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Two-fluid hydrodynamics in trapped gases

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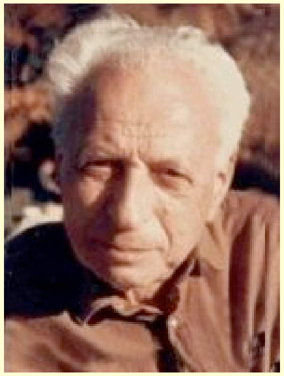
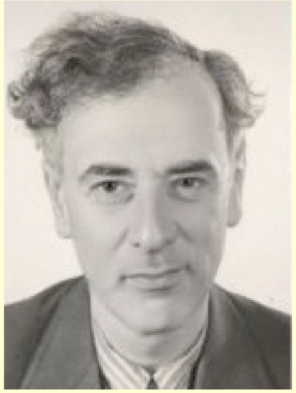
- Landau **two-fluid hydrodynamics** is the general theory for the dynamics of **all** superfluids at finite temperatures.
- Recent experiments with **trapped Fermi gases** with strong interactions (using a Feshbach resonance) should be able to access the **two-fluid regime at finite temperatures.**
- The two-fluid differential equations are difficult to solve for hydrodynamic modes in **non-uniform** trapped gases.We have developed a **variational approach** to calculate the frequencies of the hydrodyamic modes.
- Work has been done with my **Ph.D. student Ed Taylor.**

Tisza-Landau two-fluid hydrodynamics 1938-1941



Tisza



Landau

Superfluid*: *component of liquid which is associated with macroscopic occupation (BEC) of* **one single -particle** *state .Carries zero entropy, flows without dissipation with an irrotational velocity.

Normal fluid: *comprised of* ***incoherent*** *thermal excitations , behaves like any fluid at finite temperatures in* ***local thermodynamic*** *equilibrium. This requires strong collisions.*

Landau's two-fluid hydrodynamic equations describes the fluctuations of a Bose condensate in local equilibrium with the normal fluid, brought about by strong collisions . It is an extension of ordinary fluid dynamics described in terms of the usual hydrodynamic variables , now including the superfluid degree of freedom.The equations describe the dynamics of a superfluid coupled to a normal fluid at finite T, which has been brought into local hydrodynamic equilibrium by rapid collisions.

Landau's two-fluid equations describe the low frequency dynamics of superfluid ^{4}He, s-wave BCS superconductors, as well as superfluid Bose and Fermi gases. They involve the superfluid and normal fluid densities and equilibrium thermodynamic functions, but make no explicit reference to atoms or inter-atomic interactions.

Landau two-fluid equations of motion (non-dissipative limit) in linearized form:

$$\frac{\partial \delta n}{\partial t} + \nabla \cdot \delta \mathbf{j} = 0$$

$$m\frac{\partial \delta \mathbf{j}}{\partial t} = -\nabla \delta P - \delta n \nabla U_{ext}$$

All local variables are linearized around their static equilibrium values.

$$m\frac{\partial \delta \mathbf{v}_s}{\partial t} = -\nabla \delta \mu$$

This says superfluid is **irrotational.**

$$\frac{\partial \delta s}{\partial t} + \nabla \cdot (s_0 \delta \mathbf{v}_n) = 0\,,$$

This says the superfluid carries **no entropy**.

$$m\delta \mathbf{j}(\mathbf{r},t) = \rho_{s0}(\mathbf{r})\delta \mathbf{v}_s(\mathbf{r},t) + \rho_{n0}(\mathbf{r})\delta \mathbf{v}_n(\mathbf{r},t),$$
$$m\delta n(\mathbf{r},t) = \delta \rho_s(\mathbf{r},t) + \delta \rho_n(\mathbf{r},t).$$

The linearized two fluid differential equations are easy to solve for a uniform superfluid , since the solutions are plane waves. One finds two solutions which are phonon-like, with different sound velocities

$$\omega = uk$$

1. **Two components move in phase: First sound**
2. **Two components move out of phase: Second sound**

