Clebsch-Gordan Coefficients

$$|\, j_1 m_1 >|\, j_2 m_2 > = \sum_{j=|j_1-j_2|}^{j_1+j_2} C_{m\; m_1\; m_2}^{j\;\; j_1\;\; j_2} \,|\, jm > \qquad m = m_1 + m_2$$

Two systems with spin j_1 and j_2 and z components m_1 and m_2 can combined to give a system which (quantum-mechanically) is a linear combination of states having spin j from $|j_1-j_2|$ to j_1+j_2 (in integer steps) each having a z component of $m = m_1 + m_2$.

So if we make a measurement of the total angular momentum of a state made up of two spin states as defined above, the sqaure of the Clebsch-Gordan coefficient

$$C_{m\; m_1\; m_2}^{j\;\; j_1\;\; j_2}$$

represents the probability of obtaining a measurement of $j(j+1)\,\hbar^2$

Tables of Clebsch-Gordan Coefficients

j_1 x j_2 Table for combining spin j_1 with spin j_2

$$| j_1 m_1 > | j_2 m_2 > = \sum_{j=|j_1-j_2|}^{j_1+j_2} C^{j\ \ j_1\ \ j_2}_{m\ m_1\ m_2} \, | jm > \qquad m = m_1 + m_2$$

		J	J	. . .
		m	m	. . .
m_1	m_2	$C^{j\ \ j_1\ \ j_2}_{m\ m_1\ m_2}$		
m_1	m_2			
.	.			
.	.			
.	.			
.	.			

Spin ½ Formalism

Write spin up and spin down states as two-component "spinors"

"spin up" $\left|\frac{1}{2}\ \frac{1}{2}\right\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ **"spin down"** $\left|\frac{1}{2}\ -\frac{1}{2}\right\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Most general state of a spin ½ system is then †

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \alpha, \beta \text{ complex}$$

Probability that measurement of S_z yields $\hbar/2$ is $|\alpha|^2$

Probability that measurement of S_z yields $-\hbar/2$ is $|\beta|^2$

$|\alpha|^2 + |\beta|^2 = 1$

† remember, a general state is a linear superposition of the two states until a measurement is made

Spin Matrices

What about measurements of S_x and S_y ?

Must also yield either ± ħ/2, but with what probability ?

To each component of S there is an associated 2x2 matrix

$$\hat{S}_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \hat{S}_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \hat{S}_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Each has eigenvalues of ± ħ/2. Normalized eigenvectors for S_x $\quad \chi_\pm = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \pm 1 \end{pmatrix}$

$$\left.\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \frac{a}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{b}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad a, b \ \ \textit{complex} \quad \right\} \quad \begin{aligned} a &= \frac{1}{\sqrt{2}}(\alpha + \beta) \\ b &= \frac{1}{\sqrt{2}}(\alpha - \beta) \end{aligned}$$

Probability that measurement of S_x will yield ħ/2 is then $|a|^2$. $\qquad |a|^2+|b|^2 = 1$

Pauli Spin Matrices

$$\hat{S}_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \hat{S}_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \hat{S}_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\hat{S}_x = \frac{\hbar}{2}\sigma_x \qquad \hat{S}_y = \frac{\hbar}{2}\sigma_y \qquad \hat{S}_z = \frac{\hbar}{2}\sigma_z$$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The σ are the <u>Pauli spin matrices</u>

Flavour Symmetries (internal symmetries)

proton and neutron have very similar mass: m_p = 938.28 MeV/c^2 m_n = 939.57 MeV/c^2

Heisenberg: view as two different states of the same particle (the nucleon, N).

$$N = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad p = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \; n = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Reminiscent of spin ½ formalism.

p and n are "related" by a rotation in an abstract space referred to as isospin space (in analogy to spin). Adopt spin ½ formalism to deal with systems of particles with different isospin.

For any state we can define the isospin I and the third component I_3

We define the components as I_1, I_2 and I_3 to make clear that we are not talking about components in coordinate space, but in some abstract space.

Isospin

We have stated that strong and electromagnetic interactions conserve strangeness, charm etc. (see the allowed interaction vertices for the SM)

Isospin is just a version of conservation of quark number for u, d quarks, arising from the fact that their masses are so similar.

