

# Height of the Burton Tower

## Sample Formal Report

by Julien Rioux

Students were asked to measure the height of the physics building. A sample of measurements was used to obtain the best estimate for the true value. The height is estimated at  $h = 59 \pm 4$  meters with respect to the patio. The error is dominated by statistical fluctuations and could be reduced by a larger data sample.

### Introduction

The McLennan Physical Laboratories is the name of the building complex holding the physics department at the University of Toronto. Part of this structure erected in 1967 is the Burton tower. Its 14 stories is home of research across an extremely wide spectrum, from exploration geophysics to theoretical cosmology, from high temperature superconductivity to particle physics.

The true value of the height of the Burton tower can be best estimated from repetitive measurements. From a normally distributed sample of data, the average value and the standard deviation are first determined by the formulae found in the error analysis package [1].

$$\bar{x}_{est} = \frac{\sum_{i=1}^n x_i}{n} \quad (1)$$

$$\sigma_{est}^2 = \frac{\sum_{i=1}^n (x_i - \bar{x}_{est})^2}{n - 1} \quad (2)$$

However, given that each measurement  $x_i$  has a different error estimate  $\Delta x_i$ , a weighted average is more suitable to determine the true value in this case. The weight on each data point is chosen to be

$$w_i = \frac{1}{\Delta x_i^2}, \quad (3)$$

so that more precise measurements take more importance in the calculation of the average. The following formula for the weighted average (see [Appendix A](#)) will therefore replace [Equation 1](#). [2]

$$\bar{x}_{est} = \frac{\sum_{i=1}^n \frac{x_i}{\Delta x_i^2}}{\sum_{i=1}^n \frac{1}{\Delta x_i^2}} \quad (4)$$

### Experimental Method

The height of the Burton tower was measured by students in the first year physics lab in groups P0301 3D and P0302 3S. The measurements were compiled to determine the

best estimate for the actual height of the building. Most of the students used a protractor at a measured distance, making it possible to calculate the height by trigonometry. To have a statistically significant data sample, only the measurements done using this method were included in this analysis.

All measurements of the height are with respect to the patio of the building.

A spreadsheet software was used to compute averages and standard deviations. The document is shown in [Appendix B](#).

## Results

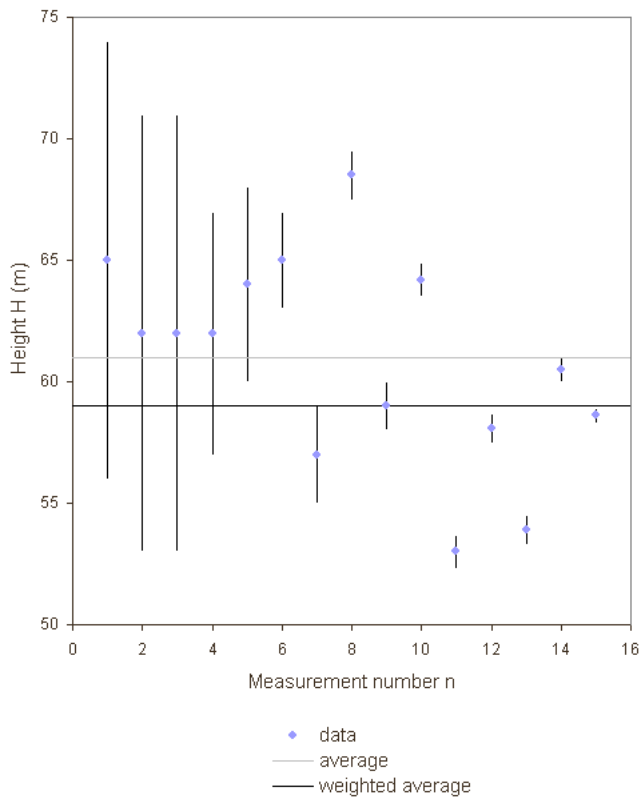


Figure 1: Measured height with errors.

The data sample is shown in [Figure 1](#). There are 15 data points ranging from 53 to 68.5 meters with uncertainties ranging from  $\pm 0.3$  to  $\pm 9$  m.

The distribution of data is shown as a histogram in [Figure 2](#). The number of measurements falling within each bin is counted and divided by the total number  $n$  to obtain the probability.

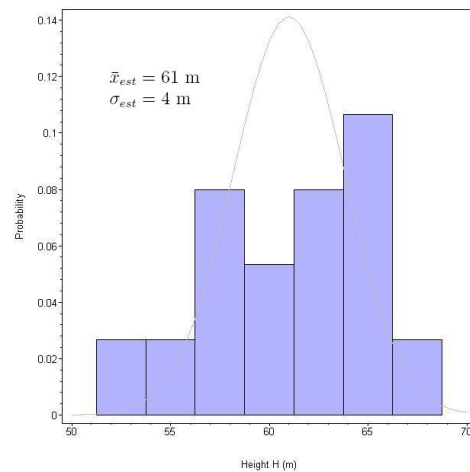


Figure 2: Data distribution, no weights.

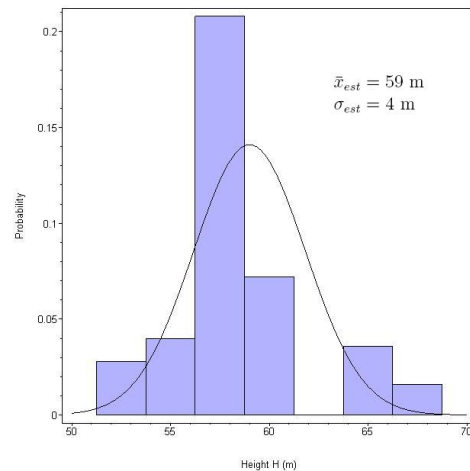


Figure 3: Weighted normal distribution.

Using Equation 1 and Equation 2, the best value for the average height and its standard deviation are estimated to be  $\bar{x}_{est} = 61$  meters and  $\sigma_{est} = 4$  meters. These parameters are used to draw the corresponding Gaussian distribution over the data.

Another histogram of the data is shown in Figure 3. This time each measurement is given a weight taken from Equation 3. Using Equation 4 and again Equation 2 we find best estimates  $\bar{x}_{est} = 59$  meters and  $\sigma_{est} = 4$  meters.

## Discussion

Purposely, the data in Figure 1 is displayed in order of decreasing error bars. This is to show that more precise measurements should have more value in determining the average.

Among the data with the biggest uncertainties, we see that most of them lie above the average shown by the horizontal grey line. Clearly, they contribute to raising the average. Giving less importance to these data points should decrease the best estimate  $\bar{x}_{est}$ .

This reasoning is confirmed when we actually compute the average of the data weighted by the inverse square of the error. The weighting procedure effectively pick the most significant measurements by changing the contribution of each one to the average. The result is an weighted average which is 2 meters lower than the previous best estimate, or  $-3.3\%$ . The weighting method does not have a significant effect on the value of the standard deviation.

## Conclusion

Using our sample data and weighting statistics, the best estimate for the value of the mean and standard deviation of a parent distribution was determined. The height of the Burton tower as measured with respect to the patio is  $h = 59 \pm 4$  meters.

The result is an impressive demonstration that initiative and inventiveness can go a long way in science. Using for the most part homemade instruments, in windy conditions, students nevertheless accomplished the required task.

It also shows that a good error analysis using weights is important for measurements where errors are uncorrelated.

Since the result is dominated by statistical errors, we can improve it with a larger number of measurements. This could easily be done given that students in the first year physics lab but outside the two groups surveyed also performed the same experiment. However, following our analysis, we realize the importance of precise measurements. Only the data with small uncertainties contribute significantly to the weighted average.

## References

1. Error Analysis in Experimental Physical Science  
<http://www.upscale.utoronto.ca/PVB/Harrison/ErrorAnalysis/>
2. John R. Taylor, *An Introduction to Error Analysis : The Study of Uncertainties in Physical Measurements*, 2nd ed., Univ. Science Books, 1997, QA278.T39

## Appendix A: Weighted Average & Normal Distribution

A weighted average of discrete values  $x_i$  with weight  $w_i$  is given by

$$\boxed{\bar{x}_{est} = \frac{\sum_{i=1}^n w_i x_i}{\sum_{i=1}^n w_i}}. \quad (\text{A.1})$$

For measured quantities the weight of  $x_i$  is given by the inverse square of its uncertainty,

$$\boxed{w_i = \frac{1}{\Delta x_i^2}}. \quad (\text{A.2})$$

Note that for a constant weight  $w$  which is the same for every  $i$  (the same error on every measurement), the above reduces to the usual formula to compute the average.

$$\bar{x}_{est} = \frac{\sum_{i=1}^n w x_i}{\sum_{i=1}^n w} = \frac{w \sum_{i=1}^n x_i}{w \sum_{i=1}^n 1} \quad (\text{A.3})$$

$$\boxed{\bar{x}_{est} = \frac{\sum_{i=1}^n x_i}{n}} \quad (\text{A.4})$$

The uncertainty in determining  $\bar{x}_{est}$  is calculated by standard error propagation. Since [Equation A.1](#) involves a sum of measured quantities, the error on the determination of the weighted average is given by

$$\begin{aligned} \Delta \bar{x}_{est}^2 &= \sum_{i=1}^n \left( \Delta \left( \frac{w_i x_i}{\sum_{i=1}^n w_i} \right) \right)^2 \\ &= \sum_{i=1}^n \left( \frac{w_i \Delta x_i}{\sum_{i=1}^n w_i} \right)^2 \\ &= \frac{\sum_{i=1}^n w_i^2 \Delta x_i^2}{\left( \sum_{i=1}^n w_i \right)^2}. \end{aligned} \quad (\text{A.5})$$

If we inverse [Equation A.2](#) we can replace  $\Delta x_i^2$  by  $\frac{1}{w_i}$  and find

$$\Delta \bar{x}_{est}^2 = \frac{\sum_{i=1}^n w_i}{\left( \sum_{i=1}^n w_i \right)^2} = \frac{1}{\sum_{i=1}^n w_i} \quad (\text{A.6})$$

or

$$\boxed{\Delta \bar{x}_{est} = \frac{1}{\sqrt{\sum_{i=1}^n w_i}}}. \quad (\text{A.7})$$

Again, if we have a constant weight  $w$  equal to  $\frac{1}{\Delta x^2}$  (the same error on every measurement), then from [Equation A.6](#) we find the formula for the standard error of the mean.

$$\Delta \bar{x}_{est}^2 = \frac{1}{\sum_{i=1}^n w} = \frac{1}{w \sum_{i=1}^n 1} = \frac{1}{wn} = \frac{\Delta x^2}{n} \quad (\text{A.8})$$

or

$$\Delta \bar{x}_{est} = \frac{\Delta x}{\sqrt{n}}. \quad (\text{A.9})$$

Whether we use a standard or a weighted average to find our best estimate  $\bar{x}_{est}$  to the true value  $x$ , the underlying statistics are the same and we still expect a normal bell-shaped distribution of the data (Figure A.1). The best estimate for the standard deviation is given by the usual formula in both cases.

$$\sigma_{est}^2 = \frac{\sum_{i=1}^n (x_i - \bar{x}_{est})^2}{n-1} \quad (\text{A.10})$$

or

$$\sigma_{est} = \sqrt{\frac{\sum_{i=1}^n (x_i - \bar{x}_{est})^2}{n-1}} \quad (\text{A.11})$$

The final physical result is always quoted as  $x = \bar{x}_{est} \pm \sigma_{est}$ .