   This mode disappears above the transition.

$$u^4 - u^2\left[\left(\frac{\partial P}{\partial \rho}\right)_\sigma + \frac{\rho_{s0}}{\rho_{n0}}(\sigma_0)^2\left(\frac{\partial T}{\partial \sigma}\right)_\rho\right] + \frac{\rho_{s0}}{\rho_{n0}}\sigma_0^2\left(\frac{\partial P}{\partial \rho}\right)_T\left(\frac{\partial T}{\partial \sigma}\right)_\rho = 0.$$

Note that velocities are expressed in terms of equilibrium thermodynamic functions.

First and second sound velocities are very different in a trapped Bose gas , compared to superfluid He4

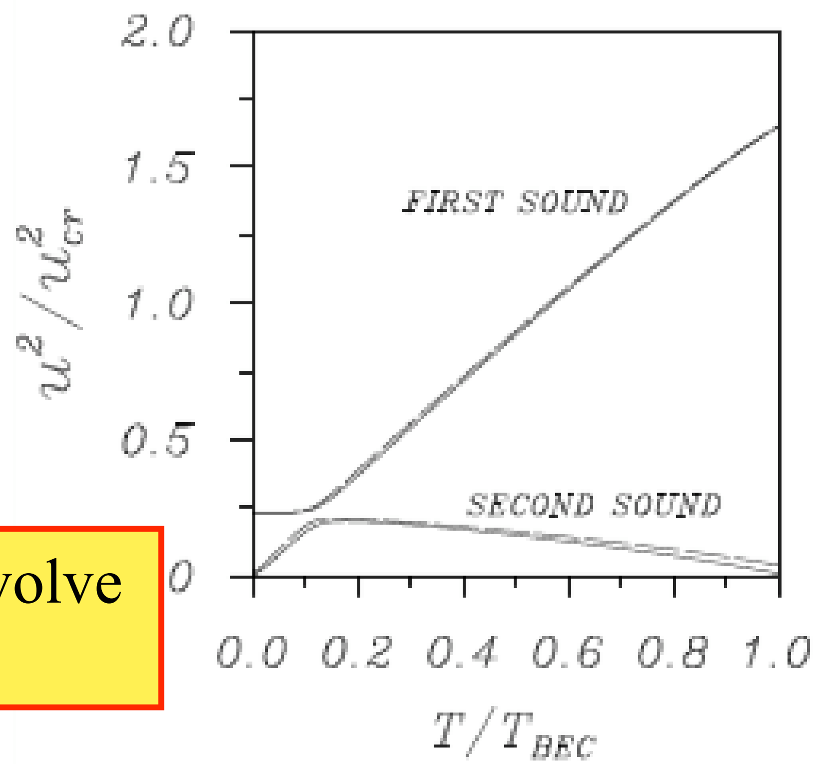
Superfluid He4 : $u_I^2 = \dfrac{\partial P}{\partial \rho}$ **First sound is a density wave**

$u_{II}^2 = \dfrac{\rho_s T s^2}{\rho_n c}$ **Second sound is a temperature wave**

Uniform Bose-condensed gas:

$$u_I^2 = \frac{5 g_{5/2}(z=1) k_B T}{3 g_{3/2}(z=1) m}$$

$$u_{II}^2 = \frac{g n_{co}(T)}{m} \qquad g = \frac{4\pi a \hbar^2}{m}$$



In a gas, both first and second sound involve temperature and density fluctuations.

Landau used the measured second sound velocity to determine the dispersion relation of the important thermal elementary excitations.

In 1946, Peshkov first measured the velocity of second sound in superfluid He4 from the propagation of temperature waves and fitted it to Landau's formula:

$$u_{II}^2 = \frac{\rho_s T s^2}{\rho_n c}$$

Landau altered his original quasiparticle excitation energies and introduced the famous phonon-roton curve in 1947. The roton energy gap and mass were fitted to give thermodynamic functions which agreed with the measured second sound velocity.

We will see that "second sound" will be similarly useful as a diagnostic tool in trapped superfluid Fermi gases.

The **structure** of the Landau equations is **universal.** All hydrodynamic modes come in **pairs**, with **in-phase and out-of-phase oscillations of the superfluid and normal components.** The **only difference** between superfluid ^{4}He and a superfluid Fermi gas is in the evaluation of the **local thermodynamic functions**, such as the pressure, entropy, etc. Only these require a **microscopic theory** for the thermal excitations of the particular system of interest.

The **corrections** to these equations give hydrodynamic damping involving transport coefficients .These are proportional to the various **transport collision times τ**, which is **small** for **strong interactions** between atoms.

What are conditions for the Landau equations?

We need the **normal fluid** dynamics to be described in terms of a few macroscopic local variables, like **local** pressure $P(\mathbf{r}, t)$, **local** temperature $T(\mathbf{r}, t)$, etc. This requires **local equilibrium**. Just as in normal fluids, this in turn requires strong collisions between the atoms.

The two- fluid equations will describe **collective oscillations** with a period T only if the appropriate **atomic collision time τ** satisfies the condition

$$\tau << T \Rightarrow \omega\tau << 1$$

This is equivalent to saying that local equilibrium requires the mean free path of the atoms to be **much smaller** than the wavelength of the collective mode.

Most experiments on collective modes in trapped Bose gases **so far** have studied have worked in the opposite **collisionless limit,** defined by

$$\tau >> T \Rightarrow \omega\tau >> 1$$

This means that the collisional mean free path is much larger than the wavelength of the the collective mode. Collisions are present but do **not lead to local equilibrium.**

How do we make the collision time small?

For a gas **above** T_{BEC}, collisions between the atoms in the thermal cloud of density $\tilde{n}_0(r)$ give rise to the classic relaxation time:

$$\frac{1}{\tau_{cl}(\mathbf{r})} = \sqrt{2}\tilde{n}_0(\mathbf{r})\sigma\bar{v},$$

$\sigma = 8\pi a^2$, where a is the s-wave scattering length.

However, when a Bose condensate forms, it turns out that it is collsions between atoms in the **high density localized condensate** and the spread-out **low density thermal cloud** are most important. The result is:

$$\frac{1}{\tau_{\kappa}(\mathbf{r})} \simeq \sqrt{2}n_{c0}(\mathbf{r})\sigma\bar{v}$$

Nikuni and Griffin, PRA, 2001

We need high density or large value of scattering length a

Trapped atomic Fermi gases -basics

- Many groups can now cool Fermi gases down to **$0.05T_F$**, where **T_F** is the Fermi temperature (10^{-6} K).

- Two different **atomic hyperfine** states $|F, m_F\rangle$ are used, where m_F = - F, .., -1, 0, 1,.., F denotes the different Zeeman levels. F denotes the **total spin** of the atom(nuclear and electronic). For **Fermions**, F must be an **odd multiple** of 1/2.