Consider proton and neutron as members of an isospin doublet (the nucleon)

$$p = \left| \frac{1}{2} \;\; \frac{1}{2} \right\rangle \qquad n = \left| \frac{1}{2} \;\; -\frac{1}{2} \right\rangle$$

Heisenberg's proposal: the strong interaction is invariant under rotations in "isospin space".

Noether's theorem $\Longrightarrow$ isospin is conserved in all strong interactions.

Other particles fall into different isospin multiplets (2I + 1 elements)

Pions have isospin 1

$$\pi^{+} = |1\;1\rangle \qquad \pi^{0} = |1\;0\rangle \qquad \pi^{-} = |1\;-1\rangle$$

More on Isospin

Really we are assigning the u and d quarks to an isospin doublet

$$u = \left| \frac{1}{2} \quad \frac{1}{2} \right\rangle \qquad d = \left| \frac{1}{2} \quad -\frac{1}{2} \right\rangle$$

All other quarks have isospin 0   (they are isospin singlet states)

Baryons made of u and d quarks have total isospin of $(1/2 \oplus 1/2) \oplus 1/2$

$\Longrightarrow$ $(0 \text{ or } 1) \oplus 1/2$ $\Longrightarrow$ $1/2$ or $3/2$

$2I + 1$ states $\Longrightarrow$ 2 states (p,n)   4 states (Δ^{++}, Δ^{+}, Δ^{0}, Δ^{-})

Perkins figure 5.1

Isospin at Quark Level

$$u = \left|\frac{1}{2}\;\;\frac{1}{2}\right\rangle \qquad d = \left|\frac{1}{2}\;\;-\frac{1}{2}\right\rangle \qquad s = \left|0\;\;0\right\rangle$$

Other quark flavours (c,b,t) are also isospin singlets, like the strange quark

Mesons containing (only) u,d quarks can therefore have isospin 0 or 1

$\pi^- = \bar{u}d \qquad \pi^0 = u\bar{u}, d\bar{d} \qquad \pi^+ = u\bar{d}$ isospin triplet I = 1

$\eta = u\bar{u}, d\bar{d}, s\bar{s}$ isospin singlet I = 0

Mesons containing only one u,d quark can only have isospin = ½

$K^0 = d\bar{s} \quad K^+ = u\bar{s}$ form an isospin doublet

$\bar{K}^0 = \bar{d}s \quad K^- = \bar{u}s$ form a *separate* isospin doublet

Some Isospin and Strangeness Assignments

			I^3				
B	**S**	I	**-1**	**-1/2**	**0**	**+1/2**	**+1**
1	0	1/2		n		p	
1	-1	0			Λ		
0	0	1	π^-		π^0		π^+
1	+1	1/2		K^0		K^+	
1	-1	1/2		K^-		$\bar{K}^0$	
1	-1	1	Σ^-		Σ^0		Σ^+
1	-2	1/2		Ξ^-		Ξ^0	
1	-3	0			Ω^-		
0	0	0			η		

Example: pion-nucleon scattering

Six elastic $\pi N \rightarrow \pi N$ processes (same particles in and out of scattering)

$$(a)\ \pi^+ + p \rightarrow \pi^+ + p \qquad (b)\ \pi^0 + p \rightarrow \pi^0 + p$$
$$(c)\ \pi^- + p \rightarrow \pi^- + p \qquad (d)\ \pi^+ + n \rightarrow \pi^+ + n$$
$$(e)\ \pi^0 + n \rightarrow \pi^0 + n \qquad (f)\ \pi^- + n \rightarrow \pi^- + n$$

and 4 charge-exchange processes

$$(g)\ \pi^+ + n \rightarrow \pi^0 + p \qquad (h)\ \pi^0 + p \rightarrow \pi^+ + n$$
$$(i)\ \pi^0 + n \rightarrow \pi^- + p \qquad (j)\ \pi^- + p \rightarrow \pi^0 + n$$

Pions carry $\mathrm{I} = 1$ and nucleons $\mathrm{I} = 1/2$ so the total isospin can be either 1/2 or 3/2

