Problem Set 5 Solutions

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1. ISOTOPIC DILUTION.

A) Mass of Exchangeable Calcium.

Model.

Here are all the things we are given in this problem:

$R(0) = 3.7 \times 10^4$ Bq : activity of the initial ^{47}Ca sample.

$\tau_n = 108$ hours: nuclear half-life of ^{47}Ca

M = 60 kg: mass of the body.

$R_{urine} = 0.56 \times 10^4$ Bq: Total activity of urine sampled by t = 20 hours.

$m_{urine} = 0.90$mg = 900 µg: mass of the stable Ca found in the urine sample.

$R_{blood} = 6.0$ Bq: activity of ^{47}Ca found in the blood sampled at t = 20 hours.

T = the time that the sample of blood is taken.

Assuming complete mixing, the ratio of the activity of ^{47}Ca in the sample of blood (R_{blood}) to the mass of stable calcium in the sample (m_{blood}), will be the same as the ratio of the activity of ^{47}Ca in the body (R_{body}) to the mass of stable calcium in the body (m_{body}) as a whole. This allows us to find the mass of stable calcium in the body IF we know the activity in the body at the time the sample was taken (T = 20 hours). But the required activity in the body is just the total activity of all the remaining ^{47}Ca MINUS the activity in the excreted urine, all measured at T=20. If you believe this, you will not need the hints, which are intended to prove this assertion. Let's use them anyway, to check that we fully understand what is going on.

Solve.

We begin by writing an expression for the total ^{47}Ca in the body and urine at time t:

$$N_{\text{total}}(t) = N_{\text{body}}(t) + N_{\text{urine}}(t)$$

The total number of ^{47}Ca decays by nuclear decay, i.e.

$$N_{\text{total}}(t) = N(0)e^{-\lambda_n t},$$ where $\lambda_n = \ln 2/\tau_n$

Now combine the above two equations to find an expression for the *activity* in the body at time t. Multiply both sides by λ_n, use the fact that $R(t) = \lambda_n N(t)$, and rearrange:

$$R_{body}(t) = R(0)e^{-\lambda_n t} - R_{urine}(t).$$ $R_{urine}(t)$ is known at t=T=20 hours.

This proves the assertion in the Model section above.

WITH CORRECT LOGIC AS SHOWN,
OR A REASONABLE EXPLANATION,
IF THE EQUATION IS GUESSED AT
BY OTHER MEANS,
THIS EQUATION GETS 10 MARKS

We then use the ratio, as described above: $\frac{\mathrm{m}_{\text{body}}}{\mathrm{m}_{\text{blood}}} = \frac{\mathrm{R}_{\text{body}}}{\mathrm{R}_{\text{blood}}}$, where the activities are calculated at t =T =20 hours. m_{body} and $\mathrm{m}_{\text{blood}}$ are, respectively, the mass of *stable* Ca in the body and the blood sample at t = T=20hours:

$$m_{body} = \frac{m_{blood} R_{body}}{R_{blood}} = \frac{m_{blood}}{R_{blood}} \left[R(0) e^{-\lambda_n t} - R_{urine}(T) \right] = \frac{900\mu g}{6Bq} \left[3.7 \times 10^4 e^{-\frac{20 \ln 2}{108}} - 0.56 \times 10^4 \right]$$

$$= \frac{900\mu g}{6Bq} \left[2.69 \times 10^4 \right] = 4040 mg = 4.0 g$$

THIS DERIVATION IS WORTH 10 MARKS

B) Long-term risk.

Model.

To understand the long-term risk of this radiation, we need to compute the total dose to the body between t = 0 and t = T, allowing for the decay and excretion of the radioactive Calcium. Dose is defined as the total energy deposited per unit mass of the organ (or, in this case, body) where it is deposited.

SNIV gives an expression for the committed dose: $D_m(0 \rightarrow \infty) = e_n R(0) / (\lambda_{\text{eff}} M)$ where e_n is the energy per decay.

REALIZING THIS IS WORTH 4 MARKS

THERE ARE TWO WAYS OF PROCEEDING FROM HERE.
THE REALIZATION THAT λ_{eff} CAN BE CALCULATED FROM THE GIVEN DATA IS WORTH 6 MARKS. THE SUCCESSFUL DERIVATION BASED ON THIS REALIZATION IS WORTH ANOTHER 10 MARKS.

λ_{eff} can be calculated from the equation $R_{\text{body}}(t) = R(0)\exp(-\lambda_{\text{eff}}\, t)$, since $R_{\text{body}}(T)$ has been calculated above (= 2.69 x 10^4 s^{-1}).

Solve

$$\lambda_{\text{eff}} = -\frac{1}{\mathrm{T}} \ln \frac{\mathrm{R}_{\text{body}}(\mathrm{T})}{\mathrm{R}(0)}$$

$$\begin{aligned}
D_m(0 \rightarrow \infty) &= \frac{e_n R(0)}{\lambda_{eff} M} = \frac{e_n R(0) T}{M \ln \left[R(0)/R_{\text{body}}(T) \right]} \\
&= \frac{(1.1\text{MeV})(3.7 \times 10^4 \text{Bq})(20\text{h})(60^2 \text{s/h})}{60\text{kg} \ln \left[3.7 \times 10^4 \text{Bq} / 2.69 \times 10^4 \text{Bq} \right]} \\
&= 1.5 \times 10^8 \frac{\text{MeV}}{\text{kg}} = 2.4 \times 10^{-5} \frac{\text{J}}{\text{kg}} = 2.4 \times 10^{-5}\ \text{Gy}
\end{aligned}$$

ALTERNATIVELY, IF IT IS NOT REALIZED THAT λ_{eff} CAN BE CALCULATED, IT WOULD BE REASONABLE TO ESTIMATE THE MAXIMUM COMMITTED DOSE BY IGNORING THE BIOLOGICAL EXCRETION, SO THAT λ_{eff} IS ESTIMATED BY λ_n . THIS IS REASONABLE, SO IS WORTH 3 MARKS. A CORRECT CALCULATION BASED ON THIS ASSUMPTION IS WORTH AN ADDITIONAL 7 MARKS.

$$\begin{aligned} D_m(0 \rightarrow \infty) &= \frac{e_n R(0)}{\lambda_n M} = \frac{e_n R(0) \tau_n}{(\ln 2) M} \\ &= \frac{(1.1\text{MeV})(3.7 \times 10^4 \text{Bq})(108\text{h})(60^2\text{s/h})}{(\ln 2)(60\text{kg})} \\ &= 3.8 \times 10^8 \frac{\text{MeV}}{\text{kg}} = 6.0 \times 10^{-5} \frac{\text{J}}{\text{kg}} = 6.0 \times 10^{-5} \text{ Gy} \end{aligned}$$

Consult SNIV for what this means for the body. Since this dose is spread out over the whole body, we have a tissue weighting factor of 1. Since we are told that the decay energy happens mainly in terms of beta and gammas, we also have a radiation factor of 1. Therfore, the equivalent and effective Dose is simply **2.4×10^{-5} Sv**. Since high dosages are defined around above 1 Sv, it looks like this dose is not a big deal.

2. BIOLOGICAL UPTAKE.

Model. In this problem we want to see how our mathematics change when we consider a previously-neglected factor: the fact that it takes an organ some time to take up the injected radioisotope.

First, gather the given information:
$N_0(t)$: the number of radioactive nuclei in the organ, in time.
$N(0)$: the number of radioactive nuclei injected into the body at time t = 0.
$R(0)$: the initial activity of the injected sample (t = 0).
λ_n = nuclear decay constant of the injected isotope.
λ_b = biological decay constant for the injected isotope and organ at hand.