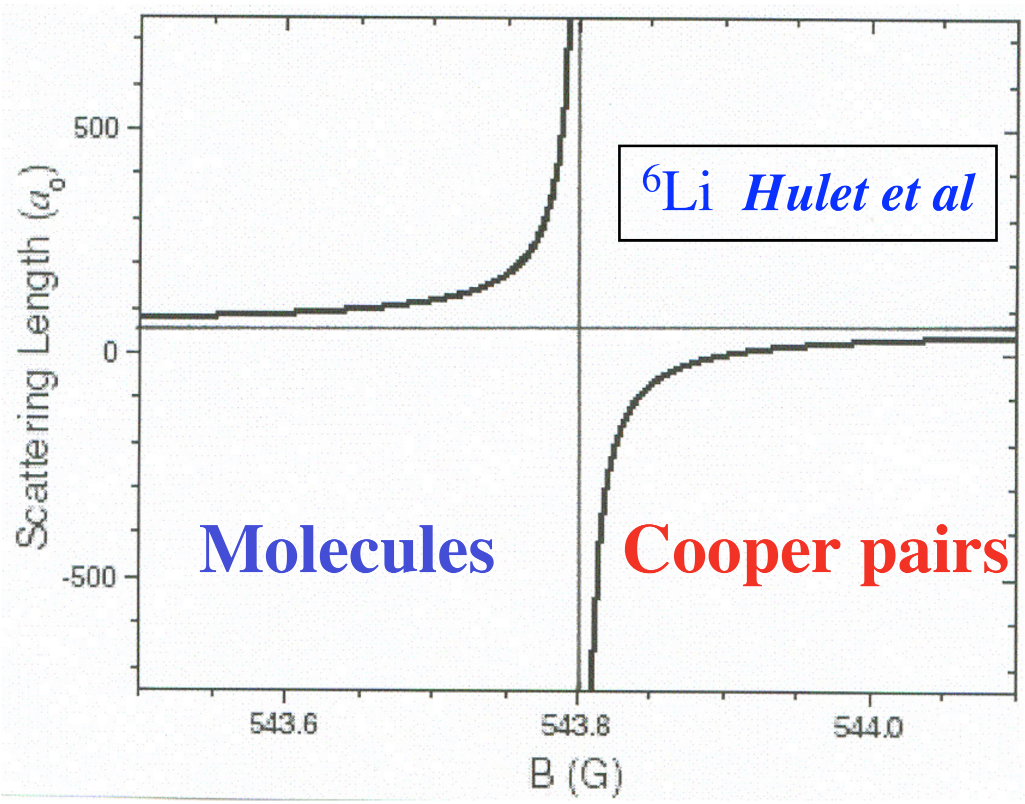
- ^{40}K (F = 9/2) - Jin (JILA, Boulder).
  ^{6}Li (F = 1/2) - Grimm (Innsbruck), Ketterle (MIT) , Hulet (Rice) , Salomon (ENS, Paris).

Why work with a two-component Fermi gas?

- At ultra-low temperatures, atoms have very low momentum and hence only the **lowest partial wave** contributions from the interaction need be kept.

- Only the **s-wave scattering** contribution is large, but this does not arise between identical Fermions because of the Pauli principle. However, it can occur between atoms with **different** values of m_F (denoted by **spin**$\Uparrow$ and **spin** $\Downarrow$).

