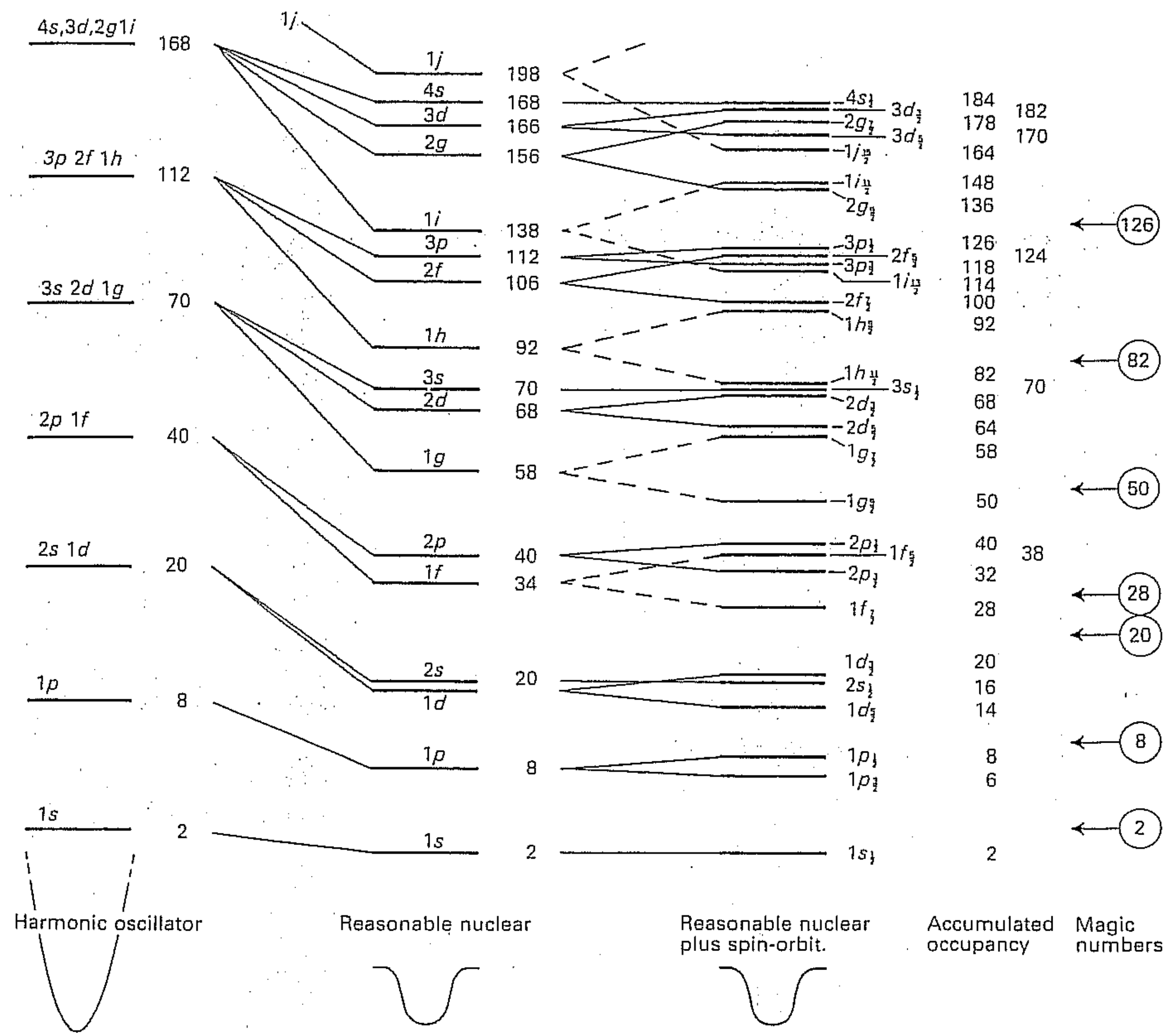
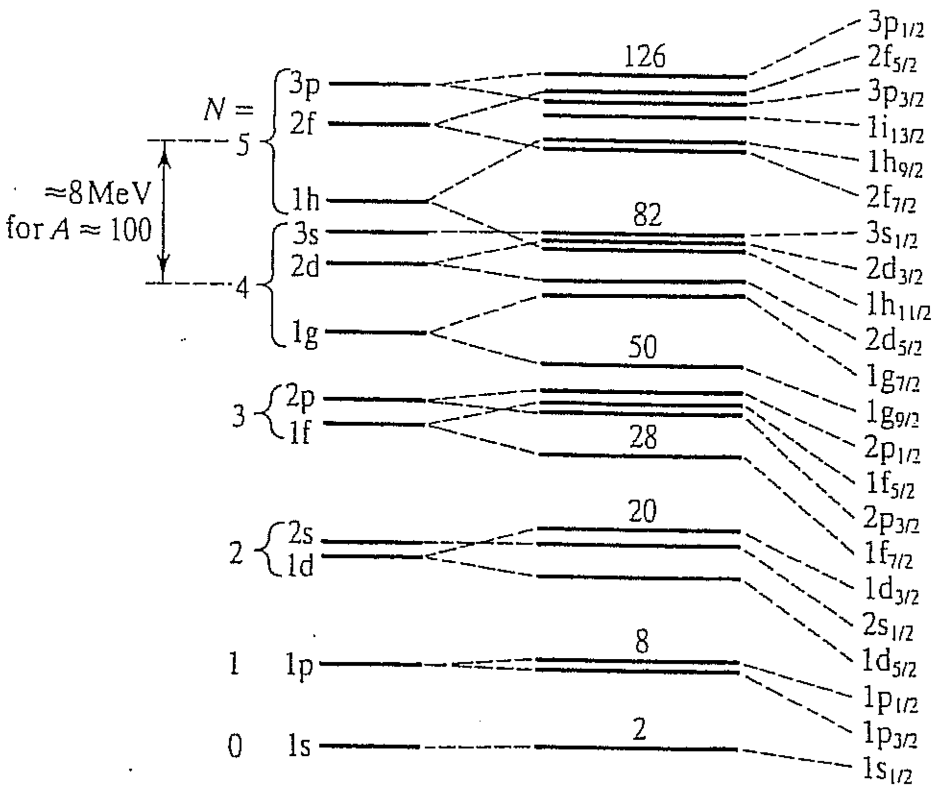
This is where we got to last time, for the energy levels predicted by the shell model