Solve

A) Simple! $N(0) = R(0)/\lambda_n$.
B) For instantanous uptake, the number of isotopes in the organ in time is simply given by the initial amount ($N_0(0) = N(0)$) times the decay functions for nuclear and biological decay:

$$N_0(t) = N(0)e^{-\lambda_n t}e^{-\lambda_{be} t} = N(0)e^{-\lambda_{eff} t}$$

C) How do we model biological uptake? The problem tells us to assume that biological uptake is exponential, while no nuclear or biological excretion happen. Since no biological uptake can have happened at t = 0, we know that

$$N_0(0) = 0$$

At infinite time (and assuming no nuclear or biological excretion) the total injected isotope will have been taken up by the organ. Therefore,

$$N_0(\infty) = N(0)$$

If uptake is exponential, that means that the amount of isotope not yet in the organ is decreasing at an exponential rate. Mathematically, it looks like this:

$$N_0(t) = N(0)\left[1 - e^{-\lambda_{bu} t}\right]$$

It is easy to verify that this function satisfies our two requirements.

D) Now combine the answers to parts (B)-(C) to find an expression for when all three mechanisms are active. The answer to (C) gives us the number of isotopes in the organ building up in time, while the answer to (B) tells us that this number is decaying in time. The effect of decay is that the exponential function multiplies the number of isotopes present, so the net effect of the decay/uptake mechanisms has to be multiplicative. Combining our answers, then, we have

$$N_0(t) = N(0)\left[1 - e^{-\lambda_{bu} t}\right] e^{-\lambda_{eff} t}$$

E) To find Dose rate, remember that *dose* is defined as energy deposited by the radiation per unit mass of the organ, whereas *activity* is the rate of decay in time. Therefore, we can write dose rate as:

$$\dot{D}_m(t) = \frac{\text{energy}}{\text{mass}}\frac{1}{\text{time}} = \frac{\text{energy}}{\text{decay}}\frac{\text{decays}}{\text{time}}\frac{1}{M} = e_n R(t)\frac{1}{M}$$

We can subsitute for the activity, knowing that activity is proportional to the number of nuclei present at a given time:

$$R(t) = \lambda_n N_0(t) = R(0)\left[1 - e^{-\lambda_{bu} t}\right] e^{-\lambda_{eff} t}$$

This gives:

$$\dot{D}_m(t) = \frac{e_n R(0)}{M}\left[1 - e^{-\lambda_{bu} t}\right] e^{-\lambda_{eff} t}$$

F) To find the total comitted dose, integrate from t = 0 to t = ∞:

$$\begin{aligned}
\text{Committed Dose} &= \frac{e_n R(0)}{M}\int \left[1 - e^{-\lambda_{bu} t}\right] e^{-\lambda_{eff} t} dt \\
&= \frac{e_n R(0)}{M}\int \left[e^{-\lambda_{eff} t} - e^{-(\lambda_{bu} + \lambda_{eff})t}\right] dt \\
&= \frac{e_n R(0)}{M}\left[\frac{1}{\lambda_{eff}} - \frac{1}{\lambda_{bu} + \lambda_{eff}}\right] \\
&= \frac{e_n R(0)}{M}\left[\frac{\lambda_{bu}}{\lambda_{eff}(\lambda_{bu} + \lambda eff)}\right]
\end{aligned}$$

F) Does this answer make sense in the limits? First, take the no-uptake case,

$$\lambda_{bu} = 0$$

In this limit, the committed dose becomes zero. This makes sense; if the organ doesn't take up any of the radioactive material, there's no dose.

Now consider the limit of instantanous uptake, or

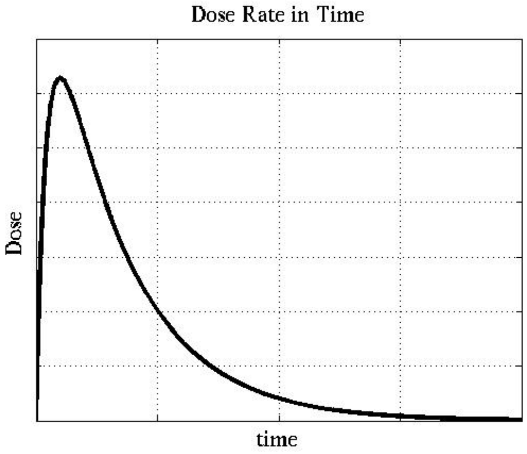
$$\lambda_{bu} = \infty$$

In this limit, the committed dose becomes

$$\text{Dose} = \frac{e_n R(0)}{M}\left[\frac{1}{\lambda_{eff}}\right] = 1.44 e_n R(0)\frac{\tau_{eff}}{M}$$

which agrees with the value quoted in SN 5.4.

G) Here's a sketch of dose rate as a function of time. Dose rate is directly proportional to activity, which we know starts at zero, then increases as the organ takes up the radioactive isotopes, and then decays again as biological excretion and nuclear decay take over. We would thus expect an initial increase which slows down slightly faster than exponentially (because decay and excretion are also working), followed by a decrease to zero. The area under this curve up to a given time is the total dose up to that time.



Model

We can do some research on average values of doses. Cosmic rays consist of lots of particles (x-rays, protons, alpha particles, pions, muons, etc) of different sizes. Let's be conservative and assume that the majority of this radiation is non-heavy stuff like beta particles and gamma rays, so that we have a radiation weighting factor of 1.

A) *Average Annual Dose – USA Today..* It's pretty tough to see where numbers such as this, quoted in newspapers, might come from. I thought I'd verify this one with the great democratizer, Wikipedia. Wiki tells me that the worldwide average dose for a human being, due to cosmic and terrestrial radiation, is about 2.4 mSv per year. That would give us an abosorbed dose of 2.4 mGy per year.

Wiki also tells me that medical exposure is somewhere between 0.4 and 1 mSv per year. Let's assume that Americans (Canadians) have more access to healthcare than most of the world's population (a discussion for another r day!) and so err on the side of more radiation. Since medical use is pretty much x-rays, we have a weighting factor of $W_R = 1$, for a whole body dose ($W_T = 1$) which gives us a dose of 1mGy per year.

So we end up with roughly 2.4+1 = 3.4 mGy per year. In a liftetime of 80 years, that's 272 mGy = .272 Gy per year.

SNIV claims that the average is 3 to 4 mSv per year. Given the variability in these figures across the US, all these numbers are in agreement.

So the *USA Today* estimate seems to be about right.

B) *Globe and Mail – comparison of Polonium and Radium.*

Model

Most of the questions involve the activity per gram, so let's calculate that first.
We want to convert find how much mass produces an activity of 1 mCi of Polonium. To do that, use the following relationship between molar mass *M*, activity *R*, Avogadro's number N_A, and the number of nuclei *N*:

$$m = \frac{N}{N_A}M = \frac{RM}{\lambda N_A} = \frac{RM\tau}{\ln 2 N_A}$$

Solve (1)

We also need to convert the given activity to Becquerel, since this is decays per second:

$$1 \times 10^{-3}\text{Ci} \times \frac{3.7 \times 10^{10}\text{Bq}}{\text{Ci}} = 3.7 \times 10^{7}\text{Bq}$$

Convert the half-life of Polonium-210 from days to seconds:

$t = 138 \times 24 \times 60^2 = 1.2 \times 10^7$ s.

Now solve for mass: 1 mCi is produced by a mass of

$$m = \frac{(3.7 \times 10^7\text{Bq})(210\text{g})(1.2 \times 10^7 s)}{(\ln 2)(6.02 \times 10^{23})} = 223 \times 10^{-9}\text{g} \simeq 0.2 \times 10^{-6}\text{g}$$

OR, the activity of 1 g is $\mathbf{(10^{-3}/(0.2\times10^{-6}) = 5{,}000\,Ci}$.

The *Globe and Mail* claims that Po-210 is 5 thousand times as radioactive per gram as radium. By definition, 1 Curie is the activity of 1 g of Radium (thank you Mme. Curie!)… So the *Globe and Mail* statement is correct.

OR, here's another way to show that the *Globe and Mail* is about right.

Solve (2)

From the above, 1 g of Polonium-210 has an activity of 5,000 Ci.

