^2\left[\left(\frac{\partial P}{\partial\rho}\right)_{\bar{s}} + \frac{\rho_{s0}}{\rho_{n0}}\frac{T\bar{s}_0^2}{\bar{c}_v}\right] + \frac{\rho_{s0}}{\rho_{n0}}\left(\frac{T\bar{s}_0^2}{\bar{c}_v}\right)\left(\frac{\partial P}{\partial\rho}\right)_T = 0}$$

We see that the velocities involve many **thermodynamic functions and derivatives**. These must be calculated and are **different** for each superfluid.

This equation for the velocities of first and second sound waves is **very general**. It applies to liquid Helium as well as to uniform **Bose and Fermi superfluid gases** , as long as the conditions for the Landau two-fluid equations are satisfied(short collision times).

It turns out that in liquid ^{4}He, the solutions simplify because the pressure and temperature are very **weakly coupled**, so that

$$\left(\frac{\partial P}{\partial T}\right)_\rho = 0$$

Thus first and second sound velocities can be approximated by:

$$\boxed{c_1^2 = \left(\frac{\partial P}{\partial \rho}\right)_{\bar{s}}, \quad c_2^2 = T\frac{\bar{s}_0^2}{\bar{c}_v}\frac{\rho_{s0}}{\rho_{n0}}}$$

One can insert these solutions into the Landau equations and check:

First sound is a simple in-phase motion $\delta \mathbf{v}_\mathrm{n} = \delta \mathbf{v}_\mathrm{s}$.

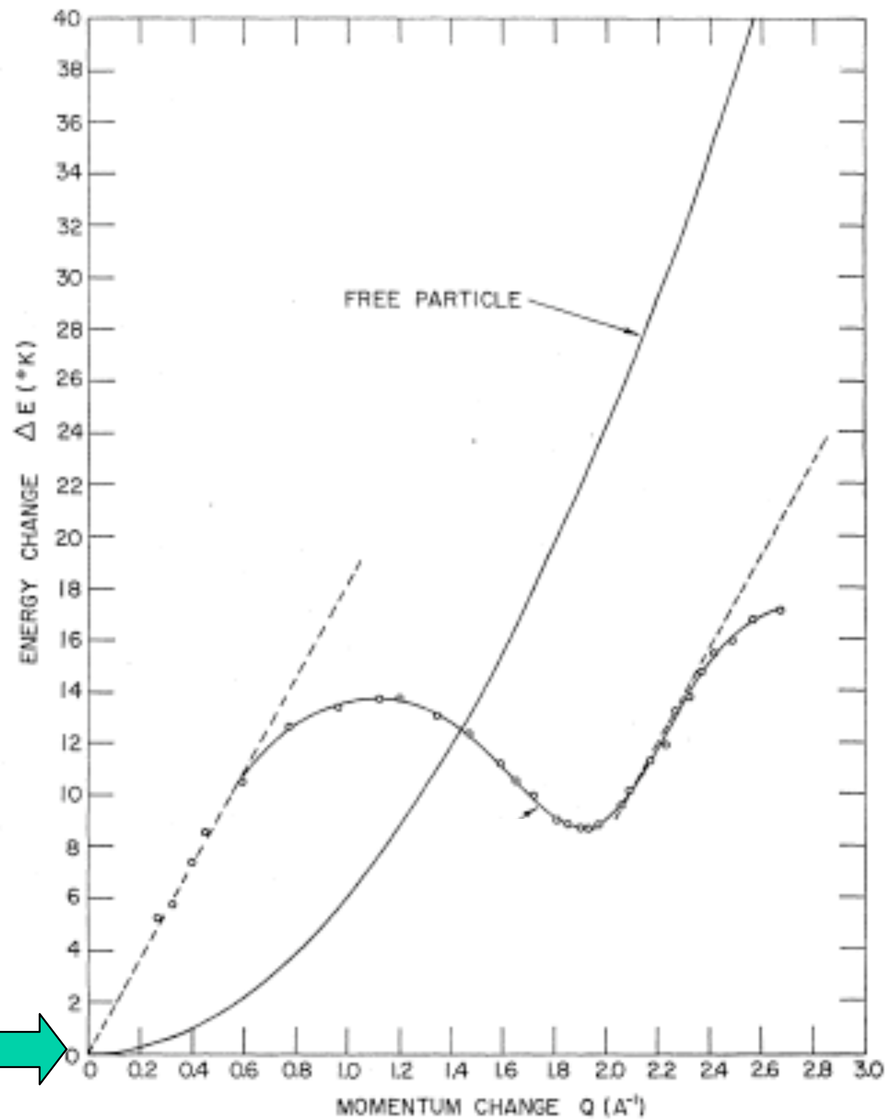
Second sound is an out-of -phase motion $\rho_{\mathrm{n}0}\, \delta \mathbf{v}_\mathrm{n} = -\rho_{\mathrm{s}0}\, \delta \mathbf{v}_\mathrm{s}$.

