Localization and absorption of waves in a weakly dissipative disordered medium

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The effect of a small imaginary part ϵ_2 to the dielectric constant on the propagation of waves in a disordered medium near the Anderson localization transition is considered. The $n \rightarrow 0$ replica-field representation of the averaged Green's function leads to a nonlinear σ model with a symmetry-breaking perturbation proportional to ϵ_2. In $d=2+\epsilon$, the renormalized energy absorption coefficient is shown to increase anomalously with frequency ω near the mobility edge ω^* as $\alpha \sim (\omega^* - \omega)^{-(d-2)\nu/2}$, $\nu = 1/\epsilon$. It is shown that the wavelength λ^* below which localization occurs is related to the elastic mean free path l by $(l/\lambda^*)^{d-1} \sim 1/\epsilon$ $(d>2)$. This may occur near the limit of attainable disorder from a quenched random array of small dielectric or metallic spheres.

I. INTRODUCTION

Waves in a strongly scattering random medium in the absence of dissipation exhibit a transition from propagating modes to localized states. In a previous paper[1] it was suggested that for phonons in a disordered elastic medium there exists a critical frequency ω^* above which all normal modes of vibration exhibit Anderson[2,3] localization, whereas at lower frequencies weaker Rayleigh scattering ensures that all states are extended. For frequencies below the mobility edge ω^*, the vibrational energy exhibits a diffusive behavior on length scales that are large compared to the phonon elastic mean free path l. Wave interference gives rise to a renormalization of this energy diffusion coefficient on still longer length scales. In $d=2+\epsilon$ dimensions the renormalized frequency-dependent diffusion coefficient vanishes as the mobility edge is approached from low frequencies according to

$$D(\omega) \sim (\omega^* - \omega)^t, \quad t=(d-2)\nu \tag{1}$$

where $\nu = 1/\epsilon$ to leading order in an ϵ expansion. In this paper it is shown that in a weakly dissipative random medium, a consequence of diffusive scattering and interference is the appearance of an anomalously large absorption of wave energy. In particular, for a plane-wave source of frequency $\omega \lesssim \omega^*$ and intensity I_0, the intensity at a distance x will decay as

$$I = I_0 e^{-\alpha x} , \tag{2a}$$

where

$$\alpha \sim (\omega^* - \omega)^{-t/2} . \tag{2b}$$

It is shown from first principles that in $d=2+\epsilon$ the mobility edge ω^* is determined by the condition $[(\omega^*/c)l]^{d-1} \sim 1/\epsilon$, where c is the average speed of wave propagation. In $d=3$ this reduces to the familiar Ioffe-Regel condition.[4]

Of experimental interest[5] is the case of electromagnetic waves propagating in a disordered medium with a small imaginary part to the dielectric constant. It is shown that a localization transition should occur near the limit of attainable disorder for a system of small dielectric spheres of dielectric constant ϵ_a immersed in an otherwise uniform background ϵ_b at a photon wavelength comparable to the sphere diameter. The absorption coefficient near the mobility edge is obtained for the case of metal spheres characterized by a simple Drude conductivity. The observation of such an anomalous absorption is suggested as a probe for the existence of a photon-mobility edge.

II. FIELD THEORY

Consider the propagation of waves in a disordered medium characterized by a complex dielectric constant $\epsilon(x)$ and satisfying the scalar wave equation

$$\nabla^2 \phi = \frac{\epsilon(x)}{c^2} \frac{\partial^2}{\partial t^2} \phi , \tag{3a}$$

where

$$\epsilon(x) = 1 + \epsilon_1(x) + i\epsilon_2 . \tag{3b}$$

Here c is the average speed of wave propagation which is hereafter set equal to unity, and it is assumed that the dissipation ϵ_2 is weak (as will be clarified later), and, for simplicity, uniform throughout the medium. The generalization to a propagating vector field is straightforward but does not alter the physics of localization.[1] In this regard, the scalar equation (3a) may be considered as a simplified model not only for sound waves but also for electromagnetic waves in a disordered system. Scattering is caused by spatial fluctuations in the real part of the dielectric constant which are characterized by ($\langle \ \rangle_{\text{ens}}$ represents the ensemble average)

$$\langle \epsilon_1(x) \rangle_{\text{ens}} = 0 \tag{4a}$$

and

$$\langle \epsilon_1(x) \epsilon_1(y) \rangle_{\text{ens}} = \gamma^2 \delta^d(x-y) . \tag{4b}$$