Feshbach resonance: two body physics

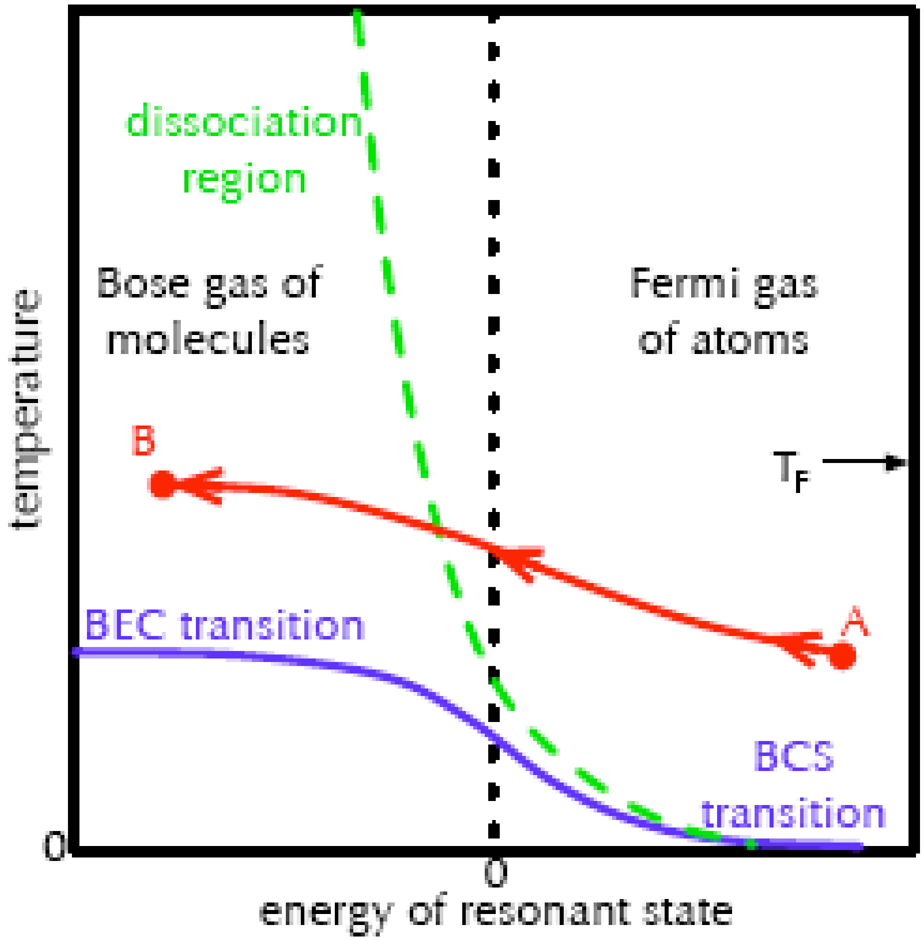


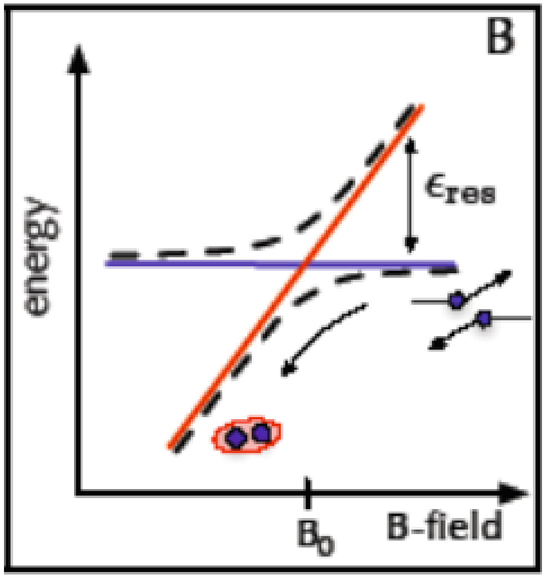
Molecules only form when $a_{2b} > 0$. This is equivalent to $\varepsilon_{res} < 0$ or $B < B_0$.

$$\varepsilon_{res} \propto B - B_0$$

$$a_{2b} = a_{bg}\left(1 - \frac{w}{B_0 - B}\right) \quad \text{--->} \quad U \equiv \frac{4\pi\hbar^2 a_{2b}}{m}$$

The molecules are very weakly bound but are **very stable** because 3-body decay is forbidden since Fermions obey the **exclusion** principle.





$$\varepsilon_R \propto B - B_0$$

The **blue curve** represents the **phase boundary** into the **superfluid state of bound pairs.**

The **bound states** in **interacting** Fermi gases are **Bosonic** in nature and hence can **Bose condense**, just like Bose atoms can. As a result, in a discussion of trapped Fermi gases, the well-known description of Bose condensate will appear once again, except that it now describes a **molecular Bose condensate** or a **Cooper pair condensate.**

The strongly interacting **Bose gases of molecules** which form in superfluid Fermi gases *via* a Feshbach resonance give us the **perfect system** to reach two-fluid behavior in gases- the **precise analogue of what Landau studied in 1941 in liquid Helium**.