Elementary excitations in superfluid ^{4}He from neutron scattering data at T = 1K

Henshaw &Woods, 1961

MODES OF ATOMIC MOTIONS IN LIQUID He 1271

ve for
its nor-
mal vapor pressure. The parabolic curve rising from the origin represents the theoretically calculated dispersion curve for free helium atoms at absolute zero. The open circles correspond to the energy and momentum of the measured excitations. A smooth curve has been drawn through the points. The broken curve rising linearly from the origin is the theoretical phonon branch calculated from a velocity of sound of 237 meters sec^{-1}. The dotted curve drawn through the point at 2.27 A^{-1} has been drawn with a slope equal to the velocity of sound.



NB: First and second sound only exist down here

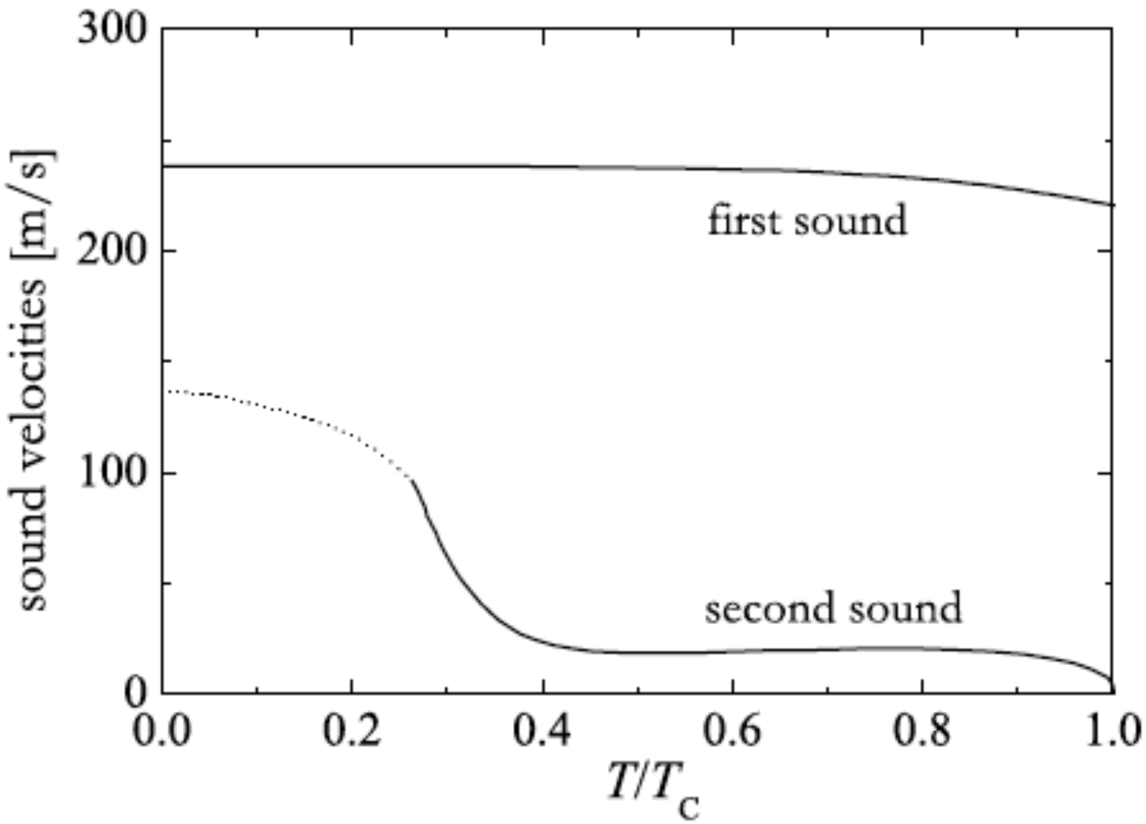
Elementary excitations vs hydrodynamic modes

It is very important in this discussion to keep clearly in mind the difference between:

- **Elementary excitations** (quasiparticles)which determine the thermodynamic properties of the system.
- **Collective modes** which involve oscillations in the density of the elementary excitations. These include the hydrodynamic modes(first and second sound) which are the solutions of the two-fluid equations.

A good example to keep in mind is the **air in this room.** The elementary excitations are well approximated as an ideal gas of **free atoms**. In addition, we know that this air sustains low freqency hydrodynamic **sound waves,** as predicted by the hydrodynamic equations. The sound velocity of these collective density oscillations is given by the usual **compressibilty** of an ideal gas of atoms.

First and second sound in superfluid ^{4}He



The detection of second sound by Peshkov in Moscow in 1944 was one of **greatest discoveries** in low temperature physics. It would be great to find **second sound in superfluid gases!**

Second sound in superfluid atomic gases?

We now come **back to atomic gases** and ask, can we achieve the conditions that validate the Landau two-fluid description?

It looks like it will be **difficult to do** this in atomic Bose gas. This is because one cannot make the interactions or the density large enough to satisfy the **local equilbrium** criterion.

However, **second sound** in Fermi superfluid gases seem much more feasible.This is because we can use a **Feshbach resonance** to adjust the size of the *s*-wave scattering length between two Fermi atoms in different hyperfine states.This allows us to study the famous **BCS-BEC crossover** in Fermi gases.

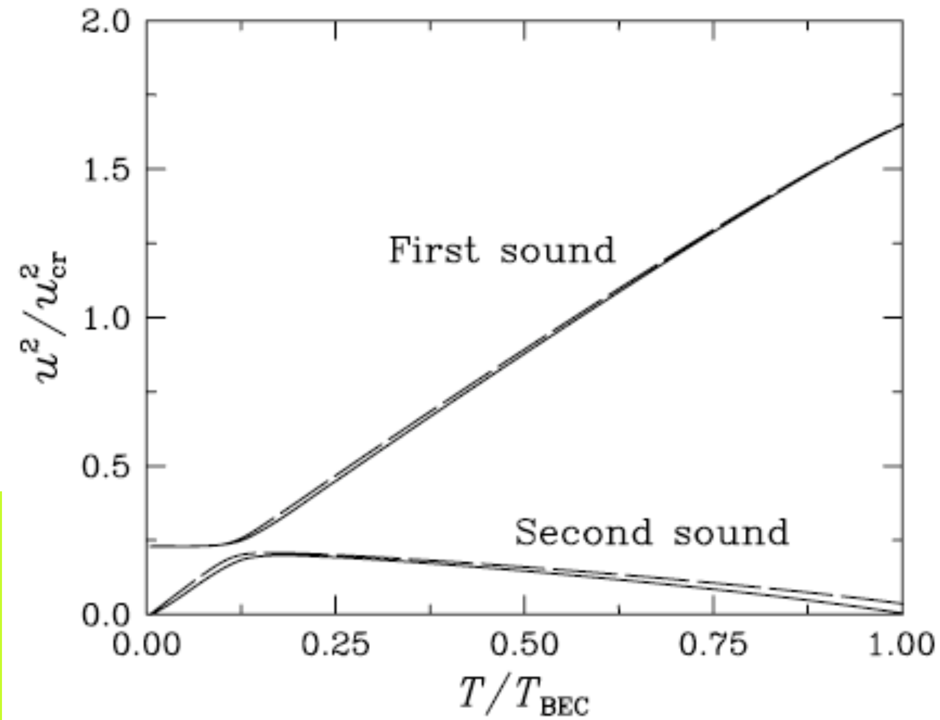
Hydrodynamic modes in dilute weakly interacting uniform Bose gas

It turns out that the hydrodynamic normal modes in a dilute gas are largely **uncoupled oscillations** of the condensate and the thermal cloud components. These are the natural extensions of the pure condensate modes at $T = 0$ and sound waves in a normal Bose gas **above** T_c

$$u_1^2 = \frac{5}{3}\frac{\tilde{P}_0}{m\tilde{n}_0} + \frac{2g\tilde{n}_0}{m}$$

$$u_2^2 = \frac{g n_{c0}}{m}$$

These velocities only depend on the equilbrium pressure, thermal atom density and the condensate density , all at temperature T.



On arXiv, Sept 4, 2009

Second sound dipole mode in a partially Bose-Einstein condensed gas

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(Dated: September 4, 2009)

We study the second sound dipole mode in a partially Bose-Einstein condensed gas. This mode is excited by spatially separating and releasing the center-of-mass of the Bose-Einstein condensate (BEC) with respect to the thermal cloud, after which the equilibration is observed. The oscillation frequency and the damping rate of this mode is studied for different harmonic confinements and temperatures. The measured damping rates close to the collisionless regime are found to be in good agreement with Landau damping. For increasing hydrodynamicity of the cloud we observe an increase of the damping.

Two-fluid hydrodynamics in the BCS-BEC crossover in Fermi superfluid gases

The most interesting recent development in ultracold gases has been the study of superfluid **Fermi atomic gases**. What makes Fermi gases so interesting is that the strength of the inter-atomic interactions can be tuned easily by using a **Feshbach resonance**. Research on this topic has exploded since 2003.

In a two-component Fermi gas, by increasing the attractive interaction, one can **smoothly** go from a **BCS** phase (Fermi quasiparticles moving in a Cooper pair condensate) to a **BEC** phase (involving a Bose condensate of molecules). A Feshbach resonance allows us to **study this crossover** and achieve the conditions for local hydrodynamic equilibrium.

This BCS-BEC crossover is beautiful physics. It has deepened our understanding of the **relation** between the BCS and BEC superfluids, showing that they are unified by the **role of a Bose condensate of bosonic pair states** in both cases. In the older literature, this BEC aspect of the BCS theory was **not** realized or emphasized.

It also allows us to produce a **strongly interacting Bose gas** of **stable** bosonic dimers composed of two Fermi atoms.

It gives us a superfluid Fermi gas with strong interactions which allow us to achieve **local hydrodynamic equilibrium**. When the scattering length is infinite (**unitarity**), we have strong collisions and hence the Landau two-fluid hydrodynamic equations will be valid. This gives a **new system where we can look for the characteristic first and second hydrodynamic oscillations.**

The main problem is that the thermodynamics at finite temperatures in the BCS-BEC crossover is quite complicated, since it involves both Fermi and Bose elementary excitations. Remember, the two-fluid equations have coefficients which involve quantities like entropy, superfluid density, etc.

In the BCS weak coupling phase, there are the expected **BCS fermi quasiparticles** plus the collective oscillations of the Cooper pair condensate. The latter are the **Anderson-Bogoliubov** Goldstone bosons.

As we go over to the BEC side, the BCS quasiparticles **disappear** as the Fermi atoms pair up to form real molecules. The end result is that we are left with a gas of interacting bosonic molecules, described in terms of **Bogoliubov excitations.**

BEC limit: **phase modes** (phonons).

Close to unitarity, both modes exists. **They are strongly coupled.**

BCS limit: **BCS quasiparticles** from breakup of Cooper pairs.

We use the Nozières/Schmitt-Rink (NSR) theory to include both kinds of excitations in a **self-consistent way**. **Pairing fluctuations beyond the BCS mean-field give the physics needed in the crossover region.**

As in the case of superfluid ^{4}He, the **superfluid density** is quite **different** from the condensate fraction of bosonic pairs. The NSR theory gives the known expressions for the normal fluid density in both the **BCS** (in terms of Fermi quasiparticles) and **BEC** limit (in terms of the Bogoliubov bosonic modes).

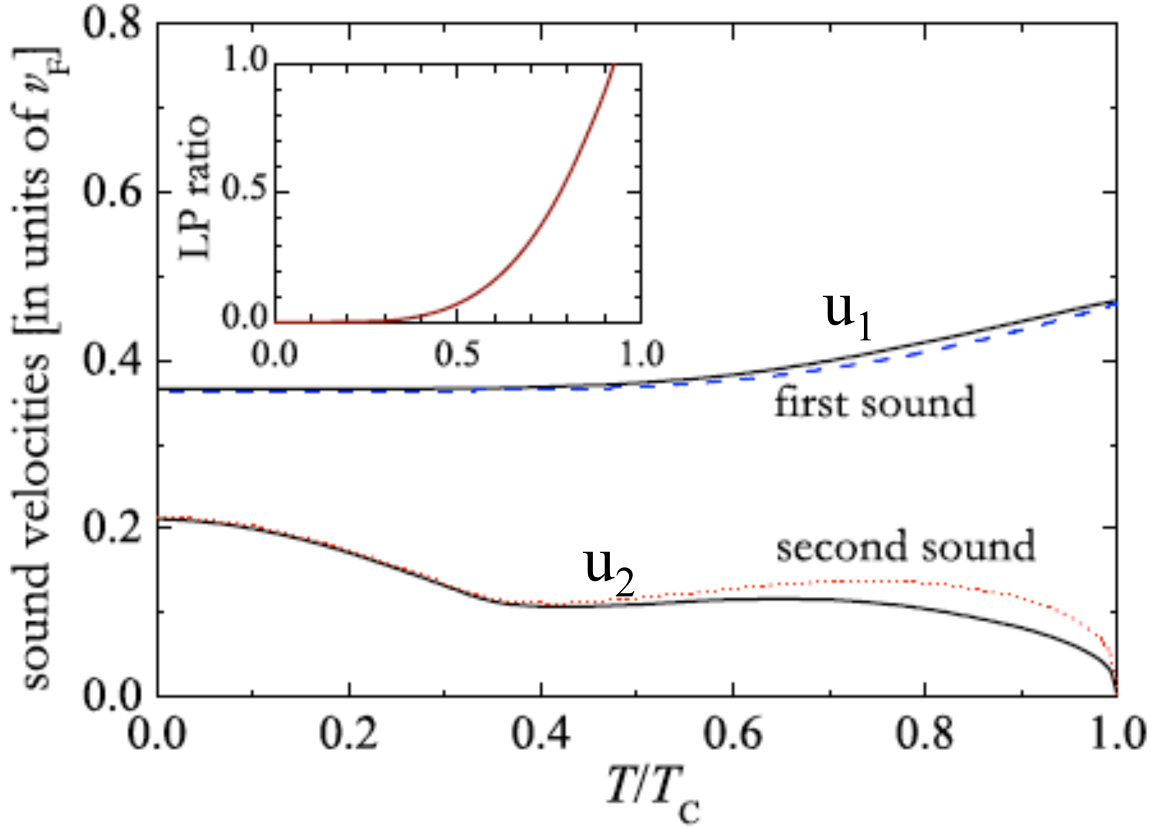
Second sound in superfluid Fermi gases at unitarity

We have evaluated **all the thermodynamic functions which appear in the two-fluid equations**. Even for a **uniform** Fermi gas at unitarity, this requires extensive numerical work. We then use these uniform gas results to obtain the thermodynamic functions in a trap within the local density approximation (LDA).

To find solutions of the Landau equations, we still have to solve these differential equations, where the **coefficients are dependent on position**.This is a difficult math problem! We have developed a **variational procedure** using an ansatz of eight powers of $\boldsymbol{r}$, the coefficients being the variational parameters. So far, we have only considered **breathing** two-fluid modes in an isotropic trap.

We believe our numerical results are **quite accurate**, because they agree with analytic solutions at **very low T** and also **near T_c**.

First and second sound velocities in a uniform Fermi gas at unitarity



Based on NSR theory

At **low *T***, the **bosonic** phonons dominate the thermodynamics.
At **higher temperatures**, the Fermi quasiparticles are the most important thermal excitations at unitarity.

Density response function for a uniform gas

The density fluctuation to first order in a small perturbation Gives the **density response** function. For a uniform gas, this is easily found by adding a time-dependent perturbation δU to the two-fluid equations. One finds

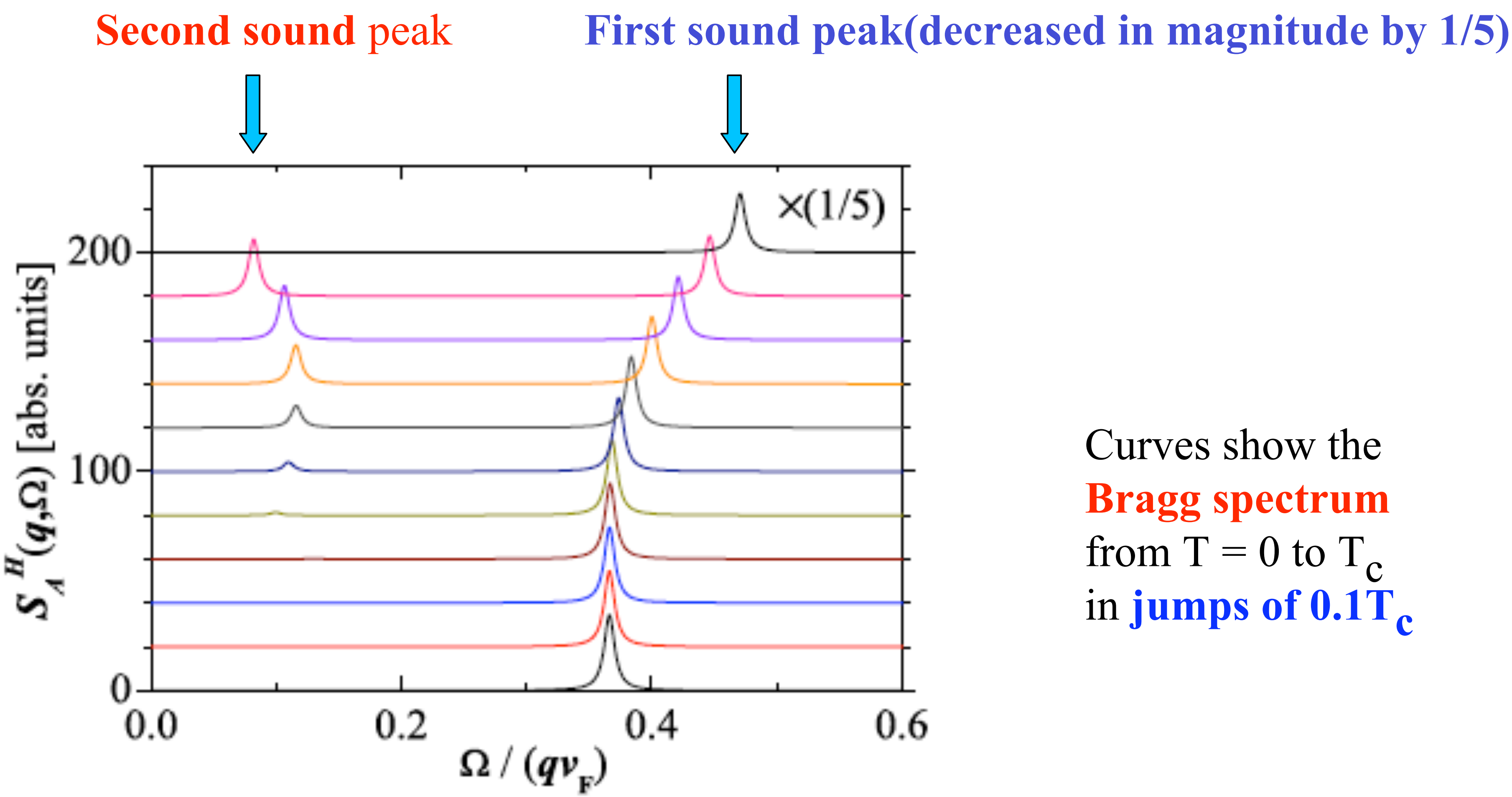
$$\boxed{\operatorname{Im}\chi_{nn}(q,\omega) = -\pi n \frac{q^2}{m}\left[Z_1\delta(\omega^2 - u_1^2 q^2) + (1 - Z_1)\delta(\omega^2 - u_2^2 q^2)\right]}$$

where

$$Z_1 = \frac{u_1^2 - v^2}{u_1^2 - u_2^2} = 1 - Z_2 \quad \text{with} \quad v^2 \equiv \bar{s}_0^2 \frac{\rho_{s0}}{\rho_{n0}} \left(\frac{\partial T}{\partial \bar{s}}\right)_\rho = T \frac{\bar{s}_0^2}{\bar{c}_v} \frac{\rho_{s0}}{\rho_{n0}}.$$

