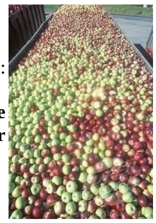
1. Error Analysis Introduction

- Almost every time you make a measurement, the result will not be an exact number, but it will be a *range* of possible values.
- The range of values associated with a measurement is described by the uncertainty, or **error**.

Exactly 3 apples (no error)

1600 ± 100 apples:

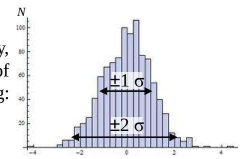
1600 is the **value**
100 is the **error**



Errors

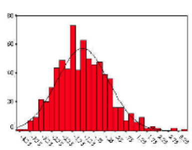
- Errors **eliminate** the need to report measurements with vague terms like “approximately” or “≈”.
- Errors give a *quantitative* way of stating your confidence level in your measurement.
- Saying the answer is 10 ± 2 means you are 68% confident that the actual number is between 8 and 12.
- It also implies that and 14 (the 2-σ range).

A histogram of many, many measurements of the same thing:



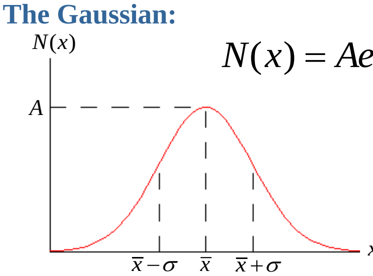
2. Normal Distribution

- A **probability distribution** is a curve which describes what the probability is for various measurements
- The most important and widely used probability distribution is called the *Normal Distribution*
- It was first popularized by the German mathematician Carl Friedrich Gauss in the early 1800s
- It is also sometimes called the **Gaussian** distribution, or the bell-curve



The Gaussian:

$$N(x) = Ae^{\frac{-(x-\bar{x})^2}{2\sigma^2}}$$



- A is the *maximum amplitude*.
- $\bar{x}$ is the *mean* or *average*.
- σ is the *standard deviation* of the distribution.

Normal Distribution

- σ is the **standard deviation** of the distribution
- Statisticians often call the square of the standard deviation, σ^2, the **variance**
- σ is a measure of the width of the curve: a larger σ means a wider curve
- 68% of the area under the curve of a Gaussian lies between the mean minus the standard deviation and the mean plus the standard deviation
- 95% of the area under the curve is between the mean minus twice the standard deviation and the mean plus twice the standard deviation

3. Estimating the Mean from a Sample

- Suppose you make N measurements of a quantity x, and you expect these measurements to be normally distributed
- Each measurement, or trial, you label with a number i, where i = 1, 2, 3, etc
- You do not know what the true mean of the distribution is, and you cannot know this
- However, you can estimate the mean by adding up all the individual measurements and dividing by N:

$$\bar{x}_{\text{est}} = \frac{1}{N}\sum_{i=1}^{N} x_i$$

Estimating the Standard Deviation from a Sample

- Suppose you make N measurements of a quantity x, and you expect these measurements to be normally distributed
- It is impossible to know the true standard deviation of the distribution
- The best estimate of the standard deviation is:

$$\sigma_{\text{est}} = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N}(x_i - \overline{x}_{\text{est}})^2}$$

- The quantity $N - 1$ is called the **number of degrees of freedom**
- In this case, it is the number of measurements minus one because you used one number from a previous calculation (mean) in order to find the standard deviation.

Finding the Statistical Error

- There is roughly a 68% chance that any measurement of a sample taken at random will be within one standard deviation of the mean
- Usually the mean is what we wish to know and each individual measurement almost certainly differs from the true value of the mean by some error
- There is a 68% chance that any single measurement lies with one standard deviation of this true value of the mean
- Thus it is reasonable to say that:

$$\Delta x_i = \sigma$$

- This error is often called *statistical*

4. Reading Error (Analog)

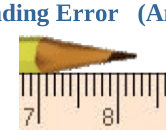
- Imagine you use a ruler to measure the length of a pencil
- You line up the tip of the eraser with 0, and the image below shows what you see over near 8 cm

- The pencil appears to be about 8.25 cm long, but what is the reading error?
- There is no fixed rule that will allow us to answer this question
- We must use our *intuition* and *common sense*!