In the **extreme limit**, all N Fermi atoms can form $N/2$ bound states. This is the **BEC limit** of an interacting Fermi gas. It is an interacting Bose-condensed gas of $N/2$ **molecules**, each with mass $M = 2m$. The molecular s-wave scattering length is $a_M = 0.6\, a_F$ AND these very weakly bound molecules are **very long-lived** (Petrov) because the **Pauli exclusion principle** prevents decay.

More generally, one expects that the dynamics in the **entire BCS-BEC crossover region** , where $|a_{2b}|k_F$ is very large, can be described by the **Landau two-fluid hydrodynamic equations at finite temperatures.**

We return to these equations. How do we solve them?

The linearized two-fluid equations are

$$\frac{\partial \delta n}{\partial t} + \nabla \cdot \delta \vec{j} = 0 \qquad m\frac{\partial \delta \vec{j}}{\partial t} = -\nabla \delta P - \delta n \nabla U_{ext}$$

$$m\frac{\partial \delta \vec{v}_s}{\partial t} = -\nabla \delta \mu \qquad \frac{\partial \delta s}{\partial t} + \nabla \cdot (s_0 \delta \vec{v}_n) = 0$$

where

$$m\delta \vec{j} = \rho_{s0}\delta \vec{v}_s + \rho_{n0}\delta \vec{v}_n$$

$$m\delta n = \delta \rho_s + \delta \rho_n$$

The T=0 limit of the Landau hydrodynamic equations

$$T = 0 \Rightarrow \rho_{n0} = 0 \Rightarrow \rho_{s0} = \rho_0 = mn_0$$

$$\boxed{\frac{\partial^2 \delta\rho(r,t)}{\partial t^2} - \nabla \cdot (n_0(r)\nabla \delta\mu(r,t)) = 0}$$

$$\delta\mu(r,t) = \frac{\partial \mu_0(n)}{\partial n} \frac{\delta\rho(r,t)}{m}$$

This reduces the problem to finding the **equilibrium equation of state**, **μ** as a function of the density **n**.
It describes a one-component **pure superfluid** at T = 0.
This approach is due to Stringari and Pitaevskii, and has been used in many papers in the last year.

The collective modes have been studied extensively in trapped Fermi superfluids in the BCS-BEC crossover region in the last two years. The region of greatest interest is in the unitarity region near the Feshbach resonance,

$$|a_{2b}|k_F \Rightarrow \infty$$

In contrast to the BEC and the BCS regions, this strong interaction region is not understood. One expects that the behavior of a Fermi gas should be universal in this strong interaction limit, since the scattering length must "drop out" of the problem. This has been confirmed by the direct Monte Carlo calculations at T = 0 by Giorgini and coworkers.

In the unitarity region of dilute Fermi gases,we are dealing with the strongest interactions in physics!!

Hydrodynamic modes may give us a way to study the unitarity region at finite temperatures!

All hydrodynamic modes come in **pairs**, involving **in-phase and out-of-phase oscillations of the superfluid and normal components.** We believe that only the **in-phase modes** have been excited in **current** experiments. These modes are **weakly** temperature-dependent and thus the T = 0 theory is pretty good.

The **out-of-phase** hydrodynamic oscillations (analogue of second sound in a uniform superfluid) will be very temperature-dependent and give a better **probe of the finite T dynamics** of the strong interaction region.

New variational approach to solving Landau equations

Our work is based on re-writing the Landau two-fluid equations in terms of a Lagrangian. We use this to find a variational solution for the normal modes of the Landau equations in a trapped superfluid gas. This is easier and more physical than to attempt to solve the Landau differential equations directly.

Our approach is based on two previous papers:

1. In 1950, Zilsel discussed Landau's equations for a superfluid Helium by using such a Lagrangian formulation.

2. In 1999, Zaremba, Nikuni and Griffin derived the Landau equations from a microscopic model for a Bose condensate (the superfluid) coupled to a thermal gas of atoms (the normal fluid).

