Multiphoton Localization and Propagating Quantum Gap Solitons in a Frequency Gap Medium

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The many-particle spectrum of an isotropic frequency gap medium doped with impurity resonance atoms is studied using the Bethe ansatz technique. The spectrum is shown to contain pairs of quantum correlated "gap excitations" and their heavy bound complexes ("gap solitons"), enabling the propagation of quantum information within the classically forbidden gap. In addition, multiparticle localization of the radiation and the medium polarization occurs when such a gap soliton is pinned to the impurity atom. [S0031-9007(97)03751-4]

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Light localization is a classical effect predicted [1] to occur in strongly scattering dielectric microstructures. In the context of photonic band gap (PBG) materials [2,3], nonclassical forms of localization such as photon-atom bound states have been predicted [4] when the resonant transition frequency of an impurity atom lies within a gap. This bound state is an eigenstate of the quantum electrodynamic Hamiltonian for a realistic PBG crystal exhibiting a general anisotropic photon dispersion relation. In this state, a virtually emitted photon may tunnel many wavelengths away from the atom before being reabsorbed, leading to non-Markov memory effects [5] in collective light emission from many atoms. It was recently shown that an effective model [6] describing both isotropic PBG systems and frequency dispersive media (DM) [7] doped with resonance atoms exhibits hidden integrability [8] and is diagonalized exactly [6,8] by means of the Bethe ansatz technique [9]. This suggests the possibility of a rich multiparticle spectrum in real physical systems exhibiting a frequency gap, when such systems are doped with impurity atoms.

In this paper, we demonstrate the existence of nonclassical states of light which may be generated, for instance, through the interaction of an external laser field with an impurity atom placed within a polariton gap [10] of a DM. In addition to ordinary polaritons and their bound complexes (ordinary solitons) occurring outside of the gap, the subgap spectrum of the system is shown to contain propagating pairs of correlated "gap excitations" and their heavy bound complexes (gap solitons). The individual gap excitations comprising the pair are correlated such that the probability amplitude of finding them far apart decreases exponentially with the ratio of their separation distance to the classical penetration length of the radiation into the medium. In addition to heavy gap solitons propagating within the gap, the spectrum contains multiphoton localized states pinned to the atom. Under external perturbations a pinned gap soliton may dissociate into propagating gap excitations. We evaluate the dispersion relations, the effective masses, and the dissociation energies of quantum gap solitons and we show that they are stable with respect to weak perturbations. Our results demonstrate a clear distinction, at the quantum level, between fermionic gap systems (such as electronic semiconductors) and bosonic gap systems. In a semiconductor, propagation within the energy gap is strictly forbidden. In the bosonic gap, however, certain nonclassical many-body gap states are allowed to propagate. This may have important consequences for the transmission of quantum information within a bosonic gap medium. Although the Bethe ansatz method made use of the *isotropic* one-particle dispersion relation, qualitative similar results may hold in a realistic, *anisotropic* PBG material which also exhibits a small nonresonant Kerr nonlinearity [11].

Consider the model Hamiltonian $\hat{H} = H_0 + V$, where

$$H_0 = \omega_{12}(\sigma^z + 1/2) + \int_C \frac{d\omega}{2\pi}\, \omega\, p^\dagger(\omega) p(\omega), \qquad (1a)$$

$$V = -\sqrt{\gamma} \int_C \frac{d\omega}{2\pi}\, z(\omega) [p(\omega)\sigma^+ + p^\dagger(\omega)\sigma^-]. \qquad (1b)$$