From Williams, Nuclear and Particle Physics

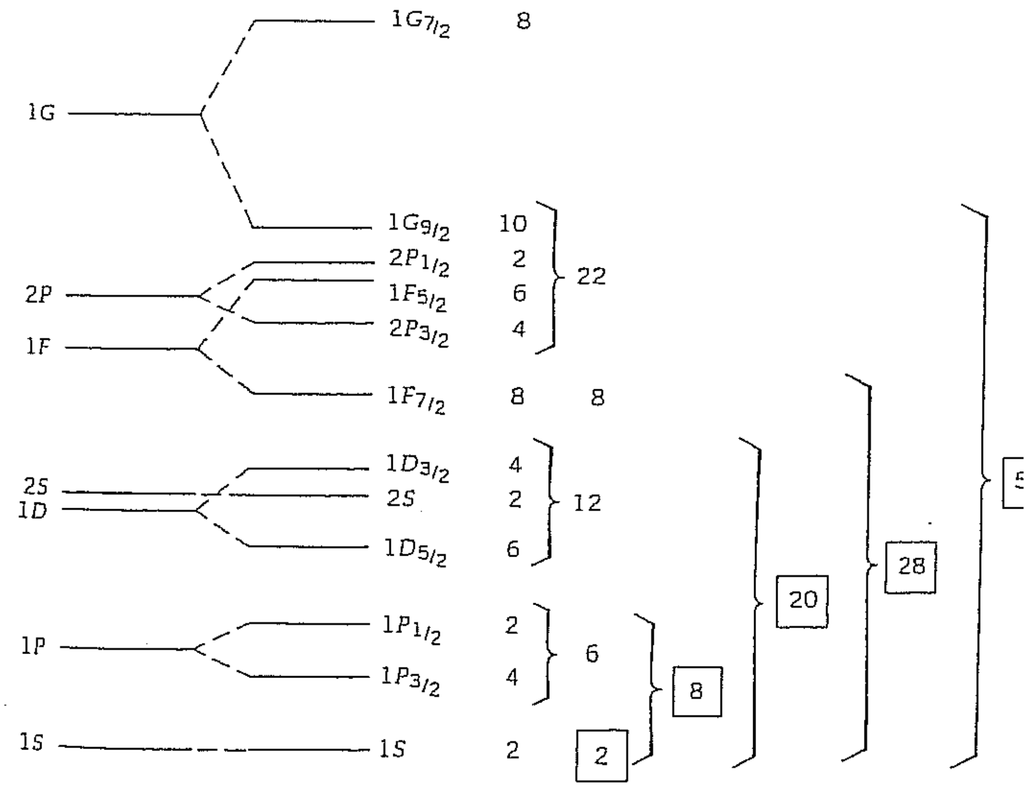


Reproduces the observed values of the magic numbers: 2, 8, 20, 28, 50, 82, 126

Some other views of this:



Saxon-Woods potential with spin-orbit term. (from Burcham and Jobes, Nuclear and Particle Physics)



Finite square-well potential with spin-orbit term. (from Das and Ferbel, Introduction to Nuclear and Particle Physics)

Visible splittings here seem more in line with expectations. In particular the $2p_{1/2} - 1g_{9/2}$ splitting does not seem comparable to those that define the magic numbers.

