Uncertainty Propagation Aid Sheet

[This is an excerpt from Uncertainty Module 4 http://uoft.me/PPupm4pdf page 8-9]

Say we have measured some quantity x with uncertainty $u(x)$ and a quantity y with uncertainty $u(y)$ and wish to combine them to get a value z with uncertainty $u(z)$. As we discussed in Module 2, we need the combination to preserve the probabilities associated with the uncertainties in x and y. We will consider a number of ways of combining the quantities. Although this Module has been discussing statistical uncertainties, this section applies to all uncertainties, including the ones you learned about in Modules 2 and 3.

Addition or Subtraction

As discussed in Modules 2 and 3, if $z = x + y$ or $z = x - y$ then the uncertainties are combined in quadrature:

$$u(z) = \sqrt{u(x)^2 + u(y)^2} \qquad (11)$$

Multiplication or Division

If $z = x \times y$ or $z = x \div y$ then the *fractional uncertainties* are combined in quadrature:

$$\frac{u(z)}{z} = \sqrt{\left(\frac{u(x)}{x}\right)^2 + \left(\frac{u(y)}{y}\right)^2} \qquad (12)$$

Multiplication by a Constant

If $z = a \times x$, where a is a constant known to a large number of significant figures, then the uncertainty in z is given by Eqn. 12 with the uncertainty in a, $u(a) = 0$. So:

$$u(z) = a\,u(x) \qquad (13)$$

Raising to a Power

If $z = x^n$ then:

$$u(z) = n x^{(n-1)} u(x) \qquad (14)$$

which can also be written in terms of the fractional uncertainties:

$$\frac{u(z)}{z} = n \frac{u(x)}{x} \qquad (15)$$

Say you are squaring *x*, so $z = x^2 = x \times x$. You may be tempted to use Eqn 12 for multiplication and division, but this is incorrect: Eqn 12 assumes that the uncertainties in the quantities *x* and *y* are independent of each other. Here there is only one quantity, *x*.

The General Case

In general *z* is some function of *x* and *y*, *z* = *f*(*x*, *y*). The uncertainty in *z* requires knowing about partial derivatives. If you don't know about these yet, you may skip this sub-section and go to the questions. Nonetheless:

$$u(z) = \sqrt{\left(\frac{\partial f(x,y)}{\partial x}u(x)\right)^2 + \left(\frac{\partial f(x,y)}{\partial y}u(y)\right)^2} \qquad (16)$$

Eqns. 11 – 15 are just applications of Eqn. 16 for various functions.