Useful to work in terms of displacement fields of the superfluid and normal fluid components

$$\mathbf{v}_s(\mathbf{r},t) \equiv \frac{\partial \mathbf{u}_s(\mathbf{r},t)}{\partial t}, \quad \mathbf{v}_n(\mathbf{r},t) \equiv \frac{\partial \mathbf{u}_n(\mathbf{r},t)}{\partial t}$$

In terms of these variables, the entropy and density fluctuations are given by

$$\delta\rho(\mathbf{r},t) = -\boldsymbol{\nabla}\cdot[\rho_{s0}(\mathbf{r})\mathbf{u}_s(\mathbf{r},t) + \rho_{n0}(\mathbf{r})\mathbf{u}_n(\mathbf{r},t)]$$

$$\delta S(\mathbf{r},t) \; = \; -\boldsymbol{\nabla}\cdot[S_0(\mathbf{r})\mathbf{u}_n(\mathbf{r},t)]\,.$$

Lagrangian quadratic in displacements of the two components , with a harmonic time dependence. The stationary value of this Lagrangian gives the normal modes of the linearized Landau two-fluid equations.

$$L^{(2)} = K[\mathbf{u}_s, \mathbf{u}_n]\omega^2 - U[\mathbf{u}_s, \mathbf{u}_n],$$

$$K[\mathbf{u}_s, \mathbf{u}_n] = \frac{1}{2}\int d^3r \ \left\{\rho_{s0}\mathbf{u}_s^2 + \rho_{n0}\mathbf{u}_n^2\right\}$$

$$U[\mathbf{u}_s, \mathbf{u}_n] = \frac{1}{2}\int d^3r \left\{ \left(\frac{\partial\mu}{\partial\rho}\right)_S [\nabla\cdot(\rho_{s0}\mathbf{u}_s + \rho_{n0}\mathbf{u}_n)]^2 + 2\left(\frac{\partial T}{\partial\rho}\right)_S [\nabla\cdot(S_0\mathbf{u}_n)][\nabla\cdot(\rho_{s0}\mathbf{u}_s + \rho_{n0}\mathbf{u}_n)] + \left(\frac{\partial T}{\partial S}\right)_\rho [\nabla\cdot(S_0\mathbf{u}_n)]^2 \right\}.$$

All the thermodynamic functions depend on position .

Procedure for finding hydrodynamic mode frequencies

1. Introduce a physical ansatz for the displacement vectors (Cartesian components $i = x, y$ *and* z)

$$u_{si}(\mathbf{r}) = a_{si} f_i(\mathbf{r}), \quad u_{ni}(\mathbf{r}) = a_{ni} g_i(\mathbf{r}),$$

   Here a_{si} and a_{ni} are our variational parameters. The choice of the functions $f_i(r)$ and $g_i(r)$ is based on the known hydrodynamic solutions in two limits:
   Pure superfluid T = 0 (Stringari , 1996 , 2004)
   Pure normal fluid $T > T_c$ (Griffin, Wu and Stringari, 1997)

2. Minimizing the Lagrangian with respect to the variational parameters reduces the problem to a set of linear algebraic equations for a_{si} and a_{ni}.

Dipole hydrodynamic modes - example

For our trial solution , we take rigid displacements :

$$\mathbf{u}_s = \hat{\mathbf{x}}_i a_s, \quad \mathbf{u}_n = \hat{\mathbf{x}}_i a_n$$

The variational minimization procedure gives

$$K[a_s, a_n] = \frac{1}{2} M_s a_s^2 + \frac{1}{2} M_n a_n^2$$

$$U[a_s, a_n] = \frac{1}{2} k_s a_s^2 + \frac{1}{2} k_n a_n^2 + \frac{1}{2} k_{sn} (a_s - a_n)^2$$