Note that both **first and second sound** appear as **poles** of the density response function, but with quite **different** weights. Using our calculated values of u_1, u_2 and v, one finds that the weight of second sound Z_2 is always smaller than **5%** of first sound.

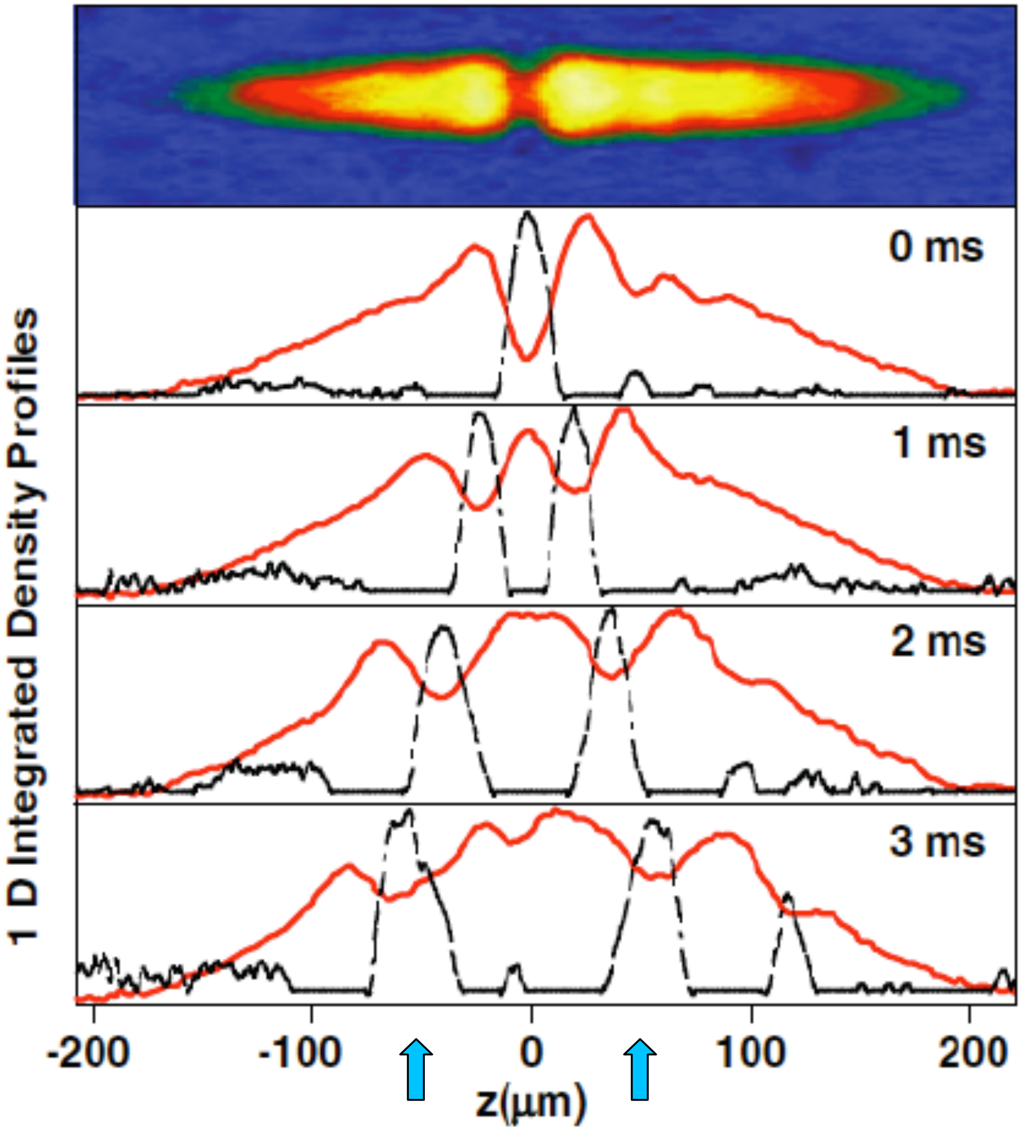
Imaginary part of density response function in a uniform gas

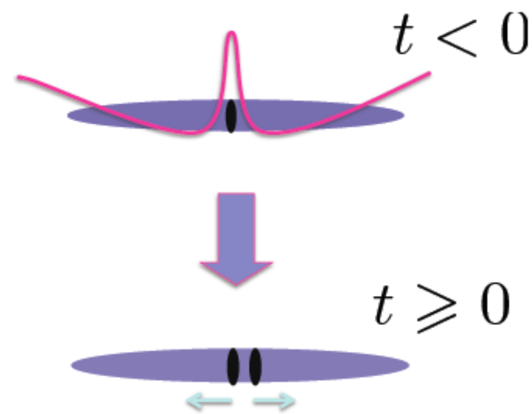


Curves show the **Bragg spectrum** from T = 0 to T_c in **jumps of $0.1T_c$**

The weight of the sound mode $\boldsymbol{i}$ is given not by Z_i but by $S_i = Z_i / u_i$.
This gives **extra weight to second sound** because $u_1 >> u_2$.