The response of the medium at x to a monochromatic source of frequency ω at the origin is given by the Green's function which satisfies the equation

$$[\nabla^2 + \omega_+^2 \epsilon(x)] G(x,0,\omega_+) = \delta^d(x), \quad \omega_{\pm} \equiv \omega \pm i\eta . \tag{5}$$

For the case of a constant imaginary part to the dielectric constant, the one-photon Green's function has the usual replica-field representation,[1,6,7]

$$G(x,0,\omega_+)=\lim_{n_+\to 0}\int D\phi\,\phi^1(x)\phi^1(0)e^{L_+}\ , \tag{6a}$$

where

$$L_+=\frac{i}{2}\int d^dx\,\phi^\alpha(x)[\nabla^2+\omega_+^2\epsilon(x)]\phi^\alpha(x),\quad \alpha=1,\ldots,n_+ \tag{6b}$$

with the contour of functional integration along the real axis in the complex ϕ plane. The transport of energy is determined by the two-particle Green's function, which when averaged over all realizations of the random medium compatible with the condition (4) has the representation

$$\langle\,|G(x,0,\omega_+)|^2\rangle_{\rm ens}=\lim_{\substack{n_+\to 0\\ n_-\to 0}}\int D\phi\,\phi_+^1(x)\phi_+^1(0)\phi_-^1(x)\phi_-^1(0)e^{L}\ , \tag{7a}$$

where

$$L=L_0+L_0^*+L_{\rm int}-\frac{\omega^2}{2}\epsilon_2\int d^dx\,\phi^\alpha(x)\phi^\alpha(x)\ ,\qquad \alpha=1,\ldots,n_++n_- \tag{7b}$$

and

$$L_{\rm int}=\frac{-\omega^4\gamma^2}{8}\int d^dx[\phi_+^\beta(x)\phi_+^\beta(x)-\phi_-^\beta(x)\phi_-^\beta(x)]^2\ ,\qquad \beta=1,\ldots,n_\pm\ . \tag{7c}$$

Here L_0 is obtained from L_+ by replacing $\epsilon(x)$ by unity. Strictly speaking, an averaging restricted by the condition that $\mathrm{Re}\epsilon(x)>0$ would generate higher-order couplings than the quartic one, (7c). Such higher-order terms, however, are irrelevant in the renormalization-group sense, and so it suffices to consider the Gaussian average (7a).

The term involving the imaginary part of the dielectric constant explicitly breaks the $O(n_+,n_-)$ replica symmetry of the Lagrangian L associated with energy conservation. The interaction term in L may be decoupled by the introduction of an intermediate tensor field $\tilde Q^{\alpha\beta}$:

$$e^{L_{\rm int}}=\int D\tilde Q\exp\left\{-\tfrac{1}{2}\int d^dx\left[\tilde Q^{\alpha\beta}\tilde Q^{\alpha\beta}+\gamma\omega^2(\phi_+,\phi_-)\begin{bmatrix}-i\tilde Q_{++} & \tilde Q_{+-}\\ \tilde Q_{-+} & i\tilde Q_{--}\end{bmatrix}\begin{bmatrix}\phi_+\\ \phi_-\end{bmatrix}\right]\right\}\ . \tag{8}$$