Here ω_{12} is the transition frequency of the two-level impurity atom placed within a bosonic frequency gap medium. The spin operators $\vec{\sigma} = (\sigma^x, \sigma^y, \sigma^z)$, $\sigma^\pm = \sigma^x \pm i\sigma^y$ satisfy the standard commutation algebra $[\sigma^i, \sigma^j] = \epsilon_{ijk}\sigma^k$ and act on the atomic variables of the system. The operators $p^\dagger(\omega)$ $[p(\omega)]$ create (annihilate) bosons of frequency ω in a specific (electric dipole) spherical harmonic state and satisfy the algebra $[p(\omega), p^\dagger(\omega')] = 2\pi\delta(\omega - \omega')$. The integration contour consists of two parts, $C = C_- \oplus C_+$, where $C_- = (0, \Omega_\perp)$ and $C_+ = (\Omega_\parallel, \infty)$ correspond to the lower and upper branches of the medium excitations, respectively. The interaction term (1b) describes emission and absorption of bosons by the atom. Here $\gamma = 4\omega_{12}^3 d^2/3$ is the inverse lifetime of the excited atom in free space with the dipole transition moment d, while the atomic form factor $z(\omega)$ contains the information about the polariton spectrum. Since the polariton gap persists for small wave vectors, $k \to 0$, we can neglect anisotropies of the underlying ionic crystal and take the dispersion relation to be isotropic. Also, we choose units in which $\hbar = c = 1$.

In the one-particle sector of the full Hilbert space, it is straightforward to verify [4,6,12] that when ω_{12} lies inside the frequency gap, one of the eigenstates of the Hamiltonian $\hat{H}$ describes a polariton-atom bound state. The multiparticle sector of the Hilbert space can be studied by the Bethe ansatz technique. The Schrödinger equation $(\hat{H} - E)|\Psi_N\rangle = 0$ is solved exactly [6,8] due to the two-polariton factorization of the multipolariton scattering. Imposing the periodic boundary conditions (PBC) on the N-polariton wave function leads to the following set of Bethe ansatz equations (BAE):

$$e^{ik_jL}\frac{h_j - i\beta/2}{h_j + i\beta/2} = -\prod_{l=1}^{N}\frac{h_j - h_l - i\beta}{h_j - h_l + i\beta}, \qquad (2)$$

which completely determine the N-particle spectrum of the model (1a) and (1b). Here, $E = \sum_{j=1}^{N}\omega_j$ is the eigenenergy and ω_j are polariton frequencies which solve Eq. (2). Also L is the radius of a sphere centered at the atom on which we apply PBC and then take the limit $L \to \infty$. In the case of a DM, the polariton momenta, $k_j \equiv k(\omega_j)$, and "rapidities," $h_j \equiv h(\omega_j)$, are expressed as

$$k(\omega) = \omega n(\omega), \qquad h(\omega) = \left(\frac{\omega_{12}}{\omega}\right)^2 \frac{\omega - \bar{\omega}_{12}}{\omega n^5(\omega)}, \qquad (3)$$

with the refractive index $n(\omega) = \sqrt{\varepsilon(\omega)}$ and the dielectric permeability of the medium $\epsilon(\omega) = (\omega^2 - \Omega_\parallel^2)/(\omega^2 - \Omega_\perp^2)$. Here $\bar{\omega}_{12} \simeq \omega_{12}$ is the Lamb shifted atomic transition frequency. The parameter $\beta = \gamma/\omega_{12}$ appears in both the polariton-atom scattering [left-hand side of Eq. (2)] and in the effective polariton-polariton coupling [right-hand side of Eq. (2)] caused by the polariton-atom scattering.

As $L \to \infty$, apart from real solutions, Eq. (2) admits complex ones, in which the rapidities h_j are grouped into the Bethe "strings." In this paper, we confine ourselves to the case when all N rapidities are grouped into a single string $h_j = H + i\frac{\beta}{2}(N + 1 - 2j)$, $j = 1, \ldots, N$ with a common real part ("carrying" rapidity) H. This is a solution of BAE if and only if the imaginary parts of rapidities h_j and corresponding momenta k_j have the same sign:

$$\mathrm{sgn}(\mathrm{Im}\, h_j) = \mathrm{sgn}(\mathrm{Im}\, k_j), \qquad j = 1, \ldots, N. \qquad (4)$$

This restricts possible magnitudes of polariton frequencies ω_j corresponding to the string rapidities. It is easy to understand that the necessary condition (NC) (4) determines the frequency intervals, in which the effective polariton-polariton coupling is attractive leading to bound many-particle complexes (quantum solitons).