Some comments on the Shell Model

The shell model (as we have been discussing it) is really better referred to as the single-particle shell model, for reasons that will become clearer soon.

It treats each individual nucleon as a single particle confined within some overall average nuclear potential.

Since the average mean free path of a non-relativistic neutron traveling through nuclear matter is about 2*fm*, how can this possibly be realistic ? What about collisions with the other nucleons orbiting in the same potential ?

More on angular momentum and spin-orbit forces

In order to explain the observed pattern of level splittings (the magic numbers) it was necessary to introduce a spin-orbit contribution to the potential

$$V(r) \rightarrow V(r) + W(r)\vec{L}\cdot\vec{S}$$

some function of r

Here $\vec{L}$ and $\vec{S}$ are the orbital and spin angular momentum operators for the nucleon.

The total angular momentum is then given by $\vec{J} = \vec{L} + \vec{S}$ so we have

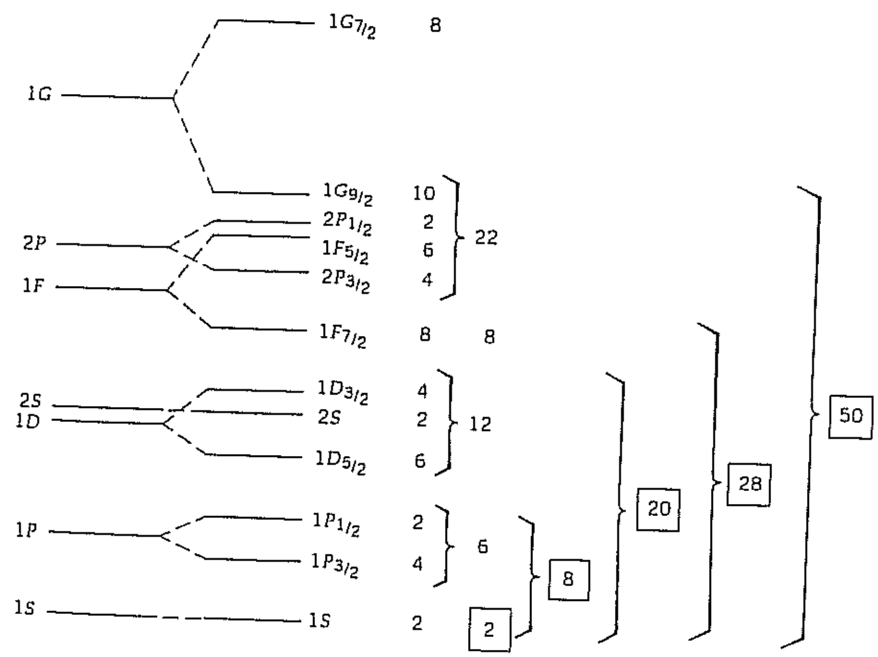
$$\vec{J}^2 = \vec{L}^2 + \vec{S}^2 + 2\vec{L}\cdot\vec{S} \quad \Rightarrow \quad \vec{L}\cdot\vec{S} = \frac{1}{2}\left(\vec{J}^2 - \vec{L}^2 - \vec{S}^2\right)$$

$$\Longrightarrow \quad \left\langle \vec{L}\cdot\vec{S} \right\rangle = \frac{1}{2}\left[j(j+1) - l(l+1) - s(s+1)\right]\hbar^2 = \frac{1}{2}l\hbar^2 \qquad (j = l + \frac{1}{2})$$

$$= -\frac{1}{2}l(l+1)\hbar^2 \quad (j = l - \frac{1}{2})$$

(as before)

As stated in the last lecture (and as visible on the energy level diagrams) it is the $j = l + 1/2$ state that is the lower energy state.



From atomic physics we know that the pairing between atomic electrons, energy levels always contain an even number of electrons. Since there are $2l+1$ states for any value of l, there are $4l+2$ fermions is a shell with orbital angular momentum l.

s-shell ($l = 0$) is occupied by 2 fermions

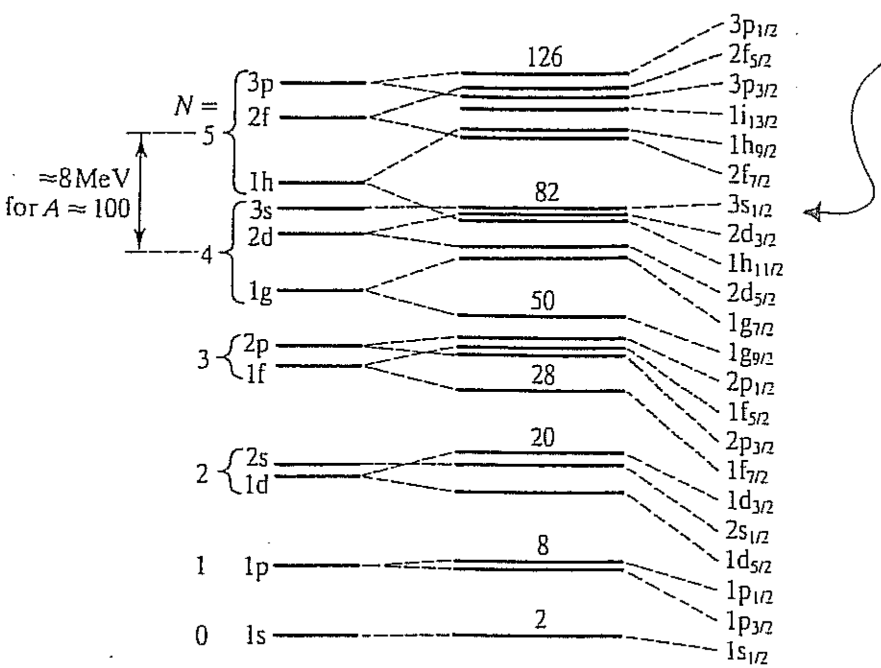
p-shell ($l = 1$) is occupied by 6 fermions

d-shell ($l = 2$) is occupied by 10 fermions etc.......

We also know from atomic physics that in a closed shell, the electrons pair in such a way that the total angular momentum associated with a closed shell is zero.

Assume that pairing effects in the nuclear shell model also result in zero contribution to the total nuclear spin (any pair, not just closed shell). This is consistent with our earlier observation that the spin-parity of all even-even nuclei is $J^P=0^+$ (though we have not discussed the parity yet).

In this model, odd-A nuclei have spin-parity assignments that depend only on the spin-parity of the unpaired nucleon, which can be determined from the energy level diagram (assuming the level ordering is correct, which may not always be the case.)



Level ordering may not be correct in densely populated energy bands, even though magic numbers are reproduced

What about parity ?

Recall that the parity of a composite system is the product of the parity of the constituents times some factor that accounts for the orbital angular momentum.

Nucleons have (by definition) positive parity (since we decided some time ago that quarks would get positive parity and anti-quarks would get negative parity).

Thus for a nucleus, the parity is given by the product of the parities of all the single-particle wave-functions.

From this, it follows in the Shell Model that all even-even nuclei have positive parity

[since each level contains an even number of particle with the same l and hence the same parity. $(-1)^n = (+1)^n = 1$ for even n]

So, let's draw some conclusions:

✓ the spin of all even-even nuclei is 0

✓ the spin of the ground state of any odd-A nucleus is the of the unpaired nucleon

✓ the parity of the ground state of all even-even nuclei is even.

✓ the parity of all odd-A nuclei is that of the wave-function of the unpaired nucleon

The first of these statements is always correct. The rest are correct, but rely on knowledge of the correct level ordering, which may not always be known. Just because we get the magic numbers correct, doesn't mean there are no other problems

Energy level notation in nuclear physics

It is useful to have a shorthand notation for the energy levels filled by the nucleons in a given nucleus.

Recall that we use letters to denote states of particular orbital angular momentum:

s, d, f, g, h, i for l = 0, 1, 2, 3, 4, 5 respectively.

Since the orbital angular momentun can couple with the spin to form $j = l \pm 1/2$ and since these levels are split by the spin-orbit interaction, we denote these separately with a subscript.

So for instance, for $l = 2$ we have either $d_{3/2}$ or $d_{5/2}$ with the latter coming first in the energy ordering.

To describe the state of a particular nucleus we can write the configuration either for protons or neutrons (or separately for both, just to be clear....each populates its own set of energy levels.

