

THE GYROSCOPE

REFERENCES

The Feynman Lectures on Physics, Chapter 20 (a very nice, intuitive description of the operation of the gyroscope). Copy available at the Resource Centre.

Most Introductory Physics texts (e.g. A.Halliday and R.Resnick, *Physics*; M.Sternheim and J.Kane, *General Physics*) discuss Angular Motion, Moments of Inertia, Simple Harmonic Motion, *etc.*

Operation and Instruction Manual for the MITAC Gyroscope (available at the Resource Centre)

INTRODUCTION

One of the most interesting areas in the science of rotational dynamics is the study of spinning solid objects, tops, hoops, wheels, *etc.* From the gyrocompass (which indicates *true* North, rather than Magnetic North) to an understanding of how a cyclist turns corners, the applications of this field of study are both practical

and fascinating. This experiment is designed to introduce you to some of these interesting and often counter-intuitive properties of rotating bodies.



The apparatus consists of a 2.7 kg cylindrical rotor that is spun at a constant rate by an electric motor. The rotor is mounted in a double gimbal arrangement which allows it to assume any orientation.

THEORY

Angular Motion. The basic equations for angular motion can often be obtained simply from those for linear motion by making the following substitutions (bold quantities are vectors):

<i>Linear variables</i>	Y	<i>Angular variables</i>
Force, F	Y	Torque, $\vec{\tau}$
Mass, <i>m</i>	Y	Moment of Inertia, <i>I</i>
Velocity, v	Y	Angular velocity, $\vec{\omega}$
Momentum, p	Y	Angular Momentum, L
Acceleration, a	Y	Angular acceleration, a

NB. The analogy needs to be treated with caution; *e.g.* *I* is not a constant property of the body, as is mass, since its value depends the axis around which it is measured.

Thus Newton's Law $\vec{F} = \frac{d\vec{p}}{dt} = \frac{d(m\vec{v})}{dt} = m\vec{a}$, becomes $\vec{\tau} = \frac{d\vec{L}}{dt} = \frac{d(I\vec{\omega})}{dt} = I\vec{\alpha}$.

So we see that a torque $\vec{\tau}$ applied to a body rotating with angular momentum \vec{L} produces a change in that angular momentum according to the relationship:

$$\vec{\tau} = \frac{d\vec{L}}{dt} \quad (1)$$

where $\vec{\tau}$ is the applied torque; $\vec{L} = I\vec{\omega}$ is the angular momentum, $\vec{\omega}$ is the angular velocity of rotation of the body and I is the moment of inertia of the body about the axis of rotation (*i.e.*, about the $\vec{\omega}$ axis).

If this torque $\vec{\tau}$ is applied perpendicular to the direction of \vec{L} (*i.e.*, perpendicular to the direction of $\vec{\omega}$) then from equation (1), \vec{L} does not change in magnitude but does change in direction (***if you do not understand why this is so, check with a demonstrator before going any further***). This change in direction of \vec{L} (and thus also of $\vec{\omega}$) is called precession and appears as a rotation of the direction of the \vec{L} vector in space with a *precession angular velocity* of $\vec{\Omega}_p$. Then it can be shown that:

$$\vec{\tau} = \Omega_p \times \vec{L} \quad (2)$$

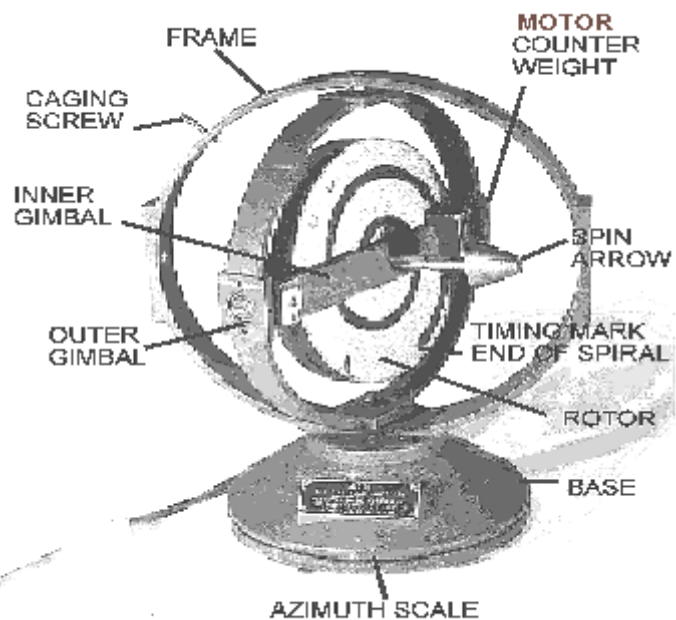
EXPERIMENTS

1. Introductory experiment

Study the gyroscope (gyro) and ensure that you can identify the outer frame, the outer gimbal (which has a vertical rotation axis), the inner gimbal (with horizontal rotation axis), and the caging screw which locks the outer gimbal to the frame, along with the associated equipment: weights, springs, supports and the scale.

Instrument alignment

- Set the gyro on a level table.
- Lock the outer vertical gimbal to the frame with the caging knob.
- Start the motor and allow a minute or so for the rotor to attain a constant operating speed.
- Adjust the arrow head and/or tail weights by turning them until the arrow remains in a horizontal plane, which indicates a balance about the inner gimbal axis.



GYROSCOPE

Free (two degrees of freedom) gyro

Unlock the outer gimbal by releasing the caging screw, then start the motor. Observe the orientation of the spin vector arrow. Carefully lift the gyro by its base. What happens to the direction of the spin vector arrow?

After returning the gyro on the table, carefully push down on the arrow or the head to produce a torque vector. Notice the opposing torque. Observe the resulting tendency to rotate (precess) about the vertical axis. Determine the directions of the spin, torque and precession vectors and verify that the spin vector precesses toward the torque vector.

Captive (one degree of freedom) gyro

Lock the outer gimbal to the base. Except for small frictional and inertial torques, there is now no directly opposing torque when the arrow is depressed as before. This can be explained by the basic gyro equation (2):

$$\vec{\tau} = \Omega_p \times \vec{L} \quad (2)$$

With the outer gimbal locked, the gyro is not free to precess and the precession angular velocity $\vec{\Omega}_p$ must be zero. Then, by Eq. (2), the angular momentum \vec{L} must be zero, which means no torque is generated to oppose the torque caused by pushing on the arrow. The gyro seems to have lost its inertial properties.

2 - Determination of rotor angular momentum

You need: 300g and 150g torquer weights; stop watch.

The gyro has to be balanced.

Unlock the outer gimbal by releasing the caging knob. Attach one of the torquer weights on the shaft of the spin vector arrow and secure the detent in one of the notches.

Orient the spin vector arrow in the horizontal plane and then release the arrow, allowing the gyro to precess about the vertical axis. Oscillations which may occur can be damped by applying a very slight torque to the outer gimbal about the precession axis.

Determine the time the outer gimbal takes to rotate five complete revolutions. Calculate the precession frequency f and the precession angular velocity $\vec{\Omega}_p$:

$$\Omega_p = 2\pi f \quad (3)$$

Measure the lever arm: the distance (along the spin vector arrow) between the inner gimbal rotational axis and the notch on which the torquer weight was placed.

Express the applied torque (in N m) by multiplying the force due to the torquer weight (in N) by the lever arm (in m).

Use (2) to determine the angular momentum \vec{L} of the gyro.

Repeat the experiment for three or four other combinations of torquer and lever arm. Two weights are supplied; each weight can be placed in any of the different positions on the shaft; more than one weight can be used at a time.

3 - Measurement of rotor moment of inertia. Analytical determination of rotor moment of inertia.

The moment of inertia of a body measures the distribution of its mass about the axis of rotation. The angular momentum \vec{L} of a rotating body is dependent upon the moment of inertia I :

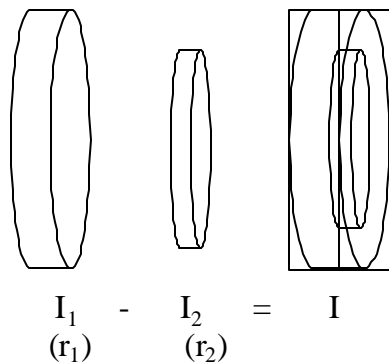
$$\vec{L} = I\vec{\omega} \quad (4)$$

\vec{L} is the angular momentum, $\vec{\omega}$ is the angular (spin) velocity of rotation of the body and I is the moment of inertia of the body about the axis of rotation which is the $\vec{\omega}$ (vector) axis.