In empty space, an effective photon-photon coupling is attractive for all frequencies of physical interest, and a Bethe string in the space of rapidities is mapped to a quantum soliton in the space of frequencies [13], $\omega_j = \Omega + i\frac{\gamma}{2}(N + 1 - 2j)$ with $E = \Omega N$. This also has a string structure. To avoid possible confusion in what follows we use the term "string" for solutions of BAE in the h space and the term "soliton" to refer to string's images in the ω and k spaces.

In the medium, analytical continuations of the functions $k(\omega)$ and $h(\omega)$ in the complex ω plane depend essentially on the position of the real part of the frequency with respect to the medium gap. We start with the case when the real part of ω lies outside the gap. Let $\omega = \lambda + i\eta$ and the real part $\lambda \in C$. For $\eta \ll \lambda$, the functions $k(\omega)$ and $h(\omega)$ are then represented as $k(\omega) = k(\lambda) + i\eta k'(\lambda)$ and $h(\omega) = h(\lambda) + i\eta h'(\lambda)$. Since $k'(\lambda) \equiv dk(\lambda)/d\lambda$ is positive for all λ, NC leads now to the condition $h'(\lambda) \equiv dh(\lambda)/d\lambda > 0$. This is met only if $\lambda \in C_-$. Therefore, the effective coupling is attractive only between polaritons of the lower branch, and the soliton image of the Bethe string is given by $\omega_j = \Omega + \frac{i}{2}\Gamma(\Omega)(N + 1 - 2j)$ and $k_j = K(\Omega) + \frac{i}{2}Q(\Omega)(N + 1 - 2j)$. Here $K(\Omega) = \Omega n(\Omega)$, $\Gamma(\Omega) = \beta/h'(\Omega)$, $Q(\Omega) = \beta k'(\Omega)/h'(\Omega)$, and $E = \Omega N$ is the soliton eigenenergy. The common real part of the polariton frequencies is found from the equation $h(\Omega) = H$, which has a root lying in C_- only if $H < 0$. The soliton obtained is quite similar to a vacuum soliton, and we will use the phrase "ordinary soliton" to refer to this solution, despite its inordinate behavior on different polariton branches. Polaritons of the upper branch are described by one-particle Bethe strings with real positive rapidities and do not form any bound complexes. The results obtained are clearly valid if $\omega_{12} \in C_- \oplus G$, where $G = (\Omega_\perp, \Omega_\parallel)$. If ω_{12} lies above the gap, $\omega_{12} \in C_+$, the effective coupling, including interbranch one, becomes attractive and admits both ordinary solitons in each branch and unusual "composite solitons" containing polaritons of different branches.

Now let us look for an image of a Bethe string, provided the real parts of all the frequencies ω_j lie inside the gap. Let $\omega = \xi + i\eta$ and $\xi \in G$. To find the analytical continuations of the functions $k(\omega)$ and $h(\omega)$ to the complex ω space, we need first to fix an appropriate branch of the function $n(\omega)$. Let $n(\xi \pm i0) = \pm i\nu(\xi)$, where $\nu(\xi) = \sqrt{|\varepsilon(\xi)|}$. In this case $k(\omega) = \mathrm{sgn}(\eta)[-\eta\kappa'(\xi) + i\kappa(\xi)]$, where $\kappa(\xi) = \xi\nu(\xi)$. Also $h(\omega) = \mathrm{sgn}(\eta)[\eta\phi'(\xi) - i\phi(\xi)]$ where $\phi(\xi) = (\xi - \omega_{12})f(\xi)$, and $f(\xi) = \omega_{12}^2[\xi^3\nu^5(\xi)]^{-1}$. Since the function $\kappa(\xi)$ is positive, NC leads to the condition $\phi(\xi) < 0$. It means that allowed gap excitations exist only for $\xi \in (\Omega_\perp, \omega_{12})$. Because of a strong nonradiative relaxation in the medium in the vicinity of the frequency $\Omega_\perp$, we focus our studies on gap states of physical interest lying in the vicinity of the atomic frequency ω_{12}. The remaining analysis is simplified by linearizing the function $\phi(\xi)$ at the point $\xi = \omega_{12}$, $\phi(\xi) \simeq a(\xi - \omega_{12})$, where $a = f(\omega_{12})$.

Now we are able to map a Bethe string to corresponding gap excitations. We start with the simplest case of a two-particle string, $N = 2$. Its complex conjugated rapidities, $h = H + i\beta/2$ and $h^* = H - i\beta/2$, are mapped to the corresponding pairs of the complex conjugated frequencies, $\omega = \xi + i\eta$ and $\omega^* = \xi - i\eta$, where the imaginary part is assumed to be positive, $\eta > 0$, and momenta, $k = q + i\kappa(\xi)$ and $k^* = q - i\kappa(\xi)$, where $q = -\eta\kappa'(\xi)$. The real and imaginary parts of frequencies are expressed in terms of the string parameters, $\xi = \omega_{12} - \beta/2a$ and $\eta = H/a$. In the spherical harmonic formalism introduced previously [6], the real part of the particle momenta q must be positive. Since $\kappa'(\xi)$ is negative, it follows that $q = \eta|\kappa'|$. Consequently only a string with a positive carrying rapidity, $H > 0$, is mapped to gap states of physical interest.

The expressions obtained describe a novel, quantum correlated state of two gap excitations. Two gap particles comprising the pair are "confined" to travel together. They do not exist separately from each other, unlike polaritons of ordinary solitons. Moreover, the confined state cannot be treated as a bound state of two polaritons from different branches (like a Wannier-Mott exciton in semiconductors), because, under the condition $\omega_{12} \in G$, the interbranch polariton-polariton coupling is repulsive. The spatial size of a pair, $\delta \sim \kappa^{-1}(\xi)$, is nothing but the penetration length of the classical radiation field with the frequency $\omega = \xi \in G$ into the medium [7]. Since the wave function of a single gap particle is unnormalizable, free one-particle gap states in the bosonic gap are forbidden in exactly the same way that electronic propagation is forbidden in a conventional semiconductor gap. However, in the case of bosons, the effective particle-particle coupling allows one to construct the normalizable wave function of a pair from unnormalizable wave functions of each particle. At large interparticle separations, the wave function of a pair has the form

$$\Psi(x_1, x_2) \sim \exp\{iq(x_1 + x_2) - \kappa(\xi)\,|x_1 - x_2|\}, \quad (5)$$

where the real and imaginary parts of momenta describe, respectively, the motion of the center of gravity and the spatial size of a pair. The auxiliary coordinate variables x_j are analogous to spatial coordinates along an arbitrary axis passing through the atom. The vicinities $x < 0$ and $x > 0$ correspond, respectively, to ingoing and outgoing spherical harmonics of the polariton field. An angular distribution of the field is determined by the specific spherical harmonic polariton state.

Let us consider now the mapping of a string containing an even number of particles ("even string"), $N = 2l$, to gap excitations. A pair of complex conjugated rapidities, h_j and h_j^*, $j = 1, \ldots, l$, is mapped to a pair of frequencies $\omega_j = \xi_j^{(0)} + i\eta$ and $\omega_j^* = \xi_j^{(0)} - i\eta$ and corresponding momenta $k_j = q_j + i\kappa(\xi_j^{(0)})$ and $k_j^* = q_j - i\kappa(\xi_j^{(0)})$, where the real parts of frequencies are given by $\xi_j^{(0)} = \omega_{12} - (\beta/a)(l + 1/2 - j)$. The upper index, (0), indicates that these expressions are derived within the linear approximation for $\phi(\xi)$ in the vicinity of $\xi = \omega_{12}$. The expressions obtained describe a bound complex of l pairs of confined gap particles (Fig. 1) with the eigenenergy

$$E_l^{(0)} = 2\sum_{j=1}^{l} \xi_j^{(0)} = 2\omega_{12}l - (\beta/a)l^2. \quad (6)$$

