Back to nuclear form factors:

Electron scattering: electrons interact via Coulomb interaction. Gives information on the spatial extent of the nuclear charge

Neutron scattering: neutrons interact only via strong interaction. Gives information on the spatial extent of the nuclear matter.

Recall the experimental procedure:

$$F(\vec{q}^{\,2}) = \frac{1}{Ze}\int \rho(r) e^{i\vec{q}\cdot\vec{r}/\hbar} d^3 r \qquad \rho(r) = \frac{1}{(2\pi)^3}\int F(\vec{q}^{\,2}) e^{-i\vec{q}\cdot\vec{r}/\hbar} d^3 q$$

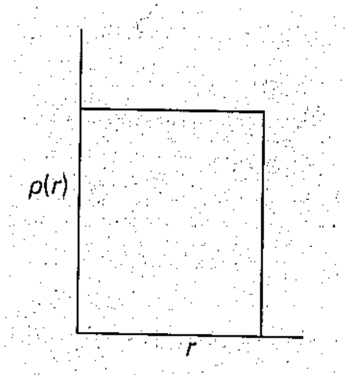
Measure this only over a finite range of $\vec{q}^{\,2}$

Cannot do this integral since need to integrate over all q

Extract nuclear charge distribution by postulating a model, caculating the corresponding form factor and comparing it to the data.

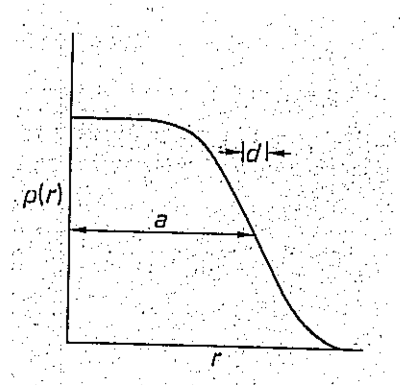
More precisely, such a model typically has free parameters that are fitted to the data over the available range of $\vec{q}^{\,2}$

Given what we know of the nucleus, two plausible models are those below:



(a)

$$\rho(r) = \rho_0 \quad r < a$$
$$0 \quad r > a$$



(b)

$$\rho(r) = \frac{\rho_0}{1 + e^{(r-a)/d}}$$ Saxon-Woods form

This model is unrealistic but calculable:

here *a* is the ½ density radius of the nucleus.

$t = (4\ln 3)d$ is the surface or skin thickness

Caculate Form Factor for Model A:

We showed in assignment 4 that for a spherically symmetric potential (e.g. a spherically symmetric charge distribution) that the form factor is given by

$$F(\vec{q}^2) = \frac{1}{Ze}\int \rho(r) e^{i\vec{q}\cdot\vec{r}/\hbar} d^3 r \longrightarrow F(\vec{q}^2) = \frac{1}{Ze}\int \rho(r)[\sin(qr/\hbar)/(qr/\hbar)] 4\pi r^2 dr$$

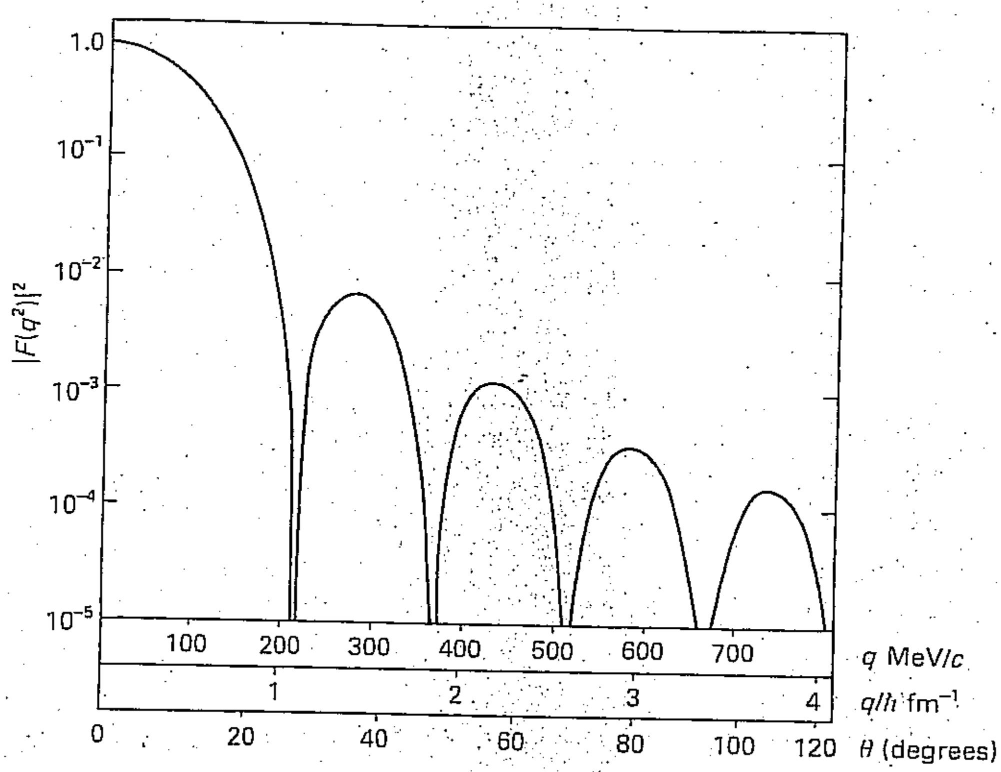
$$Ze = \int \rho(r) d^3 r = 4\pi \int \rho(r) r^2 dr$$

$$F(\vec{q}^2) = \frac{\int_0^a [\sin(qr/\hbar)/(qr/\hbar)] 4\pi r^2 dr}{\int_0^a 4\pi r^2} = \frac{\int_0^{qa/\hbar} x \sin x dx}{\int_0^{qa/\hbar} x^2 dx}$$

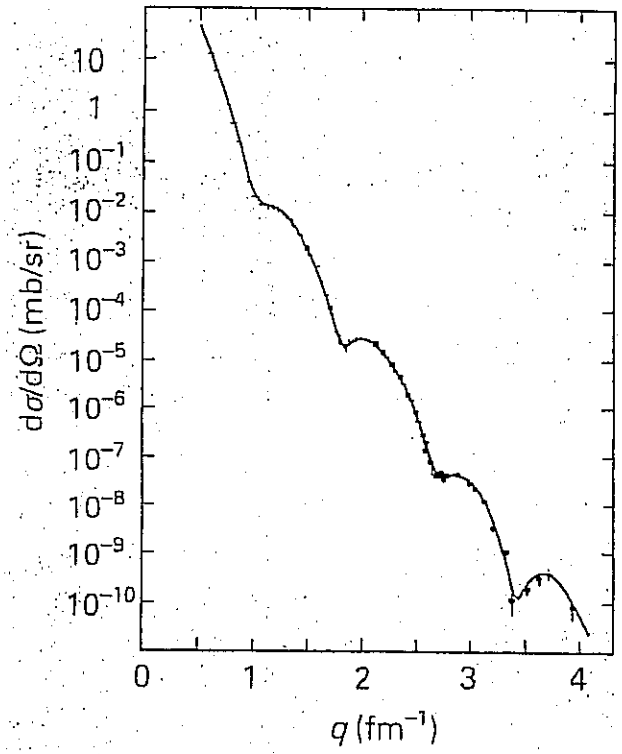
$$F(\vec{q}^2) = \frac{3\left[\sin(qa/\hbar) - (qa/\hbar)\cos(qa/\hbar)\right]}{(qa/\hbar)^3}$$

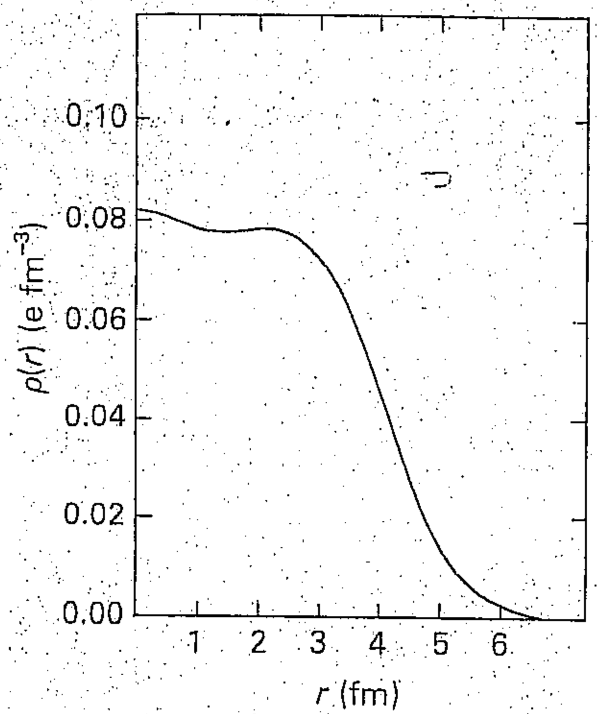
This is actually the spherical Bessel function $j_1(x)$