Consider ${}^{17}_{8}\mathrm{O}$, which has one neutron outside a closed shell. This configuration of neutrons is denoted:

$$\left[(1s_{1/2})^2 (1p_{3/2})^4 (1p_{1/2})^2\right](1d_{5/2})^1$$

Where the square brackets denote a closed shell

Let's dissect this:

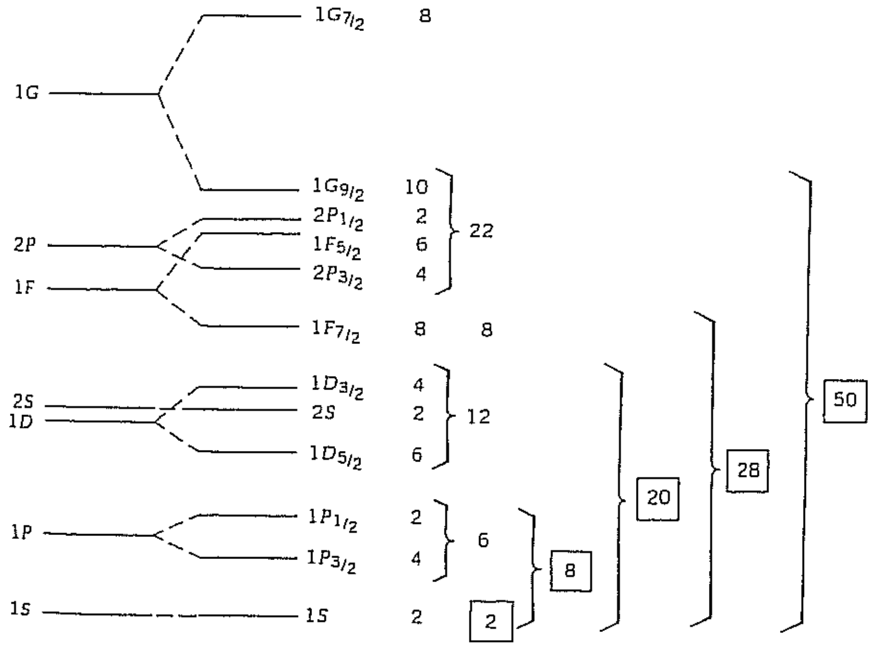
Populations of each energy level

$$\left[(1s_{1/2})^2(1p_{3/2})^4(1p_{1/2})^2\right](1d_{5/2})^1$$

$p_{3/2}$ comes before $p_{1/2}$

square brackets for closed shell

Energy level ordering is the one associated with the potential used to explain the magic numbers:



Based on our model, expect ${}^{17}_{8}\mathrm{O}$ to have $J^P = 5/2^+$

Spin-parity assignments for light nuclei

Nucleus	Binding energy (MeV)	Binding energy of last nucleon (MeV)	Binding energy per nucleon (MeV)	Spin and parity
$^{2}_{1}H$	2.22	2.2	1.1	1^+
$^{3}_{1}H$	8.48	6.3	2.8	$\frac{1}{2}^+$
$^{4}_{2}He$	28.30	19.8	7.1	0^+
$^{5}_{2}He$	27.34	−1.0	5.5	$\frac{3}{2}^-$
$^{6}_{3}Li$	31.99	4.7	5.3	1^+
$^{7}_{3}Li$	39.25	7.3	5.6	$\frac{3}{2}^-$
$^{8}_{4}Be$	56.50	17.3	7.1	0^+
$^{9}_{4}Be$	58.16	1.7	6.5	$\frac{3}{2}^-$
$^{10}_{5}B$	64.75	6.6	6.5	3^+
$^{11}_{5}B$	76.21	11.5	6.9	$\frac{3}{2}^-$
$^{12}_{6}C$	92.16	16.0	7.7	0^+
$^{13}_{6}C$	97.11	5.0	7.5	$\frac{1}{2}^-$
$^{14}_{7}N$	104.66	7.6	7.5	1^+
$^{15}_{7}N$	115.49	10.8	7.7	$\frac{1}{2}^-$
$^{16}_{8}O$	127.62	12.1	8.0	0^+
$^{17}_{8}O$	131.76	4.1	7.8	$\frac{5}{2}^+$

Note that much of our earlier discussion applied to medium and heavy nuclei. Arguments about the spin-parity of various nuclei rely on us getting the level-ordering correct.

Look at some other examples:

$${}^{13}_{6}\mathrm{C} \quad (1s_{1/2})^2(1p_{3/2})^4(1p_{1/2})^1 \qquad J^P = 1/2^- \quad \checkmark$$

$${}^{33}_{16}\mathrm{S} \quad (1s_{1/2})^2(1p_{3/2})^4(1p_{1/2})^2(1d_{5/2})^6(2s_{1/2})^2(1d_{3/2})^1 \qquad J^P = 3/2^+ \quad \checkmark$$

$${}^{47}_{22}\mathrm{Ti} \quad (1s_{1/2})^2(1p_{3/2})^4(1p_{1/2})^2(1d_{5/2})^6(2s_{1/2})^2(1d_{3/2})^4(1f_{7/2})^5 \qquad J^P = 7/2^- \quad \times$$

[The measured value is $J^P = 5/2^-$]

Note though that this last state has more than one extra (missing) nucleons, since the $f_{7/2}$ should contain 8 nucleons, and here has only 5. The agreement for such nuclei can be improved by making additional assumptions about how nucleons in *partially* filled shells interact with one another.