Isospin conservation → two distinct amplitudes: $\mathcal{M}_1$ ($\mathrm{I} = 1/2$) and $\mathcal{M}_3$ ($\mathrm{I} = 3/2$)

$$\mathcal{M}_1 \equiv \left\langle \psi_f\left(\frac{1}{2}\right) \middle| H_1 \middle| \psi_i\left(\frac{1}{2}\right) \right\rangle \qquad \mathcal{M}_3 \equiv \left\langle \psi_f\left(\frac{3}{2}\right) \middle| H_3 \middle| \psi_i\left(\frac{3}{2}\right) \right\rangle$$

Clebsh Gordan Coefficients for $1 \oplus 1/2$

		3/2
		+3/2
+ 1	+ 1/2	1

		3/2	1/2
		+1/2	+1/2
+ 1	- 1/2	1/3	2/3
0	+ 1/2	2/3	- 2/3

		3/2	1/2
		- 1/2	- 1/2
0	- 1/2	2/3	1/3
- 1	+ 1/2	1/3	- 2/3

		3/2
		-3/2
- 1	- 1/2	1

e.g. $$\pi^0 + n: \quad |1 \quad 0\rangle \left|\frac{1}{2} \quad -\frac{1}{2}\right\rangle = \sqrt{\frac{2}{3}}\left|\frac{3}{2} \quad -\frac{1}{2}\right\rangle + \sqrt{\frac{1}{3}}\left|\frac{1}{2} \quad -\frac{1}{2}\right\rangle$$

Pion-Nuclon Scattering Amplitudes

$$\pi^+ + p: \quad |1 \quad 1\rangle\left|\frac{1}{2} \ \frac{1}{2}\right\rangle = \left|\frac{3}{2} \ \frac{3}{2}\right\rangle$$

$$\pi^0 + p: \quad |1 \quad 0\rangle\left|\frac{1}{2} \ \frac{1}{2}\right\rangle = \sqrt{\frac{2}{3}}\left|\frac{3}{2} \ \frac{1}{2}\right\rangle - \sqrt{\frac{1}{3}}\left|\frac{1}{2} \ \frac{1}{2}\right\rangle$$

$$\pi^- + p: \quad |1 \ -1\rangle\left|\frac{1}{2} \ \frac{1}{2}\right\rangle = \sqrt{\frac{1}{3}}\left|\frac{3}{2} \ -\frac{1}{2}\right\rangle - \sqrt{\frac{2}{3}}\left|\frac{1}{2} \ -\frac{1}{2}\right\rangle$$

$$\pi^+ + n: \quad |1 \quad 1\rangle\left|\frac{1}{2} \ -\frac{1}{2}\right\rangle = \sqrt{\frac{1}{3}}\left|\frac{3}{2} \ \frac{1}{2}\right\rangle + \sqrt{\frac{2}{3}}\left|\frac{1}{2} \ \frac{1}{2}\right\rangle$$

$$\pi^0 + n: \quad |1 \quad 0\rangle\left|\frac{1}{2} \ -\frac{1}{2}\right\rangle = \sqrt{\frac{2}{3}}\left|\frac{3}{2} \ -\frac{1}{2}\right\rangle + \sqrt{\frac{1}{3}}\left|\frac{1}{2} \ -\frac{1}{2}\right\rangle$$

$$\pi^- + n: \quad |1 \quad 1\rangle\left|\frac{1}{2} \ -\frac{1}{2}\right\rangle = \left|\frac{3}{2} \ -\frac{3}{2}\right\rangle$$

$$(a)\ \pi^+ + p \rightarrow \pi^+ + p$$

$$(f)\ \pi^- + n \rightarrow \pi^- + n$$

Both are pure I = 3/2 on either side of scatter. Both proceed purely via the I = 3/2 amplitute $\mathcal{M}_3$

Pion-Nucleon Scattering Amplitudes

$$\pi^- + p: \quad \left|1\ -1\right\rangle \left|\frac{1}{2}\ \frac{1}{2}\right\rangle = \sqrt{\frac{1}{3}}\left|\frac{3}{2}\ -\frac{1}{2}\right\rangle - \sqrt{\frac{2}{3}}\left|\frac{1}{2}\ -\frac{1}{2}\right\rangle$$