This is plotted below for a = 4.1 *fm* ($^{58}_{28}\mathrm{Ni}$ nucleus)



The x axis is shown in units of *q (MeV/c)*, *q/ħ* (fm^{-1}) and θ (the scattering angle for incident 450 *MeV* electrons. Zeros are so-called diffraction zeros. The fact that a real charge density distribution will cut off more gradually softens these in the experimental distribution:





Measured differential cross-section for scattering of 450 MeV electrons from $^{58}_{28}\mathrm{Ni}$.

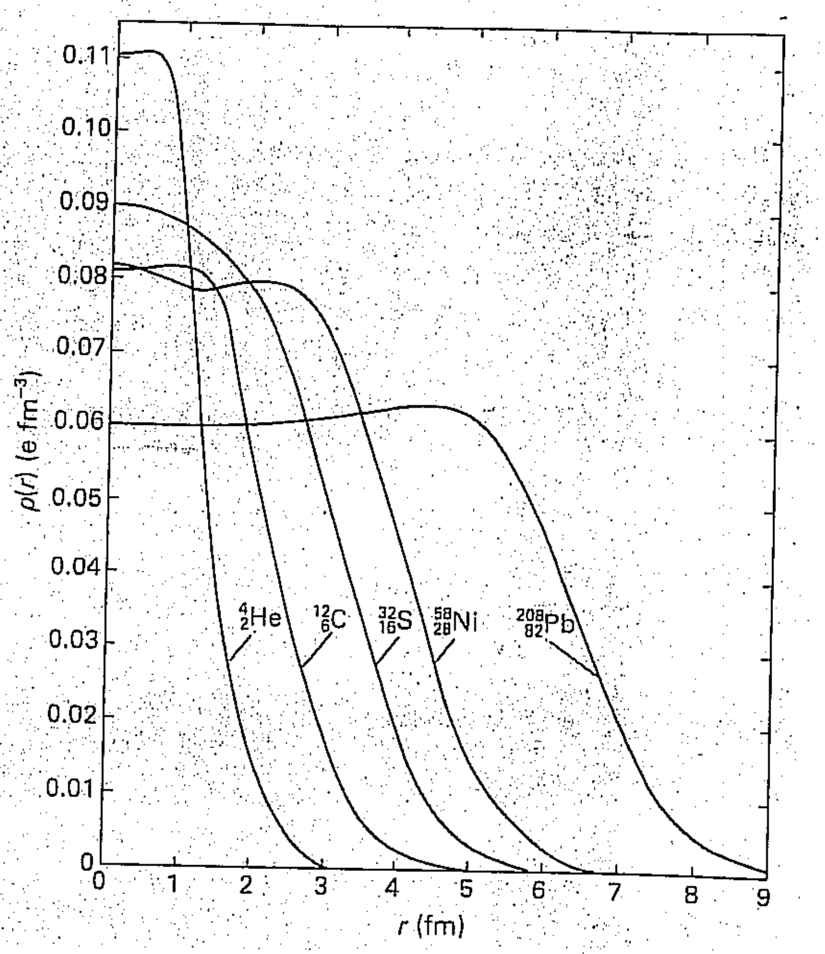
The second plot shows the charge distribution based on a model similar to the density distributions of model B.