Nuclear Magnetic Moments

As was the case (as we saw) for protons and neutrons, nuclei will have magnetic moments that are related to their total angular momentum J, so we expect that the shell model arguments that we have just applied to the J^P of certain nuclear states may also yield information about nuclear magnetic moments.

Recall for a particle of mass *m* and charge e we had $\vec{\mu} = g \frac{e}{2mc} \vec{J}$

where g is the Landé *g*-factor. For *m* = the nucleon mass, the quantity *e/2mc* is the nuclear magneton μ_N .

Given our (successful) assumptions about pairing effects on the total angular momentum, we might assume that the nuclear magnetic moment may be obtained from the magnetic moments of any unpaired constituents, plus contributions from any orbital angular momentum associated with an unpaired proton (since protons are charged). The proton and neutron magnetic moments are well known:

$$\mu_p = 2.79\mu_N \qquad \mu_n = -1.91\mu_N$$

Unpaired protons will contribute an amount $\mu_N l$

Can test this assumption by looking at the deuteron: assume that the proton and neutron are in an $1s_{1/2}$ state so $l = 0$ and we get the magnetic moment from the sum of the proton and neutron magnetic moments:

$$\mu_d = 2.79\mu_N - 1.91\mu_N = 0.88\mu_N$$

This is approximately correct. The true value is 0.86 μ_N . The difference can be accounted for by adding a small admixture (~4%) of $l = 2$ to the deuteron wave-function (this is needed for other reasons as well, though we have not discussed this)

Now consider $^{10}_{5}\mathrm{B}$. This has shell structure $(1s_{1/2})^2 (1p_{3/2})^3$ for both the protons and the neutrons so there is one unpaired neutron and one unpaired proton. The unpaired proton has $l = 1$ so contributes an amount μ_N so the total (predicted) magnetic moment is:

$$\mu_d = 2.79\mu_N - 1.91\mu_N + \mu_N = 1.88\mu_N$$

Which compares favourably with the measured value of 1.80 μ_N .

Let's consider more generally odd-*A* nuclei. We expect no contributions except from the unpaired nucleon which has two contributions, one from its intrinsic spin and another from it's orbital angular momentum (for protons only).

We can write the orbital contribution as $\mu_l = g_l l$ with $g_l =$ either 0 (n) or 1 (p)

and the the spin contribution as $\mu_s = g_s s$ with $g_s =$ -3.826 (n) or 5.586 (p)

and we can write $\mu = g_j j$ where μ is the magnetic moment in units of the nuclear magneton and j is the nucleons total angular momentum, with

$$g_j = g_s \frac{j(j+1) - l(l+1) + s(s+1)}{2j(j+1)} + g_l \frac{j(j+1) - l(l+1) - s(s+1)}{2j(j+1)}$$

