PHY138Y – N&R, Problem Set #5, Written Assignment. Solutions
(part marks shown in curly brackets, total mark in square brackets).

A. The number of ^{238}U and ^{235}U atoms remaining after time T, where λ_1 and λ_2 are the decay constants for ^{238}U and ^{235}U respectively is given by:

$$N(^{238}U) = N(0)e^{-\lambda_1 T}$$

$$N(^{235}U) = N(0)e^{-\lambda_2 T}$$

Take the ratio of these values:

$$N(^{238}U)/ N(^{235}U) = N(0)e^{-\lambda 1*T} / N(0)e^{-\lambda 2*T}$$

.. and solve for T, noting that N(0) is the same for both ^{238}U and ^{235}U, because we had the same amount of ^{238}U and ^{235}U at the supernova explosion: **{10}**

$$N(^{238}U)/ N(^{235}U) = e^{-\lambda_1 T + \lambda_2 T}$$

$$\ln(N(^{238}U)/ N(^{235}U) = -\lambda_1 T + \lambda_2 T = (-\lambda_1 + \lambda_2)T$$

$$T = \ln(N(^{238}U)/ N(^{235}U))/ (-\lambda_1 + \lambda_2)$$ **{5}**

Convert from decay constants to the half-lives given on the wiki Decay_chain page by the relationship $\lambda_1 = \ln(2)/t_1$ and $\lambda 2 = \ln(2)/t_2$, where T_1 and T_2 are the half-lives.

$$T = \ln(N(^{238}U)/ N(^{235}U)) / [\ln 2 * (-1/T_1 + 1/T_2)]$$

$$T = \ln(99.28 / 0.72) / [\ln 2 * (- 1/4.47*10^9 y + 1/7.04*10^8 y)]$$ **{2}**

[17]

Answer: $\underline{T = 5.9*10^9\ y}$

B. What is the kinetic energy of the emitted alpha?
This problem is solved in the Worked Examples, so an excellent solution is expected!

The decay chain for ^{238}U to ^{234}Th involves the release of an alpha particle.

$$Q/c^2 = m_p - m_d - m_\alpha = (M_U – 92me) – (M_{Th} – 90me) – (M_{He} – 2me)$$ **{3}**

$$Q/ c^2 = M_U – M_{Th} – M_{He}$$ **{1}**

Plugging in values of $M_U = 238.05u$, $M_{Th} = 234.04u$, and $M_{He} = 4.00u$, we find that

$$Q/c^2 = 0.0046u$$

Converting to MeV:

$Q = 0.0046u * 931.494\ MeV/c^2/u = 4.27\ MeV$ **{5}**

Since the alpha particle has very low mass, the energy of the alpha particle is, to a good approximation, equal to its kinetic energy:

Answer: **$KE_\alpha \approx Q = 4.27MeV$**

This is usually a good answer; however, most quoted values are for the Q value of the decay, not the alpha energy. Without this discussion this answer is worth .. **{2}**

If the discussion of this point was good, you can award an extra point, yielding **{3}** ***for this part. However, for the full marks, the calculation below is necessary.***

However, due to the energy taken by the daughter nucleus, we need another step to find the alpha energy. Here it is!

The momenta of the alpha and the daughter must be equal. We can express Q in terms of a ratio of energies. We then use energy = $p^2/2m$, assuming that we do not need to use relativistic formulae.

Let $M_N({}^{4}_{2}He) = m(\alpha)$

$E({}^{234}_{90}Th)/E(\alpha) = [p^2/2\ M_N({}^{234}_{90}Th)]/\ [p^2/2m(\alpha)] = m(\alpha)/\ M_N({}^{234}_{90}Th)$

$Q = E(\alpha) + E({}^{234}_{90}Th) = E\ (\alpha)\ [\ 1 + E({}^{234}_{90}Th)/E(\alpha)] = E\ (\alpha)\ [\ 1 + m(\alpha)/\ M_N({}^{234}_{90}Th)]$

So $E\ (\alpha) = Q/[\ 1 + m(\alpha)/\ M_N({}^{234}_{90}Th)] \approx Q/[\ 1 + 4/234]$

$E\ (\alpha) \approx 4.28/[\ 1 + 4/234]\ MeV = 4.20\ MeV$

Answer: **$KE_\alpha \approx Q = 4.20MeV$**

Since this was given in the Worked Examples, I think we can demand a perfect answer here – so this correct answer is worth more than simply making the usual assumption above. Of course, if the value is quoted from the Worked Examples, with a good discussion, 4 full marks should be awarded. Marks for this solution are **{4}**

[13]