Data from electron-nucleus scattering for a variety of nuclei give the following results from fits to the Saxon-Woods form:

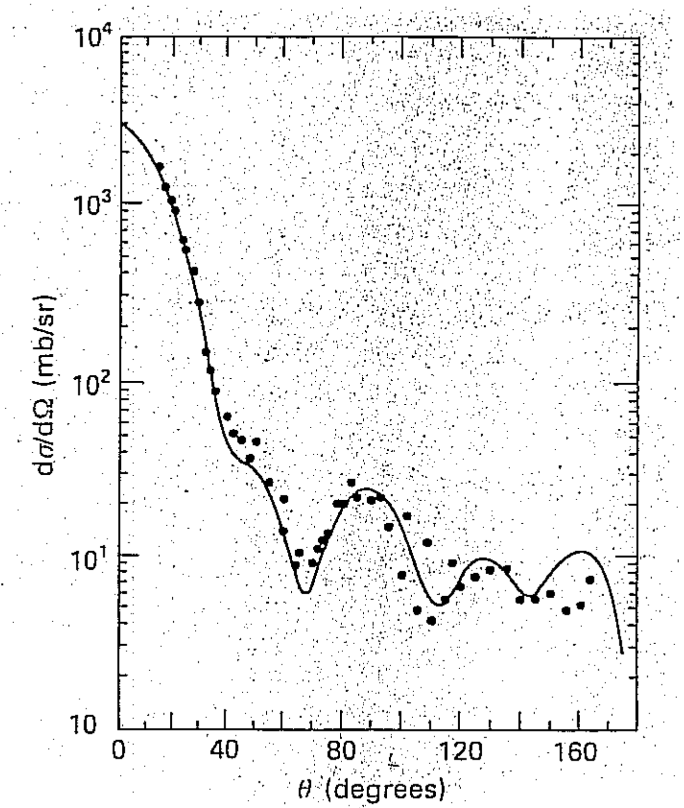
$$a = 1.18A^{1/3} - 0.48\,fm \qquad t = 2.4\,fm$$

Note that the volume depends on A, but the charge density depends on Z. We will see that as A increases the fraction Z/A decreases, so for heavy nuclei, expect the core charge density to decrease relative to lighter nuclei.

Also in lighter nuclei, the "skin" region forms a much larger fraction of the overall nuclear volume. The quoted results for the fit to the Saxon-Woods form (which from below, is clearly an approximation) are for $A > 40$.



Neutron scattering takes place via the strong interaction. More complicated since the probe is also a composite object, but on the whole expect the similar features as in electron scattering. Since the particles that carry the charge in the nucleas also carry its mass, expect the mass-density distribution to be similar in shape to the charge density distribution:



Scattering of 14 *MeV* neutrons from Ni. Solid line represents the prediction of a potential model which contains a Saxon-Woods component as well as additional parts, describing for instance, neutron absorption and the spin-orbit interaction.

Saxon-Woods parameters (from fit)

$$a = 1.2A^{1/3}\,fm$$
$$t = 3.3\,fm$$

Relatively good agreement with the results of electron-scattering

Nuclear Masses, Nuclear Stability

With very few exceptions, naturally occurring elements up to lead ($A = 209$) are stable and lie near or on a so-called "line of stability" in the (N,Z) plane where N is the number of neutrons $N = A - Z$

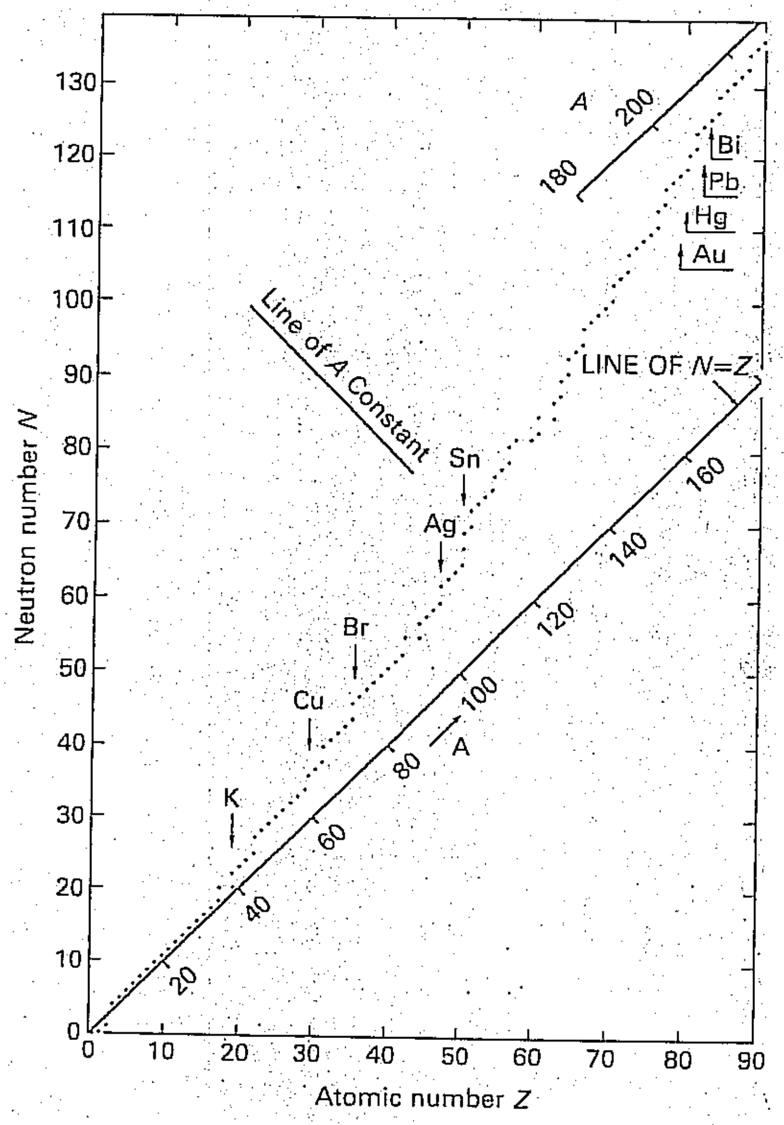
For low-A nuclei ($A < 35$) this line is $N = Z-1$, Z or $Z+1$