Introducing the change of variables

$$\tilde Q=Q-\frac{i\epsilon_2}{\gamma}\begin{bmatrix}I & 0\\ 0 & -I\end{bmatrix} \tag{9}$$

and integrating over the ϕ fields yields

$$\langle\,|G(x,0,\omega_+)|^2\rangle_{\rm ens}=\frac{4}{\gamma^2\omega^4}\lim_{\substack{n_+\to 0\\ n_-\to 0}}\int DQQ_{+-}^{11}(x)Q_{+-}^{11}(0)e^{L[Q]}\ , \tag{10a}$$

where

$$L[Q]=L_0[Q]+\frac{i\epsilon_2}{\gamma}\int d^dx(Q_{++}^{\alpha\alpha}-Q_{--}^{\alpha\alpha})\ , \tag{10b}$$

and

$$L_0[Q]=-\tfrac{1}{2}\mathrm{tr}\ln\left[\begin{bmatrix}\nabla^2+\omega_+^2 & 0\\ 0 & \nabla^2+\omega_-^2\end{bmatrix}+\gamma\omega^2Q\right]-\tfrac{1}{2}\int d^dx\,Q^{\alpha\beta}Q^{\alpha\beta}+\text{const}\ . \tag{10c}$$

The $O(n_+,n_-)$ symmetry among the ϕ fields leads to a pseudo-orthogonal $\tilde O(n_+,n_-)$ symmetry in the $\tilde Q$ fields as discussed previously.[8,6] The $\tilde O(n_+,n_-)$-symmetric part $L_0[Q]$ of the Lagrangian has a replica diagonal saddle point Q^0 which defines a coherent-potential approximation (CPA) to the averaged one-photon Green's function in the absence of dissipation:[1]

$$Q^\pm\equiv Q_{\pm\pm}^0=-\frac{\gamma\omega^2}{2}\int\frac{d^dk}{(2\pi)^d}G^\pm(k)\ , \tag{11a}$$

$$G^\pm(k)=\frac{1}{\omega_\pm^2(1+\gamma Q)-k^2}\ . \tag{11b}$$

The imaginary part of the self-energy defines a phase coherence length over which the one-photon Green's function decays in coordinate space as a result of averaging over all possible realizations of the disorder:

$$\Lambda\equiv(\gamma\omega^2\mathrm{Im}Q^+)^{-1/2}\ . \tag{12}$$

Unlike the case of phonons, in which the crystal lattice spacing introduces a natural ultraviolet cutoff to the integral (11a), the divergence of this integral for the case of photons must be regulated by the introduction of a finite correlation length to the disorder. The generalization of the coherent-potential approximation to correlated disorder has been discussed in a previous paper.[8] A finite correlation length removes the unphysical property of the correlation function (4b), namely that arbitrarily short-wavelength waves are scattered strongly.

Expanding to quadratic order in the fluctuations $\hat Q\equiv Q-Q^0$ about the saddle point leads, in the usual manner,[1,6,7,9] to a nonlinear σ model with the addition of a symmetry-breaking perturbation. In terms of the Fourier transform

$$Q(k)\equiv\int d^dx\,e^{ik\cdot x}Q(x)\ , \tag{13}$$

this effective Lagrangian may be expressed as

$$L[Q]\simeq -\frac{1}{2}\int \frac{d^dk}{(2\pi)^d} C(k) Q^{\alpha\beta}(k) Q^{\alpha\beta}(-k) + \frac{i\epsilon_2}{\gamma}\int d^dx (Q^{\alpha\alpha}_{++} - Q^{\alpha\alpha}_{--}) , \tag{14a}$$

where

$$C(k)=1-\frac{\gamma^2\omega^4}{2}\int \frac{d^dp}{(2\pi)^d} G^+(p+k/2) G^-(p-k/2) , \tag{14b}$$

and is subject to the constraint

$$Q(x)=i\,\mathrm{Im}Q^+ U^T(x)\begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} U(x),\quad U\in\tilde{O}(n_+,n_-) \tag{14c}$$

The frequency-dependent diffusion coefficient $D(\omega)$ (which in the units $c=1$ is precisely the photon mean free path l) may be obtained from the low-momentum expansion of (14b):

$$C(k)\simeq C_0(\omega)+C_2(\omega)k^2+\cdots . \tag{15a}$$

Direct evaluation yields (see Appendix)

$$C_0(\omega)=2\omega\eta\Lambda^2 \tag{15b}$$

and

$$C_2(\omega)\simeq 2\omega^2\Lambda^4 \quad (d=3) , \tag{15c}$$

where Λ is the phase coherence length defined by (12). In obtaining the above expressions, an assumption of weak disorder has been made by the neglect of the real part of the averaged one-photon self-energy (11a) ($\gamma\,\mathrm{Re}Q^+ \ll 1$). In deriving (15c), the integral in (14b) has been evaluated in the limit of photon wavelength that is short compared to the phase coherence length ($\Lambda\omega \gg 1$), and use has been made of the relation

$$\mathrm{Im}Q^+ \simeq (\pi/4)\gamma\omega\rho(\omega) , \tag{16a}$$

where

$$\rho(\omega)=\frac{S_d}{(2\pi)^d}\omega^{d-1} \tag{16b}$$

is the photon density of states and S_d is the surface area of the d-dimensional unit sphere. It follows that

$$D(\omega)\,(\equiv lc)=\lim_{\eta\to 0}\eta\frac{C_2(\omega)}{C_0(\omega)} \tag{17a}$$

$$=\omega\Lambda^2 . \tag{17b}$$

The renormalized diffusion coefficient which takes into account the effect of wave interference from different scattering events may be obtained by performing a momentum shell integration[10] or otherwise[11,12] in $d=2+\epsilon$ of the nonlinear σ model. The choice of the ultraviolet momentum cutoff in (14a) is dictated by the large-momentum behavior of the function $C(k)$. Physically, one would expect that contributions to the renormalization of the diffusivity would occur over length scales larger than the mean free path l, this being the shortest length over which the coupling constant $D(\omega)$ is well defined. Direct evaluation of the integral (14b) reveals that $k\simeq l^{-1}$ is indeed the required ultraviolet cutoff. For example, in $d=1$, a straightforward contour integration ($\gamma\,\mathrm{Re}Q^+ \ll 1$) yields (see Appendix)

$$C(k)=\frac{k^2}{k^2+l^{-2}} \quad (d=1) , \tag{18}$$

with the photon mean free path as defined in (17a). By virtue of the constraint (14c), it follows that contributions to the integral in (14a) from wave vectors that are large compared to the inverse mean free path yield a constant. This suggests the replacement

$$C(k)=\begin{cases} C_2(\omega)k^2, & kl\le 1 \\ 0, & kl>1 \end{cases} \tag{19}$$

in the nonlinear σ model. The constant which is neglected by this truncation may be rendered finite, as in the case of the one-photon self-energy, by the introduction of a nonzero correlation length to the disorder.[8]

Rescaling the Q fields by a factor of $\mathrm{Im}Q^+$ and expressing all lengths in units of the mean free path l yields a nonlinear σ model in the new dimensionless variables x, k, and Q:

$$L[Q]=\frac{-1}{2g}\int_{|k|<1}\frac{d^dk}{(2\pi)^d}k^2 Q^{\alpha\beta}(k)Q^{\alpha\beta}(-k) + ih\int d^dx (Q^{\alpha\alpha}_{++}-Q^{\alpha\alpha}_{--}) , \tag{20a}$$

subject to the constraint

$$Q(x)=iU^T(x)\begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} U(x) , \tag{20b}$$