$$(c)\ \pi^- + p \rightarrow \pi^- + p$$

$$\sqrt{\frac{1}{3}}\sqrt{\frac{1}{3}}\left\langle\frac{3}{2}\ -\frac{1}{2}\right| H_3 \left|\frac{3}{2}\ -\frac{1}{2}\right\rangle + \sqrt{\frac{2}{3}}\sqrt{\frac{2}{3}}\left\langle\frac{1}{2}\ -\frac{1}{2}\right| H_1 \left|\frac{1}{2}\ -\frac{1}{2}\right\rangle$$

$$\mathcal{M}_c = \frac{1}{3}\mathcal{M}_3 + \frac{2}{3}\mathcal{M}_1$$

Pion-Nucleon Scattering Amplitudes

$$\pi^- + p: \quad |1\ -1\rangle\left|\frac{1}{2}\ \frac{1}{2}\right\rangle = \sqrt{\frac{1}{3}}\left|\frac{3}{2}\ -\frac{1}{2}\right\rangle - \sqrt{\frac{2}{3}}\left|\frac{1}{2}\ -\frac{1}{2}\right\rangle$$

$$\pi^0 + n: \quad |1\ \ 0\rangle\left|\frac{1}{2}\ -\frac{1}{2}\right\rangle = \sqrt{\frac{2}{3}}\left|\frac{3}{2}\ -\frac{1}{2}\right\rangle + \sqrt{\frac{1}{3}}\left|\frac{1}{2}\ -\frac{1}{2}\right\rangle$$

$$(j)\ \pi^- + p \to \pi^0 + n$$

$$\mathcal{M}_J = \sqrt{\frac{2}{3}}\sqrt{\frac{1}{3}}\mathcal{M}_3 - \sqrt{\frac{2}{3}}\sqrt{\frac{1}{3}}\mathcal{M}_1 = \frac{\sqrt{2}}{3}\left(\mathcal{M}_3 - \mathcal{M}_1\right)$$

$\mathcal{M}$ is a quantum mechanical amplitude.

Cross-section σ (i.e.. probability) for some process is proportional to $\mathcal{M}^2$

Scattering of charged pions and protons

$$(a)\ \pi^+ + p \to \pi^+ + p \qquad (b)\ \pi^0 + p \to \pi^0 + p$$

$$(c)\ \pi^- + p \to \pi^- + p \qquad (d)\ \pi^+ + n \to \pi^+ + n$$

$$(e)\ \pi^0 + n \to \pi^0 + n \qquad (f)\ \pi^- + n \to \pi^- + n$$

$$(g)\ \pi^+ + n \to \pi^0 + p \qquad (h)\ \pi^0 + p \to \pi^+ + n$$

$$(i)\ \pi^0 + n \to \pi^- + p \qquad (j)\ \pi^- + p \to \pi^0 + n$$

Compare the cross-section predictions for scattering of charged pions from protons

Three processes to consider: a), c) and j)

$$\mathcal{M}_{\mathcal{A}} = \mathcal{M}_3 \qquad \mathcal{M}_{\mathcal{C}} = \frac{1}{3}\mathcal{M}_3 + \frac{2}{3}\mathcal{M}_1 \qquad \mathcal{M}_{\mathcal{J}} = \frac{\sqrt{2}}{3}\left(\mathcal{M}_3 - \mathcal{M}_1\right)$$

Relative Cross-sections for πp Scattering

$$\sigma_a : \sigma_c : \sigma_j = 9\left|\mathcal{M}_3\right|^2 : \left|\mathcal{M}_3 + 2\mathcal{M}_1\right|^2 : 2\left|\mathcal{M}_3 - \mathcal{M}_1\right|^2$$

At E_{CM} = 1232 MeV there is a dramatic bump (increase) in the pion-nucleon scattering cross-section, due to a π N (pion-nucleon) resonance, the Δ(1293).