Reading Error (Analog)

- Could the pencil actually be as long as 8.3 cm? …no, I don't think so
- Could it be 8.28 cm? …maybe
- And it could be as short as 8.23 cm, but, in my opinion, no shorter
- So the range is about 8.23 to 8.28 cm

Reading Error (Analog)



- The range is about 8.23 to 8.28 cm
- A reasonable estimate of the reading error of this measurement is half the range: ± 0.025 cm
- To be cautious, we might round up to 0.03 cm
- We say "The length of the pencil is 8.25 ± 0.03 cm."
- Meaning: if we get a collection of objective observers together to look at the pencil above, we expect most (ie more than 68%) of all observers will report a value between 8.22 and 8.28 cm

Reading Error (Digital)

- For a measurement with an instrument with a digital readout, the reading error is usually "± one-half of the last digit."
- This means one-half of the power of ten represented in the last digit.
- With the digital thermometer shown, the last digit represents values of a tenth of a degree, so the reading error is ½ × 0.1 = 0.05°C
- You should write the temperature as 12.80 ± 0.05 °C.

Choosing between Statistical and Reading Error

- In most cases, when you have both a standard deviation and a reading error, one is much larger than the other, and then you should choose the **larger** to be the error
- For example, if every time you measure something you always get the same numerical answer, this indicates that the reading error is dominant
- However, if every time you measure something you get different answers which differ more than the reading error you might estimate, then the standard deviation is dominant

5. Significant Figures

- Imagine you have a set of 30 timing measurements for which the statistical error is clearly dominant
- You use an equation to estimate that the standard deviation is 0.102933971 seconds
- Consider one of these measurements, the 5th one, for which we measured 5.49 seconds
- Using the standard deviation as the error, this measurement should be written as 5.49 ± 0.102933971
- What this means is that there is about a 68% chance that the true value is somewhere between 5.387066029 and 5.592933971 seconds….

????

Significant Figures

5.49 ± 0.102933971 seconds ????

- Clearly we are using WAY too many significant figures here!
- It would be just as instructive to say that there is about a 68% chance that the true value is somewhere between 5.4 and 5.6 seconds
- Or, you could say the measurement is: **5.5 ± 0.1 s**
- In fact it is not only more concise to report this, but it is more honest

Significant Figures

- There are two general rules for significant figures used in experimental sciences:

1. Errors should be specified to one, or at most two, significant figures.
2. The most precise column in the number for the error should also be the most precise column in the number for the value.

- So if the error is specified to the 1/100th column, the quantity itself should also be specified to the 1/100th column.

6. Propagation of Errors of Precision

- When you have two or more quantities with known errors you may sometimes want to combine them to compute a derived number
- You can use the rules of Error Propagation to infer the error in the derived quantity
- We assume that the two directly measured quantities are x and y, with errors Δx and Δy respectively
- The measurements x and y must be independent of each other.
- The fractional error is the value of the error divided by the value of the quantity: $\Delta x / x$
- To use these rules for quantities which cannot be negative, the fractional error should be much less than one

Propagation of Errors

- Rule #1 (sum or difference rule):
- If $z = x + y$
- or $z = x - y$
- then

$$\Delta z = \sqrt{\Delta x^2 + \Delta y^2}$$

- Rule #2 (product or division rule):
- If $z = xy$
- or $z = x/y$
- then

$$\frac{\Delta z}{z} = \sqrt{\left(\frac{\Delta x}{x}\right)^2 + \left(\frac{\Delta y}{y}\right)^2}$$

Propagation of Errors

- Rule #2.1 (multiply by exact constant rule):
- If $z = xy$ or $z = x/y$
- and x is an exact number, so that Δx=0
- then

$$\Delta z = |x|(\Delta y)$$

- Rule #3 (exponent rule):
- If $z = x^n$
- then $\frac{\Delta z}{z} = n\frac{\Delta x}{x}$

7. The Error in the Mean

- Many individual, independent measurements are repeated N times
- Each individual measurement has the same error Δx
- Using error propagation you can show that the error in the estimated mean is:

$$\Delta \bar{x}_{est} = \frac{\Delta x}{\sqrt{N}}$$