Pulse propagation in long cigar-shaped traps





$$\delta U(z,t) = \delta U(z)\theta(-t)$$

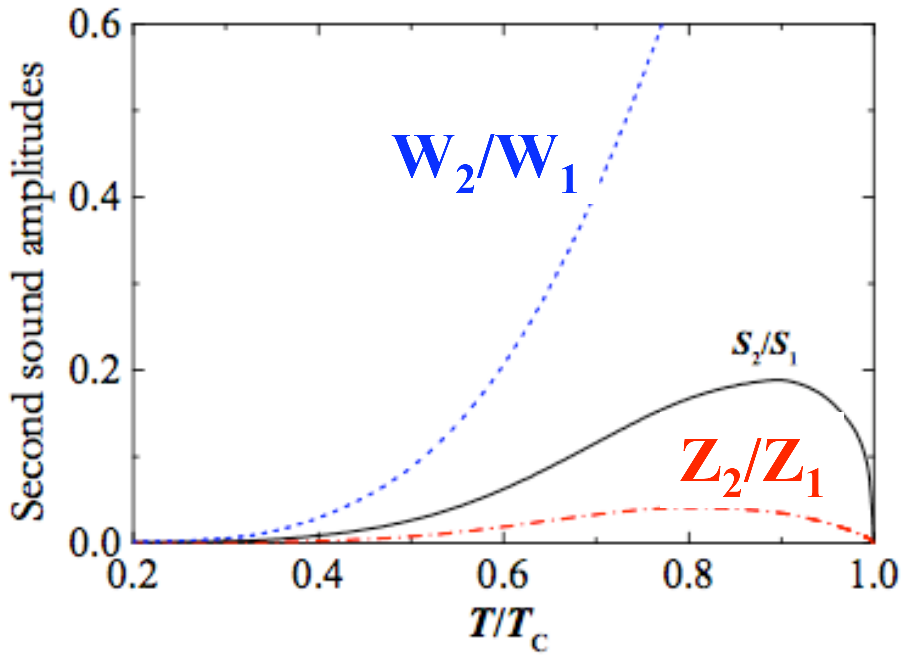
Experiments by the group of John Thomas at Duke University, USA

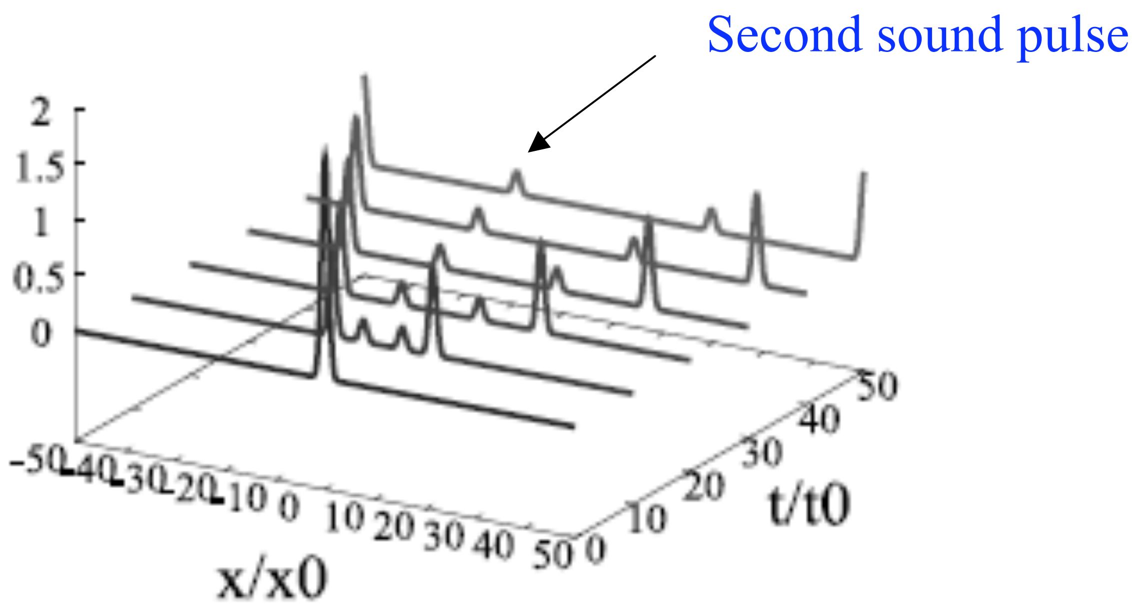
These results are at unitarity in Fermi gas but at very low T