The objective of this experiment is to calculate the rotor moment of inertia from: the angular momentum as measured in Experiment 2, and the spin angular velocity to be measured in this experiment.

Balance the gyro, apply power to the rotor and allow approximately one minute for the rotor to come up to speed. Measure the time required to complete ~50 revolutions, calculate the rotor spin velocity in rad/s. Use (4) and the previously determined \vec{L} to calculate I .

In order to calculate the rotor moment of inertia we shall consider the solid to be made of two bodies having moments of inertia I_1 and I_2 . The moment of inertia I of the rotor can be calculated by using the method of superposition. To simplify the calculation, we neglect the small hub at the centre and the rounding of all sharp edges. The rotor is a composite solid made up as the difference between two right circular cylinders. To get the solid, we have to subtract I_2 from I_1 :



$$I_1 = \frac{1}{2}M_1r_1^2 = \frac{1}{2}V_1\rho r_1^2 \quad (5)$$

$$I_2 = \frac{1}{2}M_2r_2^2 = \frac{1}{2}V_2\rho r_2^2$$

$M_{1,2}$, $r_{1,2}$, $V_{1,2}$ are masses, radii and volumes of the two cylinders; ρ is the density of the cast iron material ($7 \times 10^3 \text{ kg/m}^3$).

Calculate the value for the moment of inertia of the gyro rotor. Compare the value to that measured before. Discuss the sources of error.

4 - Measurement of system moments of inertia

In this experiment two composite moments of inertia of the system will be measured:

- the inner gimbal plus rotor moment of inertia about the inner gimbal axis of freedom;
- the inner gimbal and outer gimbal moment of inertia about the vertical axis.

When the gimbals are restrained by springs (see **Figures 1, 2a, 2b and 3**) the gyroscope would oscillate

with a natural frequency $\omega = \sqrt{\frac{K}{I}}$ (6), where K is the torque constant and

I is the composite moment of inertia of the inner gimbal plus rotor. The torque constant is equal to the ratio of the output torque to the output angular displacement:

$$K = \frac{\tau}{\theta} \quad (7)$$

In the situations shown in Figures 1, 2a, 2b and 3, the torque applied to the rotor-plus-gimbal is $\tau = \Sigma \vec{F} \times \vec{\ell}$ by definition of torque. Here $\Sigma \vec{F}$ is the sum of the forces acting at a distance $\vec{\ell}$ from the axis of rotation. These forces are provided by the two springs. If k_1 and k_2 are the spring constants of the two springs, d_1 and d_2 the extensions from the unstressed length when the rotor is at rest, and Δ the extension of one spring and the concomitant compression of the other as the rotor oscillates (vertically in Figures 2a and 2b, horizontally in Figure 3), we have:

$$\tau = \Sigma F \ell = [k_1(d_1 + \Delta) - k_2(d_2 - \Delta)] \ell \quad (8)$$

which, if $k_1 = k_2 = k$ and $d_1 = d_2 = d$, becomes :

$$\tau = 2k\ell\Delta \quad (9)$$

Now $\Delta = \ell q$ where q is the angular displacement of the rotor-plus-gimbal (see Figure 1).

Equation (7) becomes:

$$K = \frac{\tau}{\theta} = \frac{2k\ell^2\theta}{\theta} = 2k\ell^2 \quad (10)$$

and the angular frequency:

$$\omega = \sqrt{\frac{2k\ell^2}{I}} \quad (11)$$

You will assume $k_1 = k_2 = k \simeq 76.5 \text{ N/m}$

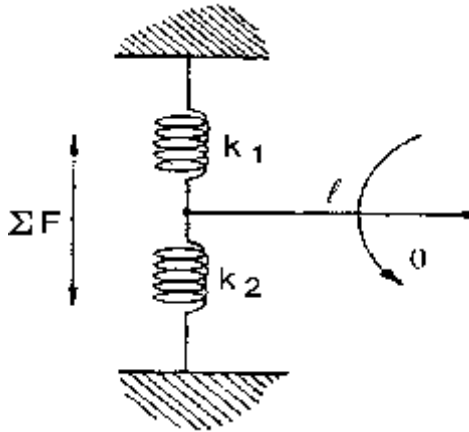


Figure 1.