and with the dimensionless conductance

$$\Sigma\equiv 1/g=C_2(\omega)(\mathrm{Im}Q^+)^2 l^{d-2} \tag{20c}$$

and symmetry-breaking field

$$h=(\epsilon_2/\gamma)(\mathrm{Im}Q^+)l^d . \tag{20d}$$

Using the relations (15c), (16a), and (17a), these may be expressed in terms of physically observable quantities:

$$1/g=(\pi/2)\rho(\omega)D(\omega)l^{d-2} \tag{21a}$$

and

$$h=(\pi/4)\epsilon_2\omega\rho(\omega)l^d . \tag{21b}$$

Momentum shell integration at the one-loop level in $d=2+\epsilon$ yields the following recursion relations in the $n_\pm \to 0$ limit:

$$\frac{dg}{d\ln(L/l)}=-\epsilon g+\frac{1}{4}\frac{S_d}{(2\pi)^d}\frac{g^2}{1+hg} , \tag{22a}$$

$$\frac{dh}{d\ln(L/l)}=dh . \tag{22b}$$

In the absence of dissipation ($h=0$) and above two dimensions ($\epsilon>0$), the mobility edge corresponding to the fixed point $g^*=4\epsilon(2\pi)^d/S_d$ of (22a) can be expressed as a simple condition on the photon mean free path. Using (21a) and (17a), this is

$$(l\omega)^{d-1}=1/2\pi\epsilon\ . \tag{23}$$

In $d=3$ this becomes precisely the condition for localization suggested by Ioffe and Regel.[4]

From the nonlinear σ model (20), it follows that the averaged two-photon propagator takes the form

$$\int d^dx\, e^{iq\cdot x}\langle\, |G(x,0,\omega_+)|^2\rangle_{\mathrm{ens}}\simeq\frac{1}{h+(1/g)q^2}\ , \tag{24}$$

where q is a dimensionless wave vector measured in units of the inverse mean free path. Comparison with (2a) suggests the identification

$$\alpha=\sqrt{hg}\,/l \tag{25}$$

for the energy absorption coefficient in the disordered medium. Substituting (21a) and (21b), this yields

$$\alpha(\omega)=(\omega\epsilon_2/2l)^{1/2} \tag{26}$$

for the unrenormalized absorption coefficient. The expression (26) describes the attenuation of electromagnetic flux as a result of diffusive scattering and absorption, and is applicable when the photon elastic mean free path l is short compared to the inelastic mean free path,

$$l_{\mathrm{inel}}=[(\omega/c)\mathrm{Im}\sqrt{1+i\epsilon_2}]^{-1}\ . \tag{27}$$

This is in contrast to the limit $l\gg l_{\mathrm{inel}}$, where it is customary to identify $2l_{\mathrm{inel}}^{-1}$ as the absorption coefficient.

In the presence of weak dissipation ($h>0$), the recursion relation (22a) may be linearized about the Anderson-Wegner fixed point $(g^*,h^*)=(4\epsilon(2\pi)^d/S_d,0)$:

$$\frac{d\,\Delta g}{d\ln(L/l)}=\epsilon\,\Delta g,\ \ \Delta g\equiv g-g^*\sim\omega-\omega^*\ . \tag{28}$$

This combined with (22b) leads to a homogeneity relation for the renormalized absorption coefficient in the vicinity of a mobility edge:

$$\alpha(\Delta g,h)=b^{-1}\alpha(b^{\epsilon}\Delta g,b^d h)\ . \tag{29}$$

Choosing $b=(\Delta g)^{-1/\epsilon}$ and using the fact that, far from the mobility edge, α has the form (25) yields the required critical behavior:

$$\alpha(\omega)\sim(\omega^*-\omega)^{-(d-2)\nu/2},\ \ \nu=1/\epsilon\ . \tag{30}$$

This behavior should be observable for frequencies close to the mobility edge, but strictly speaking for $|\Delta g|>h^{(d-2)/d}$, since the scaling form (29) is valid only for reasonably small values of its arguments. The rounding off of the divergence (30), however, should occur over a smaller and smaller frequency range in the limit of weak dissipation ($h\to0$).