As A increases, the relative number of neutrons increases as the Coulomb interaction ~ Z(Z-1) begins to grow the energy.

For heavy nuclei like lead (Pb) we have $N \sim 1.5\,Z$

Other features: there are no stable nuclei with $A > 209$

All of this tells us something about nuclear interactions.



Let's first attempt to understand the masses of nuclei:

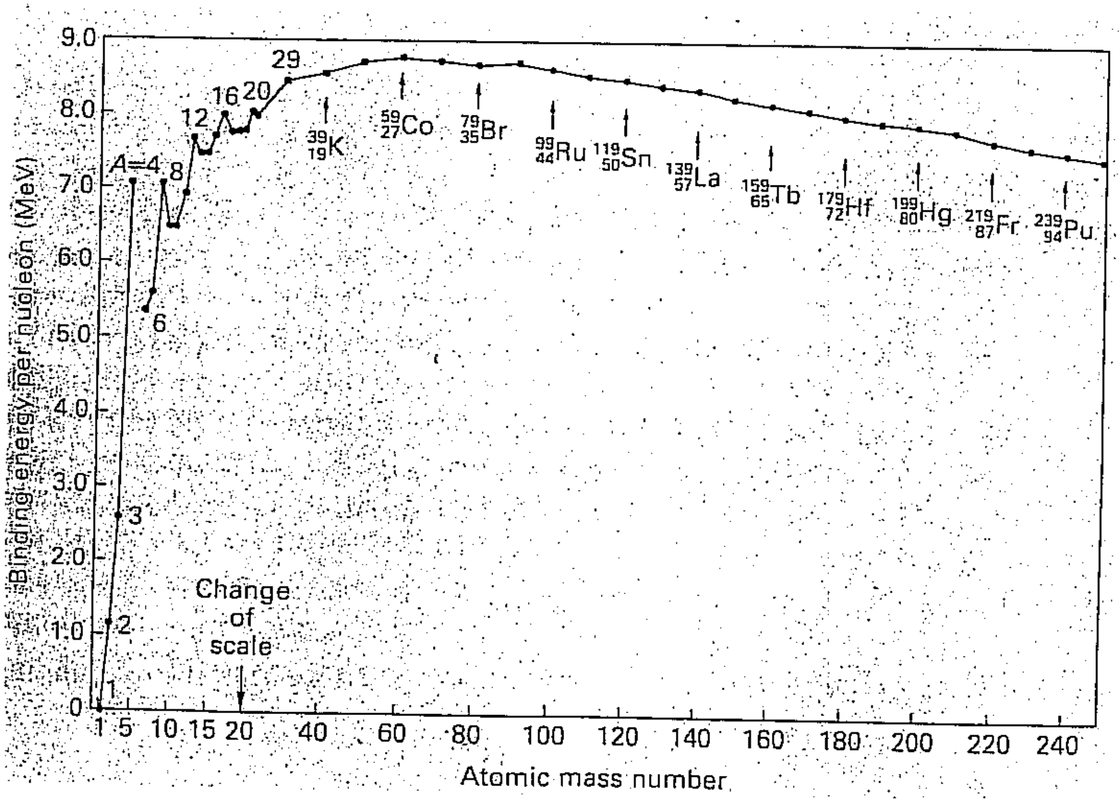
First of all, we need to understand the concept of binding energy (which should be familiar from atomic physics, and which we saw on the last assignment, for which you needed to figure out the binding energy of a deuteron).