Note that the actvity and mass of a substance are both proportional to the number of nuclei:

$$R = \lambda_n N$$
$$m = \frac{N}{N_A}M$$

where *M* is the molar mass and N_A is Avogadro's number. Thus:

$$\frac{R}{m} = \frac{\lambda_n N N_A}{NM} = \frac{N_A \ln 2}{M\tau_n} \equiv \alpha$$

where we've abbreviated the activity-mass ratio as α.

Now compare these values for the two elements. For radium, let's proceed with the most abundant radioactive isotope, which (look it up!) is ^{226}Ra, which has a half-life of 1602 years. In this case, we have

$$\begin{aligned} \frac{\alpha_P}{\alpha_R} &= \frac{N_A \ln 2}{M_P \tau_P} \cdot \frac{M_R \tau_R}{N_A \ln 2} = \frac{M_R \tau_R}{M_P \tau_P} \\ &= \frac{226 \cdot 1602\mathrm{y}}{210 \cdot 138\mathrm{d} \cdot 365^{-1}\mathrm{y/d}} = 4,560 \simeq 5,000 \end{aligned}$$

C) *Globe and Mail – mass of 1 mCi*

This has been calculated in part B) above; since 5000 Ci correspond to 1 g, 1 mCi corresponds to (1/5000)x10^{-3} g = 0.2x10^{-6} g. The *Globe and Mail* is accurate; 1 mCi has a mass of about 0.2 millionths of a gram.

D) *3 mCi of ^{210}Po would be sufficient to kill* (*Health Physics Society*)

Model. To find the damage caused by an injection of Po-210, we return to the idea of *committed dose*, the dose rate integrated over time. As quoted in SNV, the committed dose is given by: $\boldsymbol{D_m(0\rightarrow\infty) = 1.44\, e_n\, R(0)\, T_{eff}/m}$, where T_{eff} is the effective half-life, here about 36.7 days.

Solve.
Let's assume a typical body mass of 60 kg. This gives, for the total committed dose,

$$\begin{aligned} \text{Dose} &= \frac{1.44(5.41\text{MeV})(3 \times 3.7 \times 10^7\text{Bq})(3.2 \times 10^6\text{s})}{60\text{kg}} \\ &= 4.53 \times 10^{19}\text{eV/kg} = 7.70\text{J/kg} = 7.7\text{Gy} \end{aligned}$$

If we take the radiation weighting factor into account a value of 10 – 20 is appropriate. The most conservative value, $W_R = 10$, for α particles, yields a total effective dose of 77 Sv. Doses over 2 Sv can be deadly, so this is more than enough to kill (is this problem set making you depressed yet?). So the Health Physics Society is certainly correct – 3mCi would do the job! – but in fact, you'd need much less, so <u>their calculation is suspicious</u>.

E) *Fatal Dose – Wikipedia*. Finally, Wikipedia claims that a fatal dose can be caused by ingesting 8 MBq of Polonium-210, or (they claim), 50 ng.

The above equation showed that a dose of about 120 MBq (3 mCi) yields a dose of about 77 Sv. 8 MBq is about one-fifteenth of that, so it would yield a dose of about 5 Sv. This is similar to the estimate in SNIV of a fatal dose (see, e.g. the figure in 4.5.1), so it appears that Wikipedia's estimate of a <u>minimum</u> fatal dose is better than the one claimed by the Health Physics Society. It is possible that some of the difference between the two estimates comes from the use of different weighting factors, though this would still leave a discrepancy of a factor of just less than 10.

Let's check their claim for 50 ng - what activity does this mass yield?

$$\begin{aligned} R &= \frac{(\ln 2)m \cdot N_A}{\tau \cdot 210} = \frac{(\ln 2)(50 \times 10^{-9}\text{g})(6.02 \times 10^{23})}{(1.2 \times 10^7\text{s})210} \\ &= 8.28 \times 10^6\text{Bq} \end{aligned}$$

So indeed 50 ng yields the 8 MBq claimed by Wikipedia.