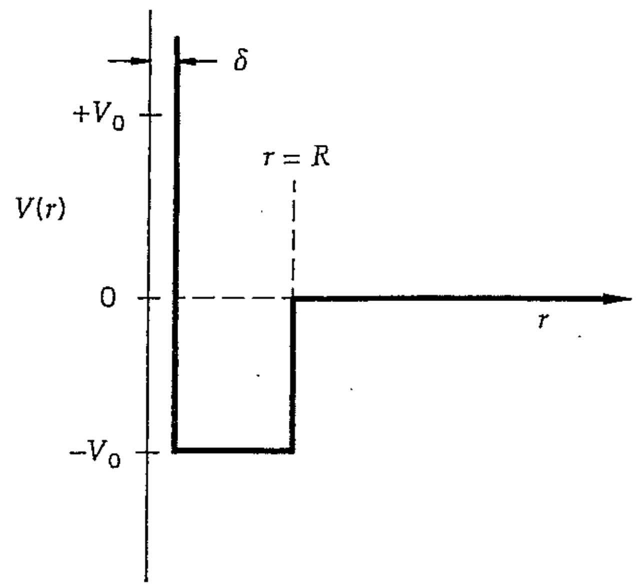
Last time.....properties of nuclear potential used for models of the nucleus:

Nuclear force is short range. There's lots of evidence for this. It's especially evident in $B/A \sim$ const. (saturation of nuclear force).

Nucleons in the nucleus interact only with their closest neighbors so force range less than nuclear size ($R \sim r_0 A^{1/3}$ fm. Density of nuclear matter constant except for surface effects......$\rho \sim 0.17$ nucleon / fm^3).

Force must be attractive (because we get bound states) but must also have a strong repulsive core to prevents collapse. This is best explained in terms of the quark model. It is generally not accounted for in simple nuclear potential models ($\delta \ll R$)



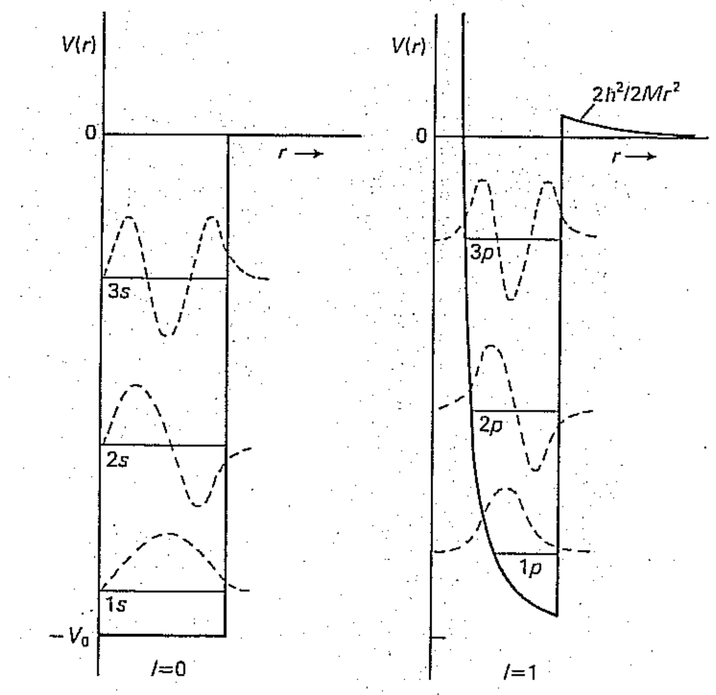
Liquid Drop Model is a classical model, though we invoked some quantum effects in justifying the form of the asymmetry term and the pairing term.

The Shell Model is inherently quantum-mechanical: start with a nuclear potential model, solve the NR Schrödinger equation, extract the energy level ordering and tune to reproduce the observed sequence of magic numbers. This requires the addition of a spin-orbit term to the potential.

Chose square-well and harmonic oscillator potentials for models of the average nuclear potential. With appropriate choices for the depth of the wells these can be made to approximate the form of the nuclear potential that is extracted from electron-nuclei and neutron-nuclei scattering experiments:

Measured form factor from scattering expts. → Charge or matter density distribution → Model for shape of nuclear potential well

Note that the potential will cut off more rapidly than the observed matter or charge density distributions since the finite depth of the potential well means that the nucleon wave-functions can penetrate (tunnel) into the classically forbidden region (i.e. they extend out past the edge of the potential.



Intermediate to the Liquid Drop Model and the Shell Model is the Fermi Gas Model, which might be familiar from solid state physics.

Treats nucleus as combination “gases” of protons and neutrons, confined to some small region of space (the nuclear radius).

This is the classic, particle(s) in a box problem. The density of states for such a system is one of the basic calculations in quantum mechanics.

Note that the (nuclear) potential wells used for the protons and the neutrons cannot be the same.

For the protons, need to first account for the Coulomb potential energy:

Coulomb potential seen by an individual proton in a nucleus is $V(r) = -\int_{\infty}^{r} \vec{E} \cdot d\vec{\ell}$

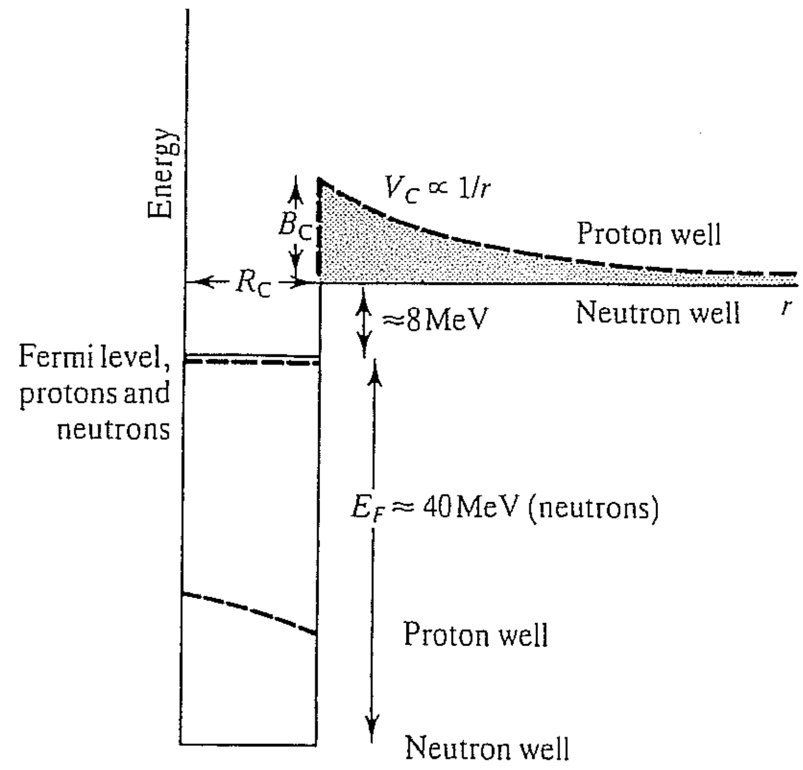
For a uniformly charged sphere, the electric field outside the sphere is trivial, and inside the sphere Gauss's Law gives us

$$\oint \vec{E} \cdot d\vec{a} = \frac{1}{\varepsilon_0} Q_{\text{enclosed}} = \frac{1}{\varepsilon_0} \frac{r^3}{R^3} (Z-1)e \qquad \left|\vec{E}\right| 4\pi r^2 = \frac{(Z-1)e}{\varepsilon_0} \frac{r^3}{R^3} \quad \Rightarrow \quad \vec{E} = \frac{(Z-1)e}{4\pi\varepsilon_0} \frac{r}{R^3} \hat{r}$$

So for a proton inside the nucleus ($r < R$) the potential seen is

$$V(r) = -\int_{\infty}^{R} \frac{(Z-1)e}{4\pi\varepsilon_0} \frac{1}{r^2} dr - \int_{R}^{r} \frac{(Z-1)e}{4\pi\varepsilon_0} \frac{r}{R^3} dr = \frac{(Z-1)e}{4\pi\varepsilon_0} \left[\frac{1}{r}\bigg|_{\infty}^{R} - \frac{r^2}{2R^3}\bigg|_{R}^{r} \right] \quad = \frac{(Z-1)e}{4\pi\varepsilon_0 R} \left(\frac{3}{2} - \frac{r^2}{2R^2} \right)$$

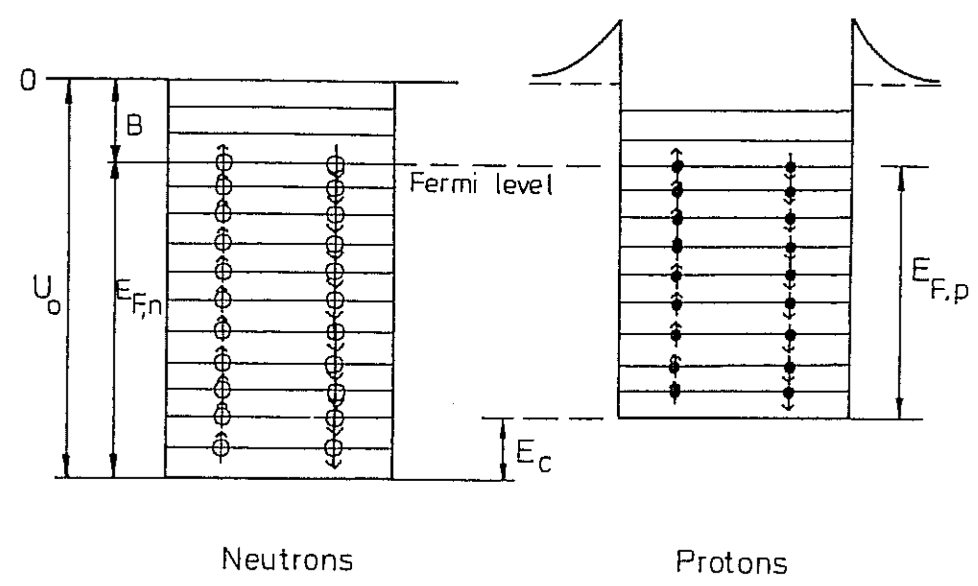
So Coulomb potential energy for a proton is given by $E_c = \frac{(Z-1)e^2}{4\pi\varepsilon_0 R} \left(\frac{3}{2} - \frac{r^2}{2R^2} \right)$



There are also additional effects that alter the depth of the proton potential well relative to that for the neutrons. In particular, we that while for low A nuclei we have N ~ Z ~ A/2, that for higher A, the number of neutrons exceeds that number of protons, which means that the nuclear potential becomes more attractive for protons than for neutrons. Leave this for the time being……

Typically also model the proton well as a square well, with energy of bottom of well raised to account for Coulomb effects (and other effects as we shall see later)

These two plots show the potential wells for protons and neutrons



The Fermi levels for *p* and *n* must be approximately equal or nucleus will be β-unstable. This means that the nuclear potential well for neutrons must be somewhat deeper that for protons. Offset here is due to repulsive Coulomb energy E_c

Density of States: Particle(s) in a box.

Consider a particle moving freely inside a cubic box of side L, volume L^3. Take the potential V = 0 inside the box and V = ∞ outside.

$$-\frac{\hbar}{2m}\nabla^2\Psi = E\Psi$$

As usual look for stationary solutions that are separable, and satisfy the boundary conditions at x = 0, L; y = 0, L; z = 0, L: these are of the form

$$\Psi(x,y,z) = K\sin(k_x x)\sin(k_y y)\sin(k_z z)$$

where $\vec{k} = (k_x, k_y, k_z)$ obeys $k_x = \frac{n_x \pi}{L}, k_y = \frac{n_y \pi}{L}, k_z = \frac{n_z \pi}{L} \quad n_x, n_y, n_z = 1,2,3,....$

[negative integer solutions differ only by a phase, which is not physical]

Allowed k values form a cubic lattice in the (+,+,+) quadrant of k-space. Counting states amounts to counting these lattice points. The spacing between them is π / L so the number of points per unit volume is given by $(L / \pi)^3$.

Number of lattice points with $k = |\vec{k}| < k_0$ is then just the number enclosed within the (+,+,+) quadrant of a sphere (in k space) of radius k_0, centred at the origin. For large k_0 we have

$$= \frac{1}{8}\left(\frac{4\pi k_0^3}{3}\right)\left(\frac{L}{\pi}\right)^3 = \frac{V}{(2\pi)^3}\left(\frac{4\pi k_0^3}{3}\right)$$

Spin-1/2 fermions can populate each k value with 2 spin states, so the number of fermion states is twice this

$$N = 2\frac{1}{8}\left(\frac{4\pi k_0^3}{3}\right)\left(\frac{L}{\pi}\right)^3 = 2\frac{V}{(2\pi)^3}\left(\frac{4\pi k_0^3}{3}\right)$$

For the NR Schrödinger equation the energy of a particle in a state of specified (n_x, n_y, n_z) and either spin is

$$E = \frac{\hbar^2}{2m}(k_x^2 + k_y^2 + k_z^2) = \frac{\hbar^2}{2m}k^2 \quad \Rightarrow \quad k = \left(\frac{2ME}{\hbar^2}\right)^{1/2}$$

So number of neutrons, *N*, and number of protons, *Z*, are given by

$$N = \frac{V}{3\pi^2}\left(\frac{2m(E_F^n)}{\hbar^2}\right)^{3/2} = \frac{V}{3\pi^2}\frac{\left(p_F^n\right)^3}{\hbar^3} \qquad Z = \frac{V}{3\pi^2}\left(\frac{2m(E_F^p)}{\hbar^2}\right)^{3/2} = \frac{V}{3\pi^2}\frac{\left(p_F^p\right)^3}{\hbar^3}$$

Where E_F^n and E_F^p are the *kinetic* energies associated with the Fermi level in the respective potential wells (so for instance they are ~ equal for the case $N = Z$)

For lighter nuclei, $A \lesssim 40$ we have $N \sim Z$ and thus

$N/V \sim$ ½ density of nuclear matter ~ ½ (0.17 nucleons / fm^3) = 0.085 / fm^3

$\Rightarrow E_F = 38$ MeV (independent of A) - this increases somewhat for heavier nuclei

Accounting for neutron separation energy (which we approximate with the average binding energy per nucleon), yields a total well depth for neutrons of

$\Rightarrow$ Neutron potential well depth ~ 46 MeV

The kinetic energy of nucleons is therefore of the same order as the well depth, so nuclei are relatively weakly bound.

Can write the expressions for the numbers of neutrons and protons in terms of the Fermi momentum (i.e. the kinetic energy, as before)

$$N = \frac{V}{3\pi^2}\left(\frac{p_F^n}{\hbar}\right)^3 \qquad Z = \frac{V}{3\pi^2}\left(\frac{p_F^p}{\hbar}\right)^3 \quad \Rightarrow \quad p_F^n = \left(\frac{3\pi^2\hbar^3}{V}N\right)^{1/3} \qquad p_F^p = \left(\frac{3\pi^2\hbar^3}{V}Z\right)^{1/3}$$

What is average kinetic energy per nucleon ? $\left\langle E_{\mathrm{kin}} \right\rangle = \dfrac{\int_0^{p_F} E_{\mathrm{kin}}\, p^2\, dp}{\int_0^{p_F} p^2\, dp} = \dfrac{3}{5}\dfrac{p_F^2}{2m}$

Total kinetic energy of the nucleus is then given by $E_{\mathrm{kin}}(N,Z) = N\left\langle E_{\mathrm{kin}}^n \right\rangle + Z\left\langle E_{\mathrm{kin}}^p \right\rangle$

$$= \frac{3}{10m}[N(p_F^n)^2 + Z(p_F^p)^2] = \frac{3}{10m}\left[N\left(\frac{3\pi^2\hbar^2 N}{V}\right)^{2/3} + Z\left(\frac{3\pi^2\hbar^2 Z}{V}\right)^{2/3}\right]$$

Using $V = \dfrac{4}{3}\pi r_0^3 A \quad \Rightarrow \dfrac{1}{V^{2/3}} = \left(\dfrac{3}{4\pi}\right)^{2/3}\dfrac{1}{r_0^2}$ this becomes:

$$E_{\mathrm{kin}} = \frac{3}{10m}\left[\left(\frac{3}{4\pi}\right)^{2/3} \frac{1}{r_0^2}\left(3\pi^2\hbar^3\right)^{2/3}\right]\left(\frac{N^{5/3}+Z^{5/3}}{A^{2/3}}\right) = \frac{3}{10m}\frac{\hbar^2}{r_0^2}\left(\frac{9\pi^2}{4\pi}\right)^{2/3}\left(\frac{N^{5/3}+Z^{5/3}}{A^{2/3}}\right)$$

$$E_{\mathrm{kin}} = \frac{3}{10m}\frac{\hbar^2}{r_0^2}\left(\frac{9\pi}{4}\right)^{2/3}\left(\frac{N^{5/3}+Z^{5/3}}{A^{2/3}}\right)$$

Here we have assumed that the radii of the proton and neutron wells are equal and set (as previously) $m_p = m_n = m$.

$$\frac{\partial E_{\mathrm{kin}}}{\partial N} \propto \frac{\partial}{\partial N}\left(\frac{N^{5/3}+(A-Z)^{5/3}}{A^{2/3}}\right) = \frac{5}{2}N^{3/2} - \frac{5}{2}(A-N)^{3/2} = \frac{5}{2}N^{3/2} - \frac{5}{2}Z^{3/2}$$

Which yields a minimum for E_{kin} at $N = Z$.

To study the behaviour around this minimum, expand this in N-Z

Define $\varepsilon = N - Z, \quad Z + N = A$ fixed $\quad \Rightarrow \quad Z = \frac{1}{2}A(1+\varepsilon/A), \quad N = \frac{1}{2}A(1-\varepsilon/A),$

Taking $\varepsilon/A << 1$, and using $(1+x)^n = 1 + nx + \frac{n(n-1)}{2}x^2 + \ldots\ldots$ we obtain

$$E_{\text{kin}} = \frac{3}{10m}\frac{\hbar^2}{r_0^2}\left(\frac{9\pi}{8}\right)^{2/3}\left(A + \frac{5}{9}\frac{(Z-N)^2}{A} + \ldots\ldots\right)$$

The first term is proportional to *A* and contributes to the volume term in the semi-empirical mass formula.

The second term is of the form we obtained through a somewhat more hand-wavy Fermi-gas-like model (See lecture from March 27)

Note that we can evaluate the coefficient of the quadratic term and compare this to the value obtained by fits to the *B/A* distribution (which gave 23.29 MeV). This yields

$$\frac{1}{6m}\frac{\hbar^2}{r_0^2}\left(\frac{9\pi}{8}\right)^{2/3}\frac{(Z-N)^2}{A} \approx 11\text{MeV}\frac{(Z-N)^2}{A}$$

This accounts for only half of the observed coefficient, the rest comes from the difference in the proton and neutron wells in the case where *N > Z*, in which case the attraction of protons is larger since the Pauli principle weakens the attraction of neutrons

Problem: ${}^{40}_{20}\mathrm{Ca}$ is the heaviest stable nucleus with $Z = N$ (it is doubly magic). The neutron separation energy is S_n = 15.6 MeV. Estimate the proton separation energy (the empirical value is 8.6 MeV).

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For the case in which $N \sim Z$, the energy due to the strong interaction should be the same for protons and neutrons. (This is the kinetic energy that comes from the solution of the NR Schrödinger equation for the square-well potential).

There is no "asymmetry" contribution to the relative depths of the potential well, so the entire difference in the neutron and proton separation energies should be attributable to the Coulomb energy.

We had $E_c(r) = \dfrac{(Z-1)e^2}{4\pi\varepsilon_0 R}\left(\dfrac{3}{2} - \dfrac{r^2}{2R^2}\right)$. The average Coulomb energy is then given by

$$\overline{E}_c = \frac{\int E_c dV}{\int dV} = \frac{(Z-1)e^2}{4\pi\varepsilon_0 R}\left(\frac{4}{3}\pi R^3\right)^{-1} \int_0^R 4\pi r^2 \left(\frac{3}{2} - \frac{r^2}{2R}\right) dr = \frac{6}{5}\frac{(Z-1)e^2}{4\pi\varepsilon_0 R}$$

For ${}^{40}_{20}\mathrm{Ca}$ $\quad \overline{E}_c = \dfrac{6}{5}\dfrac{(Z-1)e^2}{4\pi\varepsilon_0 R} = \dfrac{6}{5}\dfrac{(Z-1)}{r_0^3 A}\dfrac{e^2\hbar c}{4\pi\varepsilon_0 \hbar c} \approx 8.7\,\mathrm{MeV}$

From which we would predict $S_p = S_n - 8.7$ MeV = 6.9 MeV