A bound system always has a mass that is less than the mass it's constituents would have if they were free. This must be true, otherwise bound states would not form.

For a nucleus with Z protons and N neutrons ($A = Z + N$) we thus have:

$$M(Z,A)c^2 = ZM_p c^2 + NM_n c^2 - B$$

where B is the nuclear binding energy of the system. Clearly, the large the binding energy, the more stable the system. We can talk about the average binding energy per nucleon in a nucleus. That is, on average, the amount of energy required to remove one nucleon from the nucleus.



This plot shows the binding energy / nucleon for nuclei up to A = *239* (Pu). The stable nuclei are found at A values up to *209* and the naturally radioactive nuclei are found above this point, for reasons that we will get to.

Note that a maximum of B/A is reached for intermediate mass nuclei like $^{59}_{27}Co$ after which the value falls with A from the maximum value of about 8.7 MeV/nucleon down to about 7.5 MeV/nucleon for very high A.

Can we explain this behaviour with simple nuclear models ?

The <u>Liquid Drop Model</u> models the nucleus in analogy to a drop of incompressible fluid

From this, one can derive the so-called Semi-empirical Mass Formula (also known as the Weizsäcker formula) for nuclear masses.

The model assumes that the constituents of the drop (nucleus) are bound with some average energy a (they must be bound or the system would fall apart).

If a drop of fluid is incompressible, then its volume is proportional to the number of molecules n contained within it.

A non-rotating drop in the absence of external forces (gravitation, etc) will adjust its shape to minimize the (positive) energy associated with the surface tension.

The total energy of the drop is then

$$-an + 4\pi R^2 T$$

where T is the surface tension. We can write this as the binding energy of a drop containing n molecules:

$$B = an - \beta n^{2/3}$$

(β contains the constants of the surface term, other than the n dependence)

What if the drop carries an electric charge Q (as the nucleus does) ?

Remember your second-year electricity and magnetism. For an electric charge spread over a spherical surface (spherical volume) the energy is:

$$E = \frac{Q^2}{8\pi\varepsilon_0 R} \qquad \text{(surface)}$$

$$E = \frac{3Q^2}{20\pi\varepsilon_0 R} \qquad \text{(volume)}$$

This decreases the binding energy. Adding this contribution to our expression, we have

$$B = an - \beta n^{2/3} - \gamma Q^2 / n^{1/3}$$

Compare a nucleus:

- nucleus is (in our current approximation) spherical
- nucleons behave like molecules in a drop. There is a short-range attractive force and a shorter range repulsive force to prevent collapse.
- the nuclear density is constant (experimental observation)

Based on this analogy, postulate that for nucleus with A,Z the binding energy is

$$B(Z,A) = a_V A - a_S A^{2/3} - a_C Z^2 / A^{1/3}$$

a_V volume term

a_S surface term

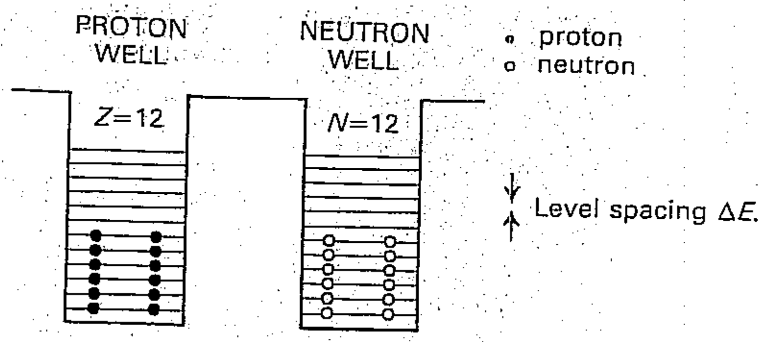
a_C Coulomb term

Does this make sense ? No, not really. Note that for fixed A it predicts that the binding energy is maximal when Z = 0. This is not very consistent with the observation that Z ~ N for lighter nuclei and 1.5 Z ~ N for heavier nuclei. So clearly we are missing something