III. SCATTERING FROM SMALL METAL SPHERES

The theory discussed in the preceding section is now applied to a simple model of a disordered composite medium consisting of a collection of small metallic spheres randomly quenched in an otherwise homogeneous dielectric host. Consider a single sphere of dielectric constant ϵ_a in a background of dielectric constant ϵ_b. From elementary electrostatics,[13] the electric field E_a within the sphere is related to the applied field E_b in the background by

$$E_a=\frac{3\epsilon_b}{\epsilon_a+2\epsilon_b}E_b\ . \tag{31a}$$

For a volume-filling fraction $f\ll1$ of metal spheres, the ensemble-average electric field is given by

$$\langle E\rangle_{\mathrm{ens}}=fE_a+(1-f)E_b\ . \tag{31b}$$

Likewise, the average displacement field satisfies

$$\langle D\rangle_{\mathrm{ens}}=f\epsilon_aE_a+(1-f)\epsilon_bE_b\ . \tag{31c}$$

The Maxwell-Garnett effective dielectric constant[14] for the medium is given by

$$\epsilon_{\mathrm{eff}}\equiv\langle D\rangle_{\mathrm{ens}}/\langle E\rangle_{\mathrm{ens}} \tag{32a}$$

$$=\epsilon_b\left[1+\frac{3f(\epsilon_a-\epsilon_b)}{(1-f)(\epsilon_a-\epsilon_b)+3\epsilon_b}\right]. \tag{32b}$$

Choosing $\epsilon_b=1$ and assuming that the metal spheres may be described by a classical Drude conductivity,

$$\epsilon_a=1+\frac{4\pi i\sigma}{\omega},\ \ \sigma=\frac{\sigma_0}{1-i\omega\tau} \tag{33}$$

where σ_0 is the dc electrical conductivity and τ the electronic mean free time, it follows that

$$\epsilon_{\mathrm{eff}}=1+\frac{4\pi\sigma_0f/\tau}{(1-f)(4\pi\sigma_0/3\tau)-\omega^2-i\omega/\tau}\ . \tag{34a}$$

For frequencies small compared to the metal plasma frequency $\omega_p\equiv\sqrt{4\pi\sigma_0/\tau}$ and small filling fractions ($f\ll1$), this may be simplified to[15]

$$\epsilon_{\mathrm{eff}}\simeq1+3f+9if\omega/4\pi\sigma_0\ . \tag{34b}$$

Here, it has also been assumed that $\omega_p\tau\gg1$. The fluctuations in the real part of the dielectric constant may be thought of as the result of fluctuations δf of the local volume-filling fraction of metal. This may be expressed as $\delta f=\delta\rho_0v$, where v is the volume of a single sphere and $\delta\rho_0$ is the local fluctuation in their number density. Comparing (34b) and (3b), it follows that

$$\langle\epsilon_1(x)\epsilon_1(y)\rangle_{\mathrm{ens}}=9v^2\langle\delta\rho_0(x)\delta\rho_0(y)\rangle_{\mathrm{ens}}\ . \tag{35}$$

Integrating over x and y for a sample of volume L^d and comparing with (4b) yields

$$9v^2\langle(\Delta N)^2\rangle_{\mathrm{ens}}=\gamma^2L^d\ . \tag{36a}$$

For a Poisson distribution of metal spheres, the mean-square fluctuation of the number of spheres $\langle(\Delta N)^2\rangle_{\mathrm{ens}}$ in a volume L^d is precisely the average number in the same volume. Therefore,

$$\gamma^2=9v^2\rho_0=9vf\ . \tag{36b}$$

Using Eqs. (12), (16a), (16b), and (17), it follows that the photon mean free path in $d=3$ for this system is

$$l=8\pi/9vf\omega^4\ . \tag{37}$$

Substituting into the mobility-edge criterion (23) suggests that photons of wavelength

$$\lambda^*/2\pi \simeq a_0 f^{1/3} \quad (d=3) \tag{38}$$

are Anderson localized in this disordered medium. Here a_0 is the radius of the metal spheres. For wavelengths that are long compared to λ^*, the effect of the dissipative term in (34b) is to give an absorption coefficient [cf. (25)]

$$\alpha(\omega) = (9/8\pi) f \omega^3 (v/\sigma_0)^{1/2} \tag{39}$$

in the diffusive scattering regime ($l \ll l_{\rm inel}$). As the wavelength approaches λ^*, this should cross over to the form (30) before rounding off to a peak value determined by ϵ_2.

IV. DISCUSSION

In $d=3$, for a volume-filling fraction $f \sim 0.1$ of metal spheres, the wavelength (38) of the photon mobility edge is estimated to be comparable to the sphere diameter. Strictly speaking, the Maxwell-Garnett dielectric constant (34b) as well as the continuum model of the disorder (4b) are based on the assumption that the wavelength of radiation is large compared to the sphere diameter and hence does not account for higher multipole scattering. As discussed in a previous paper,[8] the scattering of higher-angular-momentum components of an incident plane wave may be examined by introducing a finite correlation length for the disorder. For wavelengths that are very short compared to the correlation length, the elastic mean free path l becomes large compared to λ, resulting in classical behavior (extended states) in three dimensions. Equation (38), however, suggests that there exists an intermediate-frequency range between the classical limit and the Rayleigh scattering limit in which a photon mobility edge may be experimentally observed. This intermediate regime may be difficult to achieve for filling fractions f much smaller than the bulk percolation limit for spheres in $d=3$, since the mean free path l cannot be made smaller than the correlation length of the disorder, which in this case is the mean interparticle spacing. A mixture of insulating and conducting spheres might, however, avoid the problem of bulk conductivity associated with percolation and the consequent screening ($\omega < \omega_p$) of electromagnetic waves from the medium. Another possibility might be scattering from a dense tangle of thin metal wires coated with an insulator.[16]

The theory presented is asymptotically accurate in $d=2+\epsilon$ dimensions. For sufficiently small ϵ, both the saddle-point expansion inherent in the nonlinear σ model (20) and the approximation to the coupling constant (15c) remain asymptotically exact near the mobility edge. The difficulty of achieving a sufficiently short mean free path in the presence of a correlation length to the disorder is peculiar to three dimensions, since, for $\epsilon \ll 1$, the mobility edge occurs for wavelengths that are large compared to the sphere diameter.

In an actual experiment, effects due to nonspherical particle geometry and aggregation of metal particles can modify the unrenormalized values of both the real and imaginary part of the average dielectric constant (34b). This has been suggested[17] as a possible cause for the anomalous infrared absorption in metal smokes reported by numerous authors[5] at frequencies far below an electromagnetic mobility edge. In these experiments, the elastic mean free path is very long compared to the inelastic distance and, for that matter, the size of the sample L. In this limit, scattering may be neglected and the absorption coefficient is adequately given by $2l_{\rm inel}^{-1}$.

The mobility-edge behavior described in this paper is observable only in the limit $l \ll l_{\rm inel}$. Physically, the near divergence of the absorption coefficient (30) occurs because of the extremely slow diffusive propagation of energy in the disordered medium. However, only those wave-interference effects which occur on a length scale that is short compared to $l_{\rm inel}$ may contribute to a renormalization of the diffusion coefficient, since inelastic effects destroy the necessary phase coherence. This leads to a rounding off of the absorption peak in a small frequency range of the mobility edge, as discussed earlier. The residual diffusivity at the mobility edge caused by a finite $l_{\rm inel}$ scales as $D(\omega^*) \sim 1/(g^* l_{\rm inel}^{d-2}) \sim \epsilon_2^{d-2}$. Since $\alpha \sim \sqrt{\epsilon_2/D}$, this suggests that the peak value of the absorption coefficient should accordingly scale as $\alpha(\omega^*) \sim \epsilon_2^{(3-d)/2}$.

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APPENDIX: EVALUATION OF INTEGRALS $C_0(\omega)$ AND $C_2(\omega)$

The coefficient

$$C_0(\omega) = 1 - \frac{\gamma^2\omega^4}{2} \int \frac{d^dp}{(2\pi)^d} |G^+(p)|^2 \tag{A1}$$

may be evaluated using the CPA equations (11) by noting that

$$\int \frac{d^dp}{(2\pi)^d} |G^+(p)|^2 = \frac{2\,{\rm Im}Q^+/\gamma\omega^2}{2\omega\eta + \gamma\omega^2{\rm Im}Q^+} . \tag{A2}$$

It follows that, to leading order in η,

$$C_0(\omega) \simeq \frac{2\omega\eta}{\gamma\omega^2{\rm Im}Q^+} \quad (\eta \to 0) . \tag{A3}$$

In three dimensions,

$$C_2(\omega) = \frac{\gamma^2\omega^4}{2} \int \frac{d^3p}{(2\pi)^3} \frac{\partial G^+(p)}{\partial p} \frac{\partial G^-(p)}{\partial p} \tag{A4}$$

may be rewritten using the CPA equation for $\gamma\,{\rm Re}Q^+ \ll 1$ as

$$C_2(\omega) = (\gamma^2\omega/\pi^2) I(q_0) , \tag{A5a}$$

where

$$I(q_0) \equiv \int_0^\infty dq \frac{q^4}{[(1-q^2)^2 + q_0^4]^2} \tag{A5b}$$

and

$$q_0 \equiv (\Lambda\omega)^{-1} . \tag{A5c}$$

Here, Λ is the phase coherence length (12). The integral $I(q_0)$ may be evaluated by contour integration. Closing the contour in the upper half complex q plane encloses two double poles at $(1+iq_0^2)^{1/2}$ and $-(1-iq_0^2)^{1/2}$. Evaluating the residues yields

$$I(q_0) = \frac{\pi}{8q_0^4} \mathrm{Im} \left[(1+iq_0^2)^{1/2} \left[1 + \frac{2i}{q_0^2} \right] \right] . \tag{A6a}$$

For a phase coherence length that is long compared to the photon wavelength ($q_0 \ll 1$), this may be approximated by

$$I(q_0) \simeq \pi / 4q_0^6 . \tag{A6b}$$

Using the definition of Λ and Eqs. (16a) and (16b), it follows that

$$C_2(\omega) \simeq 2\omega^2 \Lambda^4 \quad (d=3) . \tag{A7}$$

In $d=1$ for $\gamma\,\mathrm{Re}Q^+ \ll 1$,

$$C(p) = 1 - \frac{\gamma^2\omega^4}{2} J(p) , \tag{A8a}$$

where

$$J(p) = \int_{-\infty}^{\infty} \frac{dq}{2\pi} \frac{1}{\omega^2(1+iq_0^2)+(q+p/2)^2} \times \frac{1}{\omega^2(1-iq_0^2)+(q-p/2)^2} . \tag{A8b}$$

Closing the contour in the upper half complex q plane encloses simple poles at $-p/2+\omega(1+iq_0^2)^{1/2}$ and $p/2-\omega(1-iq_0^2)^{1/2}$. Evaluating the residues in the limit $q_0^2 \ll 1$ yields

$$J(p) \simeq \frac{q_0^2}{2\omega(p^2+\omega^2 q_0^4)} . \tag{A9}$$

Using the fact that for $d=1$, $q_0^2 = \gamma^2\omega/4$, it follows that

$$C(p) \simeq \frac{p^2}{p^2 + l^{-2}} , \tag{A10}$$

where l is the photon elastic mean free path.

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