The final equations of motion describe two coupled oscillators:

$$\begin{pmatrix} M_s\omega^2 - k_s - k_{sn} & k_{sn} \\ k_{sn} & M_n\omega^2 - k_n - k_{sn} \end{pmatrix} \begin{pmatrix} a_s \\ a_n \end{pmatrix} = 0.$$

All spring constants are given as **spatial integrals** over **equilibrium** thermodynamic functions.

$$k_s = \int d^3r \left\{ \left[\left(\frac{\partial\mu}{\partial\rho}\right)_S \frac{\partial\rho_0}{\partial x_i} + \left(\frac{\partial\mu}{\partial S}\right)_\rho \frac{\partial S_0}{\partial x_i} \right] \frac{\partial\rho_{s0}}{\partial x_i} \right\} \Rightarrow \omega_i^2 M_s$$

$$k_n = \int d^3r \left\{ \left[\left(\frac{\partial\mu}{\partial\rho}\right)_S \frac{\partial\rho_0}{\partial x_i} + \left(\frac{\partial\mu}{\partial S}\right)_\rho \frac{\partial S_0}{\partial x_i} \right] \frac{\partial\rho_{n0}}{\partial x_i} + \left[\left(\frac{\partial T}{\partial\rho}\right)_S \frac{\partial\rho_0}{\partial x_i} + \left(\frac{\partial T}{\partial S}\right)_\rho \frac{\partial S_0}{\partial x_i} \right] \frac{\partial S_0}{\partial x_i} \right\} \Rightarrow \omega_i^2 M_n$$

$$k_{sn} = -\int d^3r \left\{ \left(\frac{\partial\mu}{\partial\rho}\right)_S \frac{\partial\rho_{n0}}{\partial x_i}\frac{\partial\rho_{s0}}{\partial x_i} + \left(\frac{\partial T}{\partial\rho}\right)_S \frac{\partial S_0}{\partial x_i}\frac{\partial\rho_{s0}}{\partial x_i} \right\}.$$

This has to be numerically evaluated.

$$M_s = \int d^3r\, \rho_{s0}, \quad M_n = \int d^3r\, \rho_{n0}.$$

Total superfluid and normal fluid masses

The solutions of these equations:

1. In-phase dipole (Kohn) mode:

$$\omega = \omega_i \qquad a_s = a_n$$

Note that both components move together

2. Out-of-phase dipole mode:

$$\omega^2 = \omega_i^2 + \frac{M_s + M_n}{M_s M_n} k_{sn}.$$

$$M_s a_s = - M_n a_n$$

Note that components move out-of-phase.

For the analogous formal solutions for the radial and axial breathing hydrodynamic modes, see Taylor & Griffin, cond.mat and PRA, in press.

Future work:

Use our variational formulation to calculate explicit values of the two-fluid mode frequencies in the unitarity region of the BCS-BEC crossover, as a function of the temperature. Current experiments (Thomas, Grimm) have only excited the in-phase hydrodynamic collective modes, which are weakly dependent on temperature.

This requires calculating spatial integrals of various products of equilibrium thermodynamic functions and their derivatives. We plan on using the simple Nozieres-Schmitt-Rink (NSR) theory for the thermodynamic functions for a uniform gas in the unitarity region (this has been shown by Liu and Hu to agree very well with QMC calculations) . We will use the the local density approximation(LDA) to include the trap potential.

The analogue of dilute mixtures of He^3 in He^4 ?

He^3-He^4 mixtures have been extensively studied in low temperature physics. In particular , the **two-fluid equations of such mixtures** are well known. There is an additional conservation equation. The key point is that the He^3 atoms always form part of the **normal fluid**. Thus even at T= 0, there is some normal fluid present.

Consider a **mixture of Fermi atoms (K) in a Bose-condensed gas of Bose atoms (Rb) .** Using a Feshbach resonance between the Fermi and Bose atoms, we might be able to get into the domain of collisional hydrodynamics , where the **two-fluid hydrodynamics would describe the collective modes of this quantum gas mixture**.

This is an example of the new world we can now enter!