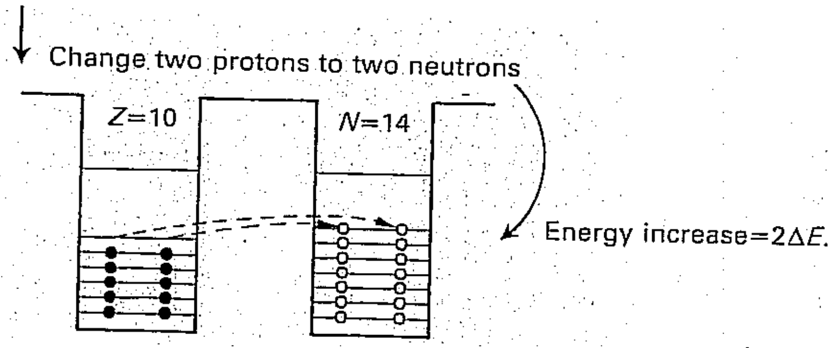
So far we have assumed that the binding energy is the same for a proton and a neutron (once the Coulomb effects are moved to a separate term).

Imagine two potential wells, each with an associated set of energy levels (one for protons and one for neutrons: these levels fill according to the Pauli-exclusion principle since the nucleons are fermions.

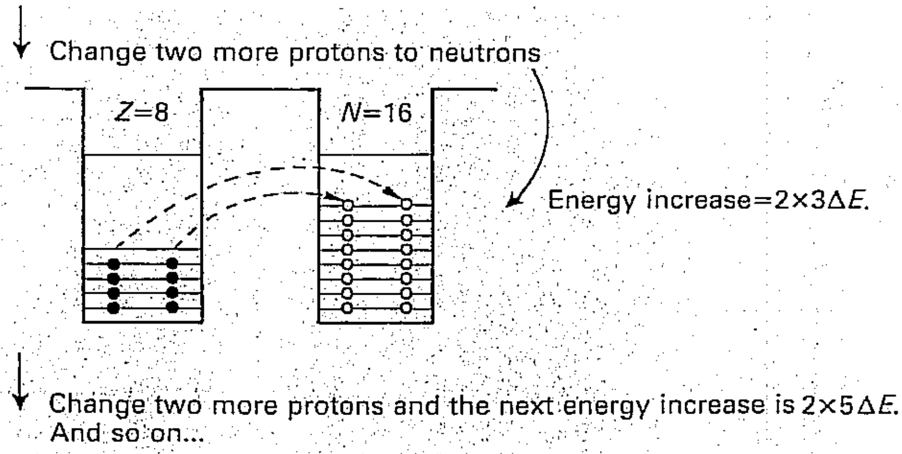
If $Z = N$, the wells are filled to the same level



If we move one step away from this scenario (in the direction of N > Z) we have



To go the next step, requires a move of three energy levels, and for the step after that it is 5 ΔE per nucleon.



Overall the numbers are (for changing a proton into a neutron, in units of ΔE)

1, 1, 3, 3, 5, 5, 7, 7, ...

The cumulative effect is 1, 2, 5, 8, 13, 18, 25, 32 ......
for N-Z = 2, 4, 6, 8, 10, 12, 14, 16, ...

This is independent of whether you are changing a proton to a neutron or vice versa, so we require an asymmetry term in the binding energy expression that accounts for this.

To change from $N\text{-}Z = 0$ to $N > Z$ with $A = Z + N$ held constant requires an additional energy of $\sim[(N-Z)^2/8]\Delta E$

So add a term which reduces the binding energy when $Z \neq N$

Energy levels in a potential well are $\propto$ (well volume) $^{-1}$ so we add a term of the form

$$-a_A(Z-N)^2/A \qquad \text{(asymmetry term)}$$

Finally, we need one last term to account for the experimental observation that the binding between two protons or two neutrons is larger that the binding between a proton and a neutron. That is, "like" nucleons pair up. For odd-A nuclei this term is zero. For even-A nuclei there are two cases:

- Z odd, N odd (oo)
- Z even, N even (ee)