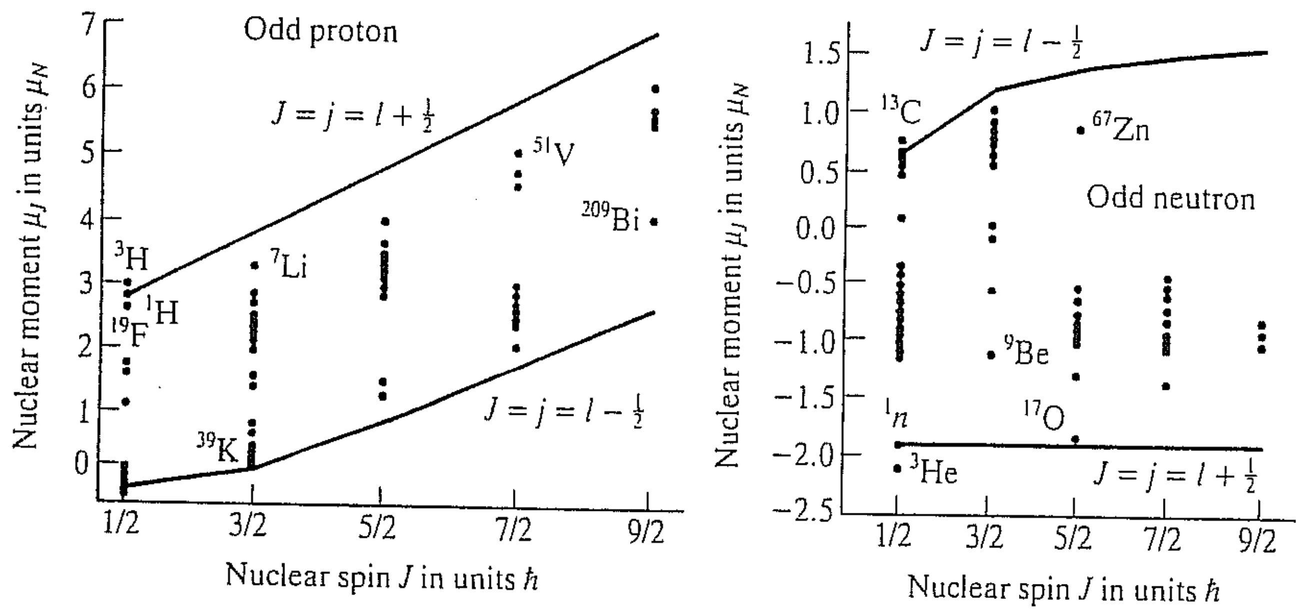
Values calculated using this method are shown on the next slide for some light nuclei

Shell Model predictions for magnetic moments of light nuclei:

Nucleus	Odd nucleon type and configuration		Nuclear spin-parity j^P	Magnetic dipole moment nuclear magnetons Calculated	Measured
$^{3}_{1}H$	p	1 $s_{1/2}$	$1/2^+$	2.793	2.9788
$^{3}_{2}He$	n	1 $s_{1/2}$	$1/2^+$	−1.913	−2.1276
$^{7}_{3}Li$	p	1 $p_{3/2}$	$3/2^-$	3.793	3.2564
$^{9}_{4}Be$	n	1 $p_{3/2}$	$3/2^-$	−1.913	−1.1776
$^{11}_{5}B$	p	1 $p_{3/2}$	$3/2^-$	3.793	2.6885
$^{11}_{6}C$	n	1 $p_{3/2}$	$3/2^-$	−1.913	−1.0300
$^{13}_{6}C$	n	1 $p_{1/2}$	$1/2^-$	0.638	0.7024
$^{13}_{7}N$	p	1 $p_{1/2}$	$1/2^-$	−0.264	−0.3221
$^{15}_{7}N$	p	1 $p_{1/2}$	$1/2^-$	−0.264	−0.2831
$^{15}_{8}O$	n	1 $p_{1/2}$	$1/2^-$	0.638	0.7189
$^{17}_{8}O$	n	1 $d_{5/2}$	$5/2^+$	−1.913	−1.8937
$^{17}_{9}F$	p	1 $d_{5/2}$	$5/2^+$	4.793	4.7224
$^{19}_{9}F$	p	1 $d_{5/2}$	$5/2^+$	4.793	2.6288

These predictions seem to become poorer as *A*. Increases. For the higher *A* values the measured moments typically fall in between the values expected for $l = j \pm ½$. This is rather unexpected since once J^P is specified only one of these two *l* values (for the unpaired nucleon) is allowed. So there appears to be no reason for this to form one of the bounding values for the empirical results.

Measured nuclear magnetic moments



Schmidt diagrams showing the measured magnetic moments for odd-*A* nuclei as a function of the nuclear spin. The lines show the expected values for a single nucleon at the spin values $j = l \pm ½$. The cases of unpaired protons and unpaired neutrons are shown separately.

How good should the agreement actually be ? Note that angular momentum is quantized, so it is more reasonable to assume that the shell model prediction the spin-parity assignment of various states than to assume that it can predict nuclear magnetic moments:

Some comments on potential difficulties

We have assumed that bound and unbound nucleons have the same intrinsic magnetic moments. There is no real reason for this to be true.

The single particle shell model which correctly predicts the spin-parity of odd-A nuclei assumes that the nuclear spin is entirely due to the unpaired nucleon. However, the same spin-parity can be constructed from other more complicated motions in which the remaining normally paired nucleons are not so paired, and contribute to the angular momentum. In this case the actual wave-function is a superposition of all states of the same total angular momentum and parity. This is called configuration mixing. Clearly this does not inspire confidence in the single-particle calculation of the magnetic moments.