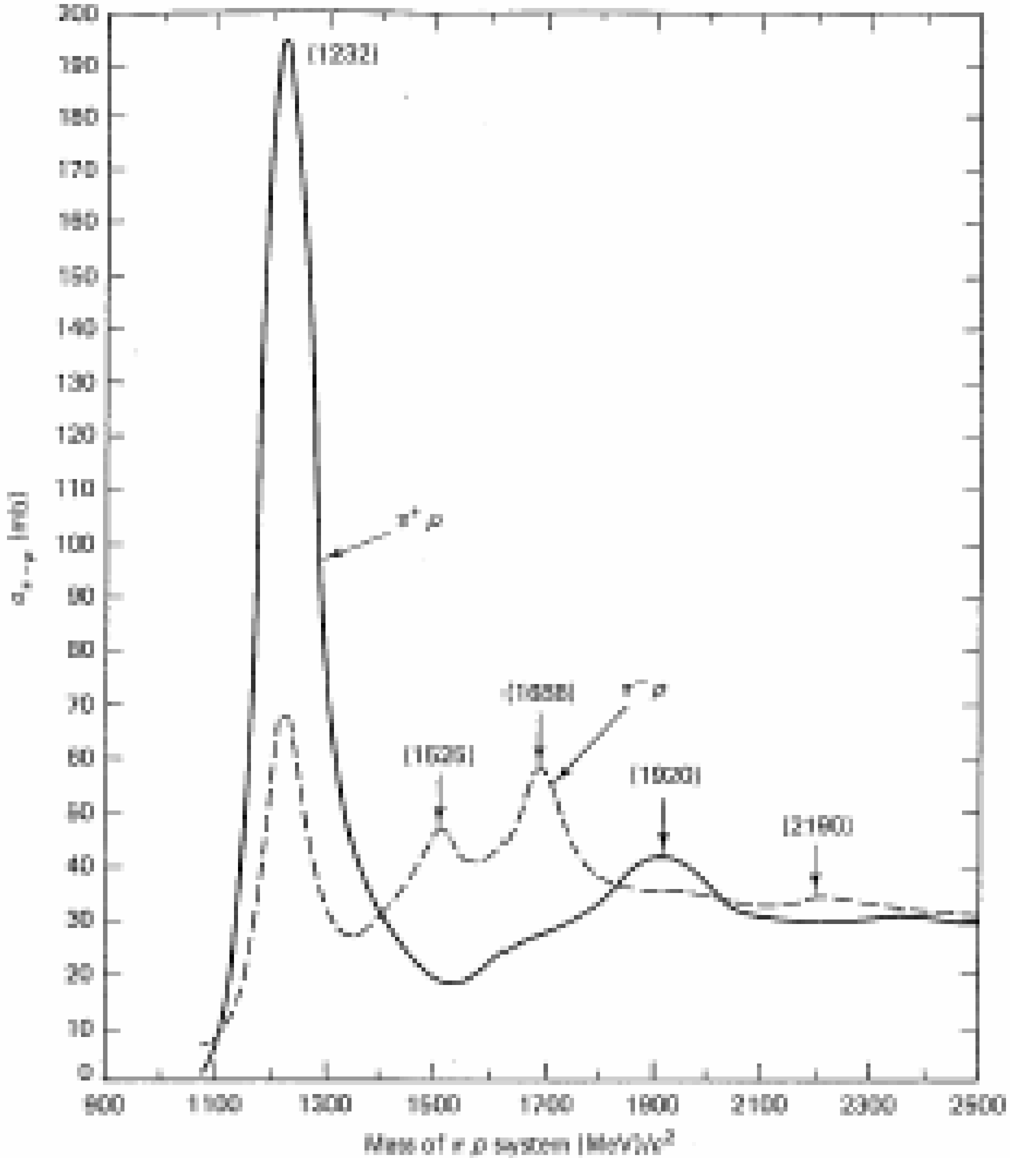
Δ carries isospin 3/2 (so there are 2 (3/2) +1 = 4 charge states)

Expect the scattering at this energy to be dominated by the isospin 3/2 channel

$$\sigma_a : \sigma_c : \sigma_j = 9 : 1 : 2$$

Generally, experimentally it is easiest to measure the total cross-section, so processes (c) and (j) are combined to yield:

$$\frac{\sigma_{tot}(\pi^+ + p)}{\sigma_{tot}(\pi^- + p)} = 3$$ (*e.g.* at E_{CMS} = 1293 MeV)



(1232)
$\pi^+ p$
$\pi^- p$
(1520)
(1688)
(1920)
(2190)
$\sigma_{\pi^{\pm} p}$ (mb)
Mass of πp system (MeV/c^2)

Strong Interaction

More general than Heisenberg's hypothesis that the nuclear force between a proton and proton is the same as that between a proton and a neutron is the following principle:

The strong interaction is flavour-independent.

That is, the force between a two quarks does not depend on what flavour of quarks you are discussing.