•The binding energy will be greater for the (ee) case than for the (oo) case. For case (oo) subtract a term and for (ee) case add it. Empirically it has been shown that the form of this term is

$$\delta(Z,A) = a_p / A^{1/2}, \quad a_p = 12 MeV$$

So, we now have, for the nuclear binding energy:

$$B(Z,A) = a_V A - a_S A^{2/3} - a_C Z^2 / A^{1/3} - a_A (A-2Z)^2 / A + \delta(Z,A)$$

This is the semi-empirical mass formula. Values of the coefficients are found by fitting the binding energy data for medium and heavy mass nuclei:

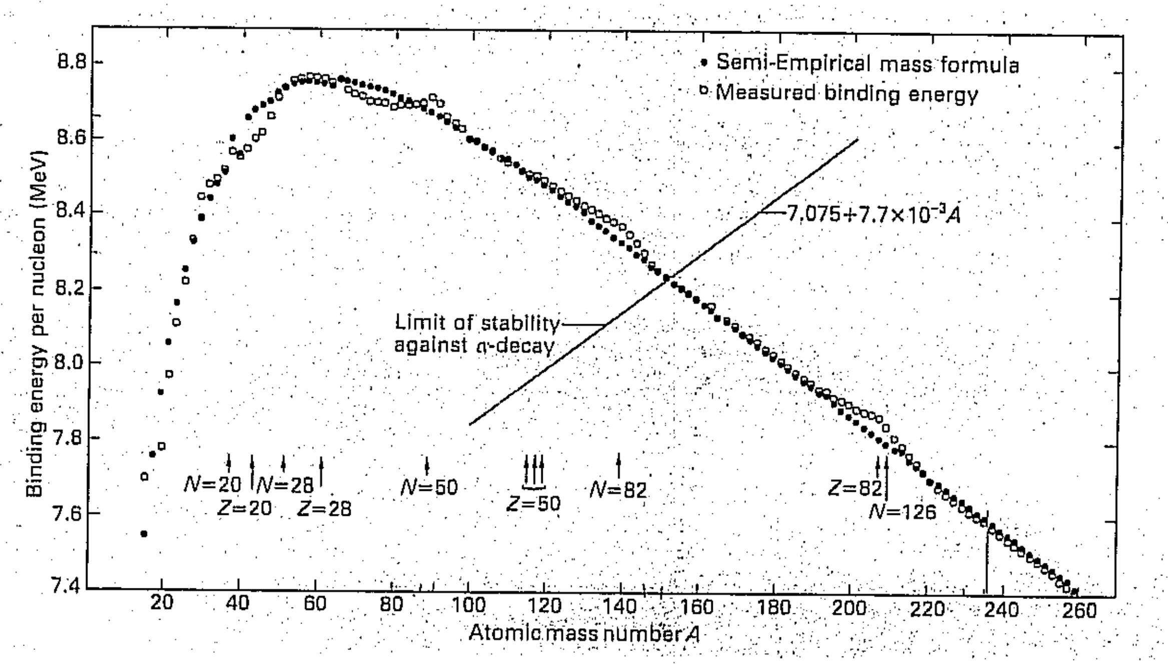
$a_V = 15.56\ MeV$

$a_S = 17.23\ MeV$

$a_C = 0.697\ MeV$

$a_A = 23.29\ MeV$

$a_P = 12.0\ MeV$



The binding energy per nucleon plotted as a function of the atomic mass number *A*.

Note the deviations at certain values of N, Z. We will come back to these later on.

The straight line shows the line of stability against α-decay, which we will also discuss later on.

The agreement between our semi-empirical formula and the data is quite good for intermediate and large values of A.

What conclusions can we draw from this ?

First, this must mean that there is some validity to the model used. In particular, the volume term is proportional to A, and not to $A(A-1) \sim A^2$ as it would be if each nucleon interacted with all other nucleons. Note the Coloumb term, which goes like Z^2 (technically this should be $Z(Z-1)$ but the best fit coefficients were extracted for Z^2).

This is evidence that indeed the nuclear force responsible for binding nucleons together is of short range.

Nuclear forces exhibit <u>saturation.</u> For a system held together by forces that saturate, the total binding energy is proportional to the total mass or to the number of constituents, except for surface effects.

Nuclear density of nucleons ~ $0.17/\text{fm}^3$ so the average separation is about 1.9 fm,

So the range of the nuclear force must be ~ 1-2 fm.