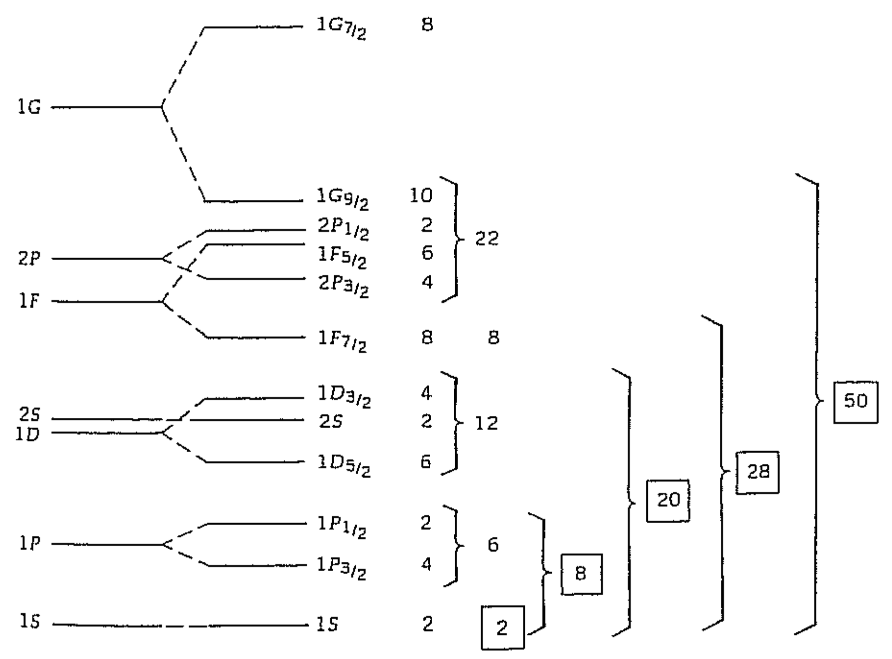
The existence of nuclear forces means that there are currents in the nucleus due to (for instance) charged pion exchange. Such currents would contribute to the magnetic moment.

More shortcomings: excited states in the shell model

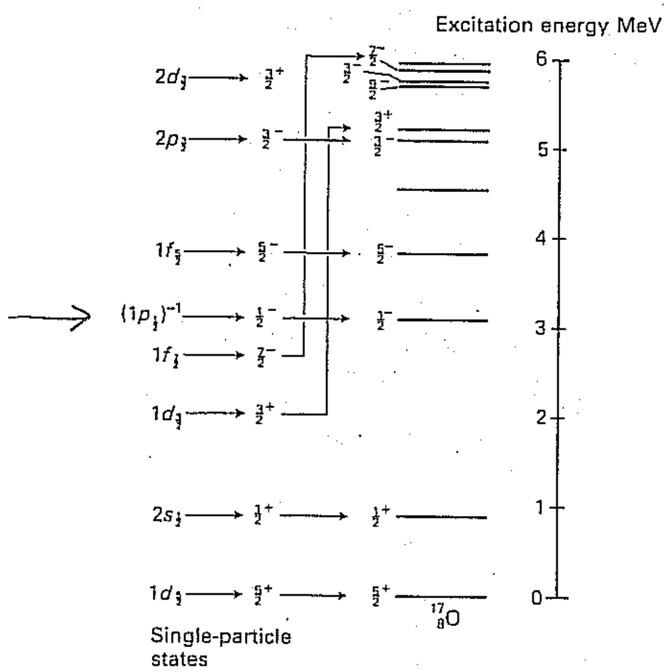
Consider a nucleus with a doubly closed shell plus one nucleon (for instance $^{17}_{8}\mathrm{O}$)

Shell model did a good job of explaining the ground state spin-parity. What about the excited states ? Recall the configuration for the neutrons is $\left[(1s_{1/2})^2(1p_{3/2})^4(1p_{1/2})^2\right](1d_{5/2})^1$

Next single-particle energy levels in the sequence are $2s_{1/2}, 1d_{3/2}, 1f_{7/2}, 2p_{3/2}$ etc, so expect that the excited states will come in the sequence $1/2^+$, $3/2^+$, $7/2^-$, $3/2^-$ etc.



Observed spectrum of excited states is more complicated indicating that there considerations other than just the properties of the unpaired nucleon.



What is the explanation for this ? One possibility is that it can be “cheaper” to create a hole than to promote a nucleon, so for instance a neutron from the 1*p* level can be promoted to pair with the previously unpaired nucleon in the outermost shell. Then there is an unpaired nucleon in the $1p_{1/2}$ shell giving an excited state with $J^P = 1/2^-$ (which is the observed J^P of the first excited state at an energy of 3.055 MeV.

One should also note that it is possible (likely) that excitations involve multiple nucleons rather than just the single unpaired nucleon