C. Consider the first decay of ^{238}U to ^{234}U and an alpha.

Let N(0) = N be the constant number of ^{238}U atoms in the body,

let m be the mass of U in the body (=$2.0 * 10^{-5}$g),

let $T_{½}$ be its half-life (= 4.5 billion years),

let λ_n be the usual decay constant,

let T be the time considered,

let $N(\rightarrow T) = N_T$ be the number of atoms that have decayed in time T,

let M be the body mass, (estimated here = 70 kg) ***(each student's will be different!)***

let m_{mol} be the molar mass of Uranium (= 238 g/mol)

let E_α be the energy of the alpha (= 4.27 MeV from previous question and look-up table)

let p be the fraction of the natural U that is ^{238}U (= 0.9928)

let N_A be Avogadro's number (= $6.022 * 10^{23} mol^{-1}$)

{clear parameter definitions 1 mark}

Assume that 1. All radioisotopes under consideration are uniformly distributed throughout the body;
2. All of the decay energy is deposited in the body (none escapes).

{1 mark for each these assumptions}

Then

$N_T = (dN/dt)\, T = \lambda_n NT = (\ln 2)NT/ T_{½}$ **{6}**

Note that N_T is just the number of ^{234}Th atoms in the body, a time T after birth. To calculate the ^{238}U dose, we need the energy deposited in a body of mass M.

$$E = NE_\alpha = (\ln 2)NT\, E_\alpha / T_{½}$$ **{6}**

The absorbed dose is:

$$D(238) = E/M = (\ln 2)NT\, E_\alpha / (MT_{½})$$ **{2}**

Since $N = pmN_A/m_{mol}$, **{2}**

$$D(238) = p(\ln 2)\, mN_A T\, E_\alpha /(m_{mol}MT_{½})$$

Answer: For T = 1 year: **D(238) = 76.1 nGy**

{2}

Why don't we need to consider contributions to the energy deposition from the later decays of ^{238}U? **In fact, we do!** Each U decay leads to a daughter that decays within a day or so. Certainly the betas are more penetrating than the alphas; however most are still stopped in the body. The sum of their energies, being slightly less than that of the first alpha decay, does lead to a lower absorbed dose, but not by much. Of course due to the alpha radiation weighting factor of 20, compared to 1 for the betas, the alpha contribution to the equivalent and effective dose dominates. So the question is misleading insofar that students are asked to justify an incorrect statement! Leniency here! I suggest awarding only 2 marks for students who attempted to justify the unjustifiable, allowing the 3 marks for those who actually got it right. Any evidence of thought should receive some marks! ***{3}***

[23]

D. Consider now the first decay of ^{235}U to ^{231}Th and an alpha. This is very similar to the previous question, except:

p is the fraction of the natural U that is ^{235}U (= 0.0072)
$T_{1/2}$ is $7.04*10^8$y
$E_\alpha = 4.68$MeV

Using the formula: $D(235) = p(\ln 2)\, mN_A T\, E_\alpha /(235MT_{½})$ **{6}**

Answer: For T = 1 year: **D(235) = 3.9 nGy** **{1}**

Why can we ignore the ^{239}Pu? Assuming that the ^{239}Pu was produced in the supernova at about the same rate as the ^{235}U, we can calculate how much ^{239}Pu must be left now. From A, we have, with obvious notation, $N(^{239}Pu)/N(^{235}U) = \exp(-\lambda_{Pu} + \lambda_U)\, T_{sn}$ (T_{sn} is the time since the supernova exploded ≈ 6 billion years). Inserting values, this ratio is about $\exp(-10^9)$ – or effectively zero. Thus, all of the contribution comes from the daughter ^{235}U.

{3 *marks for a half decent explanation*}

The next isotope in the ^{235}U decay chain, ^{231}Th , decays within a day with a beta energy of less than a tenth of the alpha energy from the ^{235}U; the next alpha decay, of ^{231}Pa , has a half-life of 33,000 years, so obviously won't contribute in a human lifetime.

{2 *marks for a half-decent explanation*}

[12]

E. The total absorbed dose is:

D(238) + D(235) = 80 nGy **{1}**

The equivalent dose is 20 times this value, as $W_R = 20$ for alphas:

20 * (D(238) + D(235)) = 1.6 μSv **{1}**

The effective dose equals the equivalent dose, since $W_T = 1$ for the whole body. **{1}**

Answer: **Total absorbed dose: 80 nGy**
Equivalent dose: 1.6 μSv
Effective dose : 1.6 μSv

[3]