We will use the phrase "gap soliton" to refer to this state of the system. Unlike an ordinary soliton, a gap soliton is stable with respect to quite weak perturbations of the system, because the dissociation energy, U_d, of a soliton with l pairs into two solitons with l_1 and $l_2 = l - l_1$ pairs is positive: $U_d \equiv E_{l_1}^{(0)} + E_{l_2}^{(0)} - E_l^{(0)} = 2(\beta/a)l_1l_2$. The radial thickness of the spherical harmonic soliton pulse is determined by the imaginary part of the momentum k_1 corresponding to the rapidity h_1. Since $\kappa(\xi)$ is a monotonically decreasing function, the gap soliton size, $\delta_l^{(0)} \simeq \kappa^{-1}(\xi_1^{(0)})$, falls with the growth of the number of pairs l.

Since the effective coupling constant is very small, $\beta \ll 1$, the linear approximation works well even for large solitons containing many pairs. But, in this approximation, the soliton energy is independent of its momentum. For what follows it is convenient to introduce the soliton energy per particle $\epsilon_l^{(0)} \equiv E_l^{(0)}/2l = \omega_{12} - (\beta/2a)l$, and the soliton momentum per particle, $q \equiv (2l)^{-1}\sum_j q_j \approx (\eta/l)\sum_j |\kappa'(\xi_j^{(0)})|$. To estimate the first corrections to Eq. (6), we have to keep the next term, $i(\eta^2/2)\phi''(\xi)$, of the Taylor series for the function $h(\omega)$ and the next term in the expansion of the function $\phi(\xi)$ at the point $\xi = \omega_{12}$: $\phi(\xi) \simeq a(\xi - \omega_{12}) + b(\xi - \omega_{12})^2$, where $b = f'(\omega_{12}) > 0$. The frequencies ξ_j are then given by $\xi_j - \xi_j^{(0)} = -(b/a)(\xi_j^{(0)} - \omega_{12})^2 + (b/a)\eta^2$, while the soliton momentum is still given by q. The term $-(b/a)(\xi_j^{(0)} - \omega_{12})^2$ leads to the first order correction to the energy of a motionless soliton and determines the width

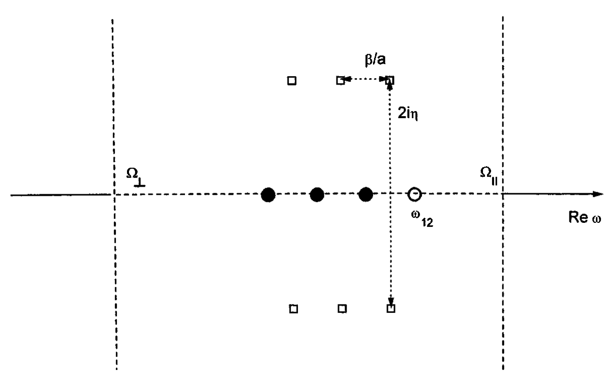

FIG. 1. The frequencies of the mobile six-particle gap soliton (open squares) which has dissociated from the pinned seven-particle gap soliton. In the latter, the particle with the frequency ω_{12} (open circle) bound to the atom and the remaining three pairs of six particles (solid circles) comprise the bound complex pinned to the atom.