Apply a **localized density pulse** for $t < 0$ given by $\delta U(z,t) = \delta U(z)\theta(-t)$
For $t > 0$, this perturbation produces a pulse in the density propagating along both + z and -z directions. This is described by **linear response theory** using the hydrodynamic **density response** function just discussed.

$$\delta n(z,t) = W_1\left[\delta U(z - u_1 t) + \delta U(z + u_1 t)\right] + W_2\left[\delta U(z - u_2 t) + \delta U(z + u_2 t)\right]$$

where the **amplitudes** of the first and second sound pulses are given by $W_i = Z_i/(u_i)^2$. Because of its **low** velocity, the second sound pulse is **amplified.**

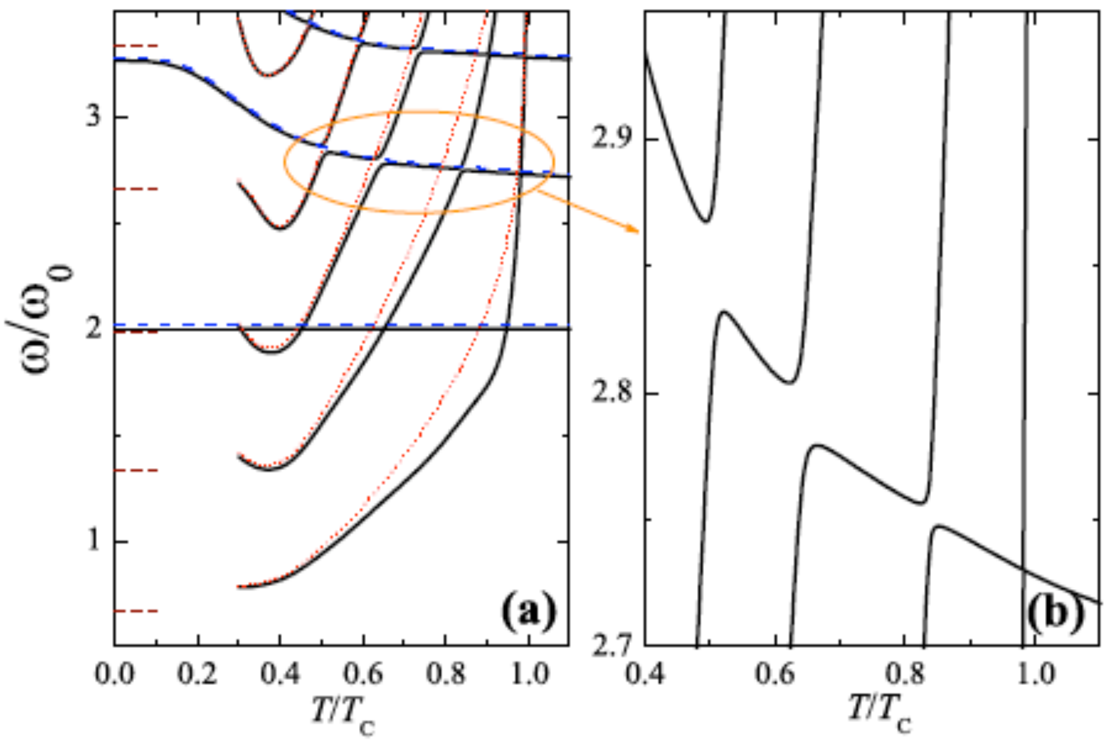




Arahata and Nikuni, arXiv, 2009

Some what to our surprise, in a **density** pulse propagation experiment carried out around $T \sim 0.8T_c$, the contribution from second sound will be **comparable** with that from first sound, even though **second sound is mainly a temperature oscillation**.

First and second sound frequencies at unitarity in a trapped gas



The **left side** shows the spectrum of first and second sound breathing modes, as a function of *T*

The **horizontal** lines are first sound branches and the **vertical** lines are second sound.

Our **variational results** for first sound frequencies agree with analytic expressions at T = 0 and at $T_{c.}$(Bruun and Clark,1999).Our second sound frequencies at low T extrapolate to analytic results at T = 0.

To date, the **only** breathing mode studied by experiment is the **lowest frequency first sound mode.** This is independent of temperature and is an exact eigenstate of a trapped gas (Castin)

Summary

For the first time, we have solved the Landau equations for a strongly interacting superfluid gas in a harmonic trap, and have calculated the frequencies of first and second sound breathing modes as a function of the temperature.

We have found that first sound is mainly an in-phase oscillation, and second sound is mainly an out-of-phase oscillation of the superfluid and normal fluid components.

We expect that experiments will soon detect second sound in superfluid Fermi gases near unitarity.

In the future, we can include the corrections to local equilbrium associated with various transport coefficients, and check recent predictions at unitarity (based on AdS/CFT)