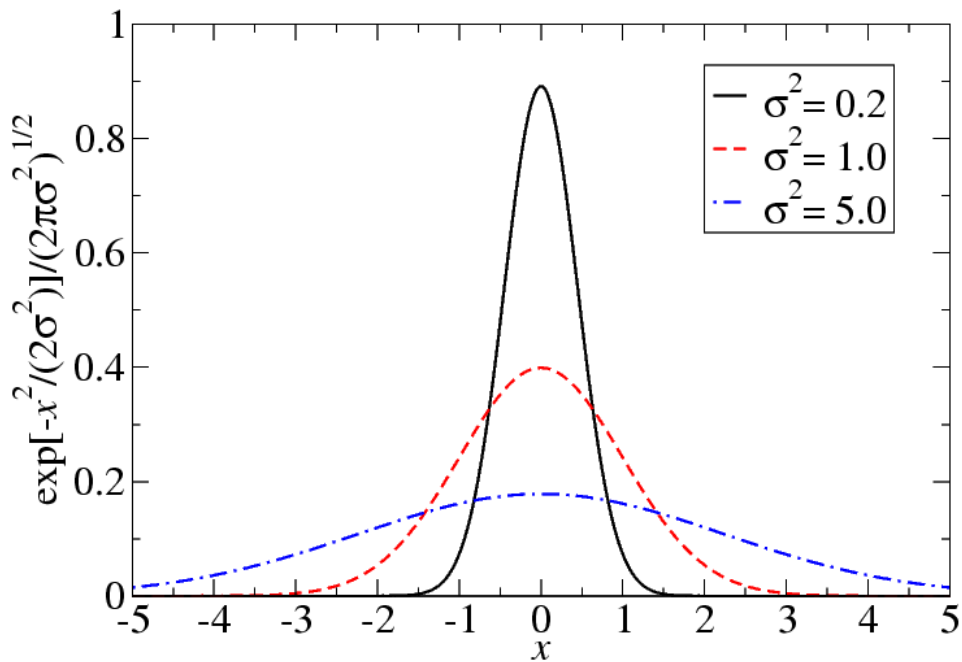


Figure A.1: Normal or Gaussian distribution. [source : wikipedia.org]

## Appendix B: Sample Calculations

	A	B	C	D	E	F	G	H
1	height (m)	error (m)		$x/dx^2$	$1/dx^2$	$(x-\bar{x})^2$		
2	65	9		0.80	0.01	38.52		
3	62	9		0.77	0.01	10.28		
4	62	9		0.77	0.01	10.28		
5	62	5		2.48	0.04	10.28		
6	64	4		4.00	0.06	27.11		
7	65	2		16.25	0.25	38.52		
8	57	2		14.25	0.25	3.22		
9	68.5	1.0		68.50	1.00	94.22		
10	59	1		59.00	1.00	0.04		
11	64.2	0.7		131.02	2.04	29.23		
12	53.0	0.7		108.16	2.04	33.56		
13	58.1	0.6		161.39	2.78	0.48		
14	53.9	0.6		149.72	2.78	23.94		
15	60.5	0.5		242.00	4.00	2.91		
16	58.6	0.3		651.11	11.11	0.04		
17								
18				sum $x/dx^2$	sum $1/dx^2$	sum $(x-\bar{x})^2$		
19				1610.22	27.39	322.66		
20								
21	average	60.85		weighted average		58.79		
22	std dev	4.30		std dev		4.49		

Figure B.1: Sample spreadsheet used to calculate averages.

	A	B	C	D	E	F
1	height (m)	error (m)		$x/dx^2$	$1/dx^2$	$(x-\bar{x})^2$
2	65	9		=A2/B2^2	=1/B2^2	=(A2-\$F\$21)^2
3	62	9		=A3/B3^2	=1/B3^2	=(A3-\$F\$21)^2
4	62	9		=A4/B4^2	=1/B4^2	=(A4-\$F\$21)^2
5	62	5		=A5/B5^2	=1/B5^2	=(A5-\$F\$21)^2
6	64	4		=A6/B6^2	=1/B6^2	=(A6-\$F\$21)^2
7	65	2		=A7/B7^2	=1/B7^2	=(A7-\$F\$21)^2
8	57	2		=A8/B8^2	=1/B8^2	=(A8-\$F\$21)^2
9	68.5	1		=A9/B9^2	=1/B9^2	=(A9-\$F\$21)^2
10	59	1		=A10/B10^2	=1/B10^2	=(A10-\$F\$21)^2
11	64.2	0.7		=A11/B11^2	=1/B11^2	=(A11-\$F\$21)^2
12	53	0.7		=A12/B12^2	=1/B12^2	=(A12-\$F\$21)^2
13	58.1	0.6		=A13/B13^2	=1/B13^2	=(A13-\$F\$21)^2
14	53.9	0.6		=A14/B14^2	=1/B14^2	=(A14-\$F\$21)^2
15	60.5	0.5		=A15/B15^2	=1/B15^2	=(A15-\$F\$21)^2
16	58.6	0.3		=A16/B16^2	=1/B16^2	=(A16-\$F\$21)^2
17						
18				sum $x/dx^2$	sum $1/dx^2$	sum $(x-\bar{x})^2$
19				=SUM(D2:D16)	=SUM(E2:E16)	=SUM(F2:F16)
20						
21	average	=AVERAGE(A2:A16)		weighted average		=D19/E19
22	std dev	=STDEV(A2:A16)		std dev		=SQRT(F19/16)

Figure B.2: The same spreadsheet showing the formulae.