F. Let M be the atomic masses, m be the nuclear masses.
Calculate Q for various steps of this reaction. For $n + {}^{235}U \rightarrow {}^{236}U$, we have

$$Q_1 = m(n) + m({}^{235}U) - m({}^{236}U) \qquad \textbf{\{4\}}$$

And for ${}^{236}U \rightarrow {}^{144}Ba + {}^{89}Kr + 3n$ we have:

$$Q_2 = m({}^{236}U) - m({}^{144}Ba) - m({}^{89}Kr) - 3m(n) \qquad \textbf{\{4\}}$$

Adding these, with obvious notation (m_l is the electron mass, m(n) is the neutron mass):

$$Q_1 + Q_2 = m({}^{235}_{92}U) - m({}^{144}_{56}Ba) - m({}^{89}_{36}Kr) - 2m(n)$$

$$Q_1 + Q_2 = [M({}^{235}_{92}U) - 92m_l] - [M({}^{144}_{56}Ba) - 56m_l] - [M({}^{89}_{36}Kr) - 36m_l] - 2m(n)$$

$$Q_1 + Q_2 = M({}^{235}_{92}U) - M({}^{144}_{56}Ba) - M({}^{89}_{36}Kr) - 2m(n) \qquad \textbf{\{3\}}$$

Now plugging in the following values:

$M({}^{144}_{56}Ba) = 143.922953u$
$M({}^{89}_{36}Kr) = 88.917630u$
$M({}^{235}_{92}U) = 235.043929u$
$m(n) = 1.008664u$
$m_l = 0.000549u$

We get that the energy per atom is:
$$Q_1 + Q_2 = 0.186018u = 173 \text{ MeV} = 2.77*10^{-11}J \qquad \textbf{\{2\}}$$

Now calculate the total energy emitted in a day at the Pickering station:
$4{,}000 \text{ MW} = 4*10^9 \text{ J/s}$

So in one day, the energy emitted is:
$$4*10^9 \text{ J/s} * 3600 \text{ s/hr} * 24 \text{ hr/day} = 3.46*10^{14} \text{ J/day} \qquad \textbf{\{4\}}$$

So the number of atoms required per day is:
$$(3.46*10^{14} \text{ J/day}) / (2.77*10^{-11} \text{J/atom}) = 1.25*10^{25} \text{atoms} \qquad \textbf{\{2\}}$$

Now convert from atoms to mass, noting the 235 g of ${}^{235}U$ contains $6.02 * 10^{23}$ atoms:

$$1.25*10^{25} \text{ atoms} * (235g / 6.02*10^{23} \text{ atoms}) = 4.88*10^3 g = 4.88kg \qquad \textbf{\{1\}}$$

Answer: **Approximately 5kg of ^{235}U is required to generate 24 hours of power at the Pickering Generating Station.**

[20]

G. Let m be the mass of K in normal milk
Let f be the fraction of ^{40}K = 0.000117
Let N(K) be the number of K atoms in normal milk
Let M_{mol} be the molar mass of K = 39 g
Let $R(^{40}K)$ be the natural activity of ^{40}K in milk
Let t be the time that it takes ^{131}I to decay to $R(^{40}K)$
Let R(t) be the activity of ^{131}I after time t
Let R(0) be the activity after the contamination = 2000 $Bq.L^{-1}$
Let T_I be the half life of ^{131}I = 8.04 days
Let T_K be the half life of ^{40}K = $1.28 * 10^9$ years (***has to be looked up***)

The number of K atoms in1L of normal milk is given by:

$N(K) = m*N_A/ M_{mol}$ $(= 2.00g * N_A / (39.1g/mol))$

The number of ^{40}K atoms in this milk is thus:

$N(^{40}K) = f*m*N_A/M_{mol} = (0.000117 * 2.00g * N_A / (39.1g/mol))$

Now calculate the activity of this milk due to ^{40}K:

$R(^{40}K) = \lambda_K N(^{40}K) = (\ln2 / T_K) * N(^{40}K) = (\ln2 / T_K) * (f*m*N_A/M_{mol}) = 61.9$ Bq

{6}

The time, t, that it takes ^{131}I to decay to $R(^{40}K)$ is given by:

$R(t) = R(^{40}K) = R(0) \exp(-\lambda_I t)$

Solving for t yields:

$t = 1/\lambda_I * \ln(R(0)/R(^{40}K)) = (T_I / \ln2)* (\ln(R(0)/R(^{40}K))$ **{5}**

With R(0) = 2000Bq, $R(^{40}K)$ = 62Bq, and T_I = 8.04 d, t = 40.3 days. **{1}**

[12]

Answer: **In 40.3 days the ^{131}I activity in the milk will have reduced to natural ^{40}K levels.**

TOTAL MARKS = [100]

Solutions Prepared by
Cristen Adams & Tony Key.
March 2008