of the soliton band, while $(b/a)\eta^2$ leads to the kinetic energy contribution to the total soliton energy in the effective mass approximation:

$$\epsilon_l = \epsilon_l^{(0)} - \Delta_l + q^2/2m_l\,. \tag{7}$$

Here $\Delta_l = \frac{b\beta^2}{12a^3}(4l^2 - 1)$ and $m_l = \frac{a}{2bl^2}(\sum_j |\kappa'(\xi_j^{(0)})|)^2$ are the band half-width and the effective mass, which increase with l. At small l, the propagating gap soliton bands are very narrow and solitons are very heavy and even motionless at $q = 0$. But the bandwidth increases as l^2, so that large l solitons are quite mobile when the momentum q becomes larger than the range of validity of the effective mass approximation. At arbitrary q, the exact equations for the soliton parameters ξ_j and η_j are given by $\mathrm{Re}\, h(\xi_j, \eta_j) = H$ and $\mathrm{Im}\, h(\xi_j, \eta_j) = \beta(l - j + 1/2)$. The solution of these equations requires simple numerical calculations.

Finally, we evaluate the pinning energy of a gap soliton to the atom. In h space, pinned solitons are described by odd strings with $H \to 0^+$. The one-particle string, $l = 0$, with $H \to 0^+$ is clearly mapped to the gap state with $\omega = \bar{\omega}_{12} \approx \omega_{12}$. This state is nothing but the polariton-atom bound state [4,6,12] in the one-particle spectrum of the system. Therefore the extra real rapidity of an odd string can be mapped to the gap state with $\xi_0^{(p)} = \omega_{12}$, while the remaining complex conjugated pairs of rapidities are mapped to a deformed motionless gap soliton with $\xi_j^{(p)} = \omega_{12} - (\beta/a)(l - j + 1)$. We used here the term "deformed" to emphasize that the soliton frequencies now contain the extra term $-\beta/2a$ due to the polariton-atom bound state, which deforms the soliton, in contrast to the mobile, even gap soliton whose frequencies are given by $\xi_j^{(0)}$. Since one of the particles of an odd motionless gap soliton is bound to the atom, a soliton as a whole is also "pinned" to the atom. Moreover, as $H \to 0^+$, the imaginary parts of frequencies and the real parts of particle momenta vanish. A pinned soliton describes a many-particle state of the system, in which the radiation and medium polarization are localized in the vicinity of the atom. To evaluate the energy of pinning, we need only to compare the energy of a soliton with $2l + 1$ particles pinned to the atom, $E_l^{(p)} = \omega_{12}(2l + 1) - (\beta/a)l(l + 1)$, with the sum of the energies of the one-particle bound state, ω_{12}, and a motionless gap soliton with l pairs, $E_l^{(0)}$. We find that the binding energy $U_l = -(\beta/a)l$ is proportional to the number of pairs. Moreover, the energy required to pull a single pair of particles out of a pinned soliton, $U_1 = (E_{l-1}^{(p)} + E_1^{(0)}) - E_l^{(p)} = (\beta/a)(2l - 1)$, is even greater than the energy required to pull off all l pairs. Therefore, the state of a pinned soliton is stable with respect to quite weak perturbations of the system and its stability increases with l. We mention finally that, for $H > 0$, it is also possible to construct composite solitons, consisting of an odd number of bosons, which correspond, physically, to a bound state of a gap soliton and a polariton in the upper branch.

In summary, we have shown that the isotropic dispersive medium doped with an impurity atom exhibits a rich many-particle spectrum containing heavy, mobile, gap solitons as well as pinned solitons. Mobile gap solitons are highly nonclassical, quantum correlated states consisting of an even number of gap particles. This suggests the remarkable possibility that a bosonic frequency gap medium, while impervious to classical linear wave propagation may allow propagation of certain correlated quantum excitations. Multiparticle gap solitons may be generated by both nonlinearly exciting an impurity atom and Dicke superradiance from a collection of these excited atoms. Unlike the single excitation which can tunnel a distance given by the classical penetration length within the gap, the paired excitations as well as the resulting heavy gap solitons can propagate freely through the gap of this harmonic medium. The Bethe ansatz solution which we have presented relied on the existence of an isotropic polariton dispersion relation. In a real PBG material, the photon dispersion relation is highly anisotropic. The propagation of quantum information within a PBG material, in this manner, would be of considerable importance in such applications as quantum computing [14].

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