Figure 2a.

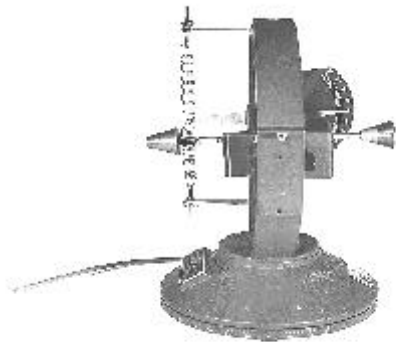


Figure 2b.



Figure 3.

Procedure

A. Moment of Inertia about Inner Gimbal Axis

Balance the gyro (see *Instrument alignment*, above). Attach the pointer and spring lever to the inner gimbal then apply power to the gyro rotor. Balance the outer gimbal by means of the adjustment weights on the shaft of the spin vector arrow. Attach the springs as shown in Figure 2a and 2b.

Deflect the spin vector arrow a small amount (5 to 10 degrees) and then release it.

Time ~ 20 oscillations with a stopwatch. Compute the natural frequency in oscillations/second. Calculate K from Eq. (10) and then use Equation (11) to compute the “inner” moment of inertia I_i .

B. Moment of Inertia about the Outer Gimbal Axis

Use the gyro configuration shown in Figure 3 without rotor excitation.

Orient the spin vector arrow in the horizontal plane.

Deflect the outer gimbal ~ 5 to 10 degrees about the vertical axis and then release it.

Time ~20 oscillations with a stopwatch; compute the natural oscillation frequency in oscillations/second.

Use equations (10) and (11) to compute the “outer” moment of inertia I_o of the rotor, inner gimbal and outer gimbal about the vertical axis.

5. Analytical study of nutation

Nutation is the oscillation of the spin arrow as it precesses. This may be observed by giving a downward tap to the head of the spin arrow as it precesses; a damped oscillatory motion will be observed. Another exercise would be to clip one of the weights to the spin arrow and observe the precession.

The angular frequency of the nutation depends on the inertial properties of the spinning rotor. If I_{ow} is the moment of inertia of the outer gimbal plus rotor *plus weight* about its vertical axis, and I_{iw} the moment of inertia of the inner gimbal plus rotor *plus weight* about its horizontal axis, it can be shown that the angular frequency of nutation of the rotor is given by:

$$\omega_n = 2\pi f_n = \frac{L}{\sqrt{I_{iw} I_{ow}}} \quad (12)$$

I_o and I_i determined in Experiment 4 become now I_{ow} and I_{iw} . They have a term added to take account of the extra contribution of the torquer mass.

Balance the gyro and unlock the outer gimbal. Apply power to the rotor and align the spin vector arrow in the horizontal plane. Clip a weight (use the maximum torque available) and time ~50 cycles of nutation. Calculate the nutation frequency in oscillations/second. Compute the nutation frequency using Equation (12) and compare the result with the measured value.

Observations

- A limitation of the apparatus comes from friction in the bearings of the gimbals. This is particularly noticeable with a large weight hung on the rotor axis. If friction did not exist, the arrow would not drop with time. You should figure out which bearing is causing this and you might work out how to move the base to eliminate this torque while you perform your experiment.
- One of the most useful features of a gyroscope is the stability of its direction of spin in space. Gyroscopic stability can be understood by rewriting Equation (1):

$$\vec{\tau} = \frac{I\mathbf{W}_f - I\mathbf{W}_i}{\Delta t}$$

where \mathbf{W}_f and \mathbf{W}_i are the final and initial angular speed (spin) of the object. If $\vec{\tau}$ is small or applied for a short time, it will have only a small effect in changing the angular momentum of the object and the spin direction will be constant.

RM Serbanescu - Feb. 2003. Previous versions of this guide sheet were written by D. Harrison (1974), J. Vise (1988) and T. Key (1995)