Phy489 Lecture 6

Reminder about invariant mass:

A system of *n* particles has a mass defined by $M_{INV}^2 c^2 = P_{TOT} \bullet P_{TOT}$ where P_{TOT} is the total four momentum of the system $P_{TOT} = p_1 + p_2 + p_3 + ...+ p_n$

$$M_{INV}^2 c^2 = \left(\frac{E_1 + E_2 + ...+ E_n}{c}, \vec{p}_1 + \vec{p}_2 + ...+ \vec{p}_n \right)^2 = \left(\frac{E_{TOT}}{c}, \vec{p}_{TOT} \right)^2 = \frac{E_{TOT}^2}{c^2} - \left| \vec{p}_{TOT} \right|^2$$

This mass is **invariant** regardless of whether the particles are correlated in some way. If they represent the final state particles in some decay:

$$A \rightarrow C_1 + C_2 + ...+ C_n \qquad \text{then} \qquad M_{INV}(\text{final state particles}) = M_A$$

For a scattering process $A + B \rightarrow C_1 + C_2 + ...+ C_n, \quad M_{INV}^2 c^2 = \dfrac{E_{TOT}^2}{c^2} - \left| \vec{p}_{TOT} \right|^2$

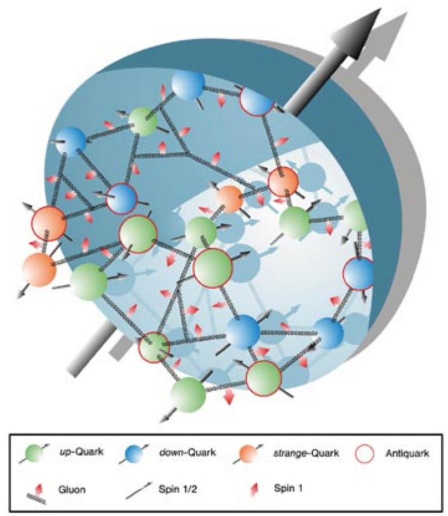
In the CM frame where $\left| \vec{p}_{TOT} \right| = 0$ this becomes

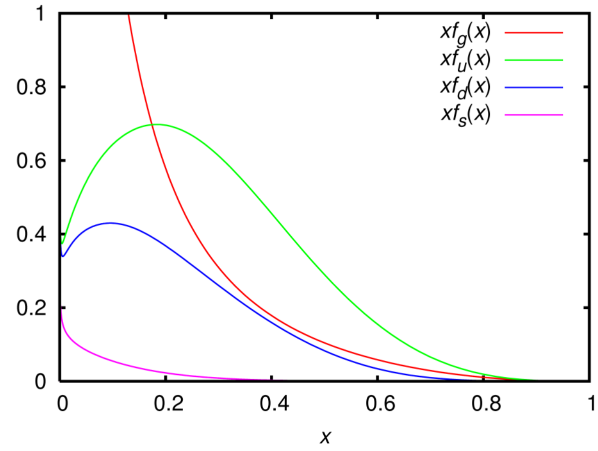
$$\frac{E_{TOT}^2}{c^2} = \frac{(E_1 + E_2 + ...+ E_n)^2}{c^2} = \left(\sqrt{\left| \vec{p}_1 \right|^2 + m_1^2 c^2} + \left(\sqrt{\left| \vec{p}_2 \right|^2 + m_2^2 c^2} \right) + ...+ \left(\sqrt{\left| \vec{p}_n \right|^2 + m_n^2 c^2} \right) \right)^2$$

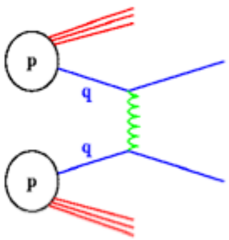
$= (m_1 + m_2 + ...+ m_n)^2 c^2$ at threshold where $\vec{p}_1 = \vec{p}_2 = ...= \vec{p}_n = 0$

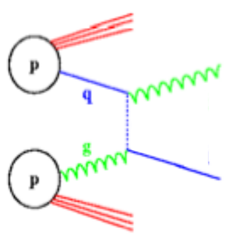
A brief aside about the LHC

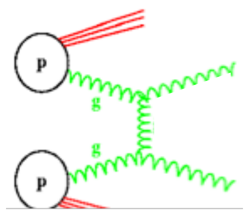
A proton is not a fundamental particle, so high energy (14TeV) collisions between protons are not really collisions between two protons, but rather collisions between the constituents of the protons, the quarks and gluons (sometimes collectively referred to as partons) each of which carries some fraction x of the total proton energy and momentum.











Since the colliding partons (qq, qg, gg) typically carry different fraction of the momentum of the two protons, the parton-parton collision is NOT typically in the CM frame, nor is the CM energy of this collision 14TeV.

LHC pp collisions continued

For the pp system we have a (square of the) centre-of-mass energy of

$$s = (p_1 + p_2)^2 = \cancel{p_1^2} + \cancel{p_2^2} + 2p_1 \bullet p_2 \approx 2p_1 \bullet p_2$$

At large energies (relevant for LHC), we can ignore the first two terms since $p_1^2 = p_2^2 = m_p^2 c^2$

For the parton-parton system let's assume that the constituents involved in the fundamental collision each carry only a fraction the total proton momentum: call these fractions x_1 and x_2. The four momenta of the two colliding partons are then $x_1 p_1$ and $x_2 p_2$ and the total four momentum of the system is $x_1 p_1 + x_2 p_2$. The corresponding invariant

$$\hat{s} = (x_1 p_1 + x_2 p_2)^2 = \cancel{x_1^2 p_1^2} + \cancel{x_2^2 p_2^2} + 2x_1 x_2 p_1 \bullet p_2$$

represents the square of the effective centre-of-mass energy of the parton-parton collision.

Here s is the square of the center-of-mass energy of the pp system while $\hat{s}$ represents the square of the centre of mass energy of the parton-parton collision, *e.g.* the energy available for production of new particles in the final state (so, in fact, the energy scale that is probed by the collision).

More on LHC pp collisions

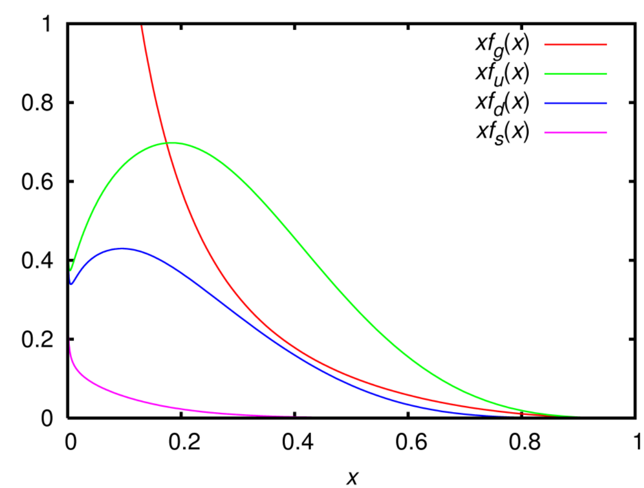
So what is the average parton-parton CM energy of an LHC pp collision?

Estimate the fraction of the proton energy and momentum which is carried by the individual partons. For a very rough estimate, just take peak values for quarks and assume qq,qg and qq collisions are equally probable:

$$\hat{x}_q = \frac{1}{3}(0.2) + \frac{1}{3}(0.2) + \frac{1}{3}(0.1) = 0.17 \qquad \hat{x}_g \sim 0.05$$

$$\langle x_1 x_2 \rangle \approx \frac{1}{3}(0.17)(0.17) + \frac{1}{3}(0.17)(0.05) + \frac{1}{3}(0.05)(0.05) = 0.013$$

$$\langle \hat{s} \rangle = .013s \quad \Rightarrow \quad \hat{E}_{CM} \sim \sqrt{.013}E_{CM} = (0.11)(14\ TeV) \approx 1.5\ TeV$$



Note that this is probably not a very reliable calculation, but it illustrates the point that the average parton-parton centre-of-mass energy at the LHC will be significantly less than the 14TeV pp CM energy.

Of course the collisions will sweep out all possible values of the effective CM energy, from 0-14TeV, but the bulk of the collisions will be at CM energies in the 1-2 TeV region (and will NOT be in the parton-parton CM frame).

Symmetries and Conservations Laws

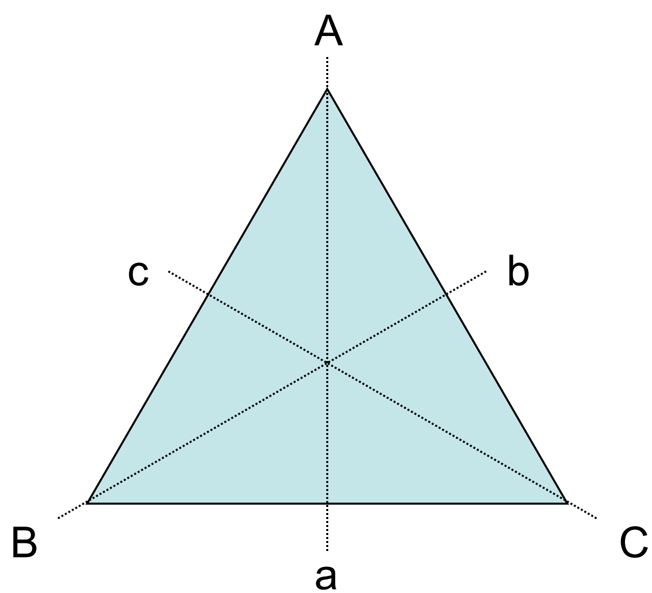
Please read through §4.1 and attempt some of problems 4.1- 4.7. We will not really discuss these issues in the lectures.

We have come across SU(3) in the context of the original quark model.

We will see SU(2) in the context of spin-1/2 particles and isospin.

In general, we will not discuss group theory, or rely much on knowledge of it.

However, discussion of the symmetry properties of the equilateral triangle in Griffiths §4.1 is a useful introduction to this:



This shape is invariant under rotations of ±120º as well as under reflections about aA, bB and cC. These are *discrete* symmetry transformations and along with the identity operation form a *group*.

A circle is symmetric under arbitrary rotations about its centre. This is called a *continuous* symmetry.

Symmetries: Noether's Theorem

Every symmetry of nature is associated to a conservation law, and *vice versa*.

- ✓ Translations in time ➔ conservation of energy
- ✓ Translations in space ➔ conservation of linear momentum
- ✓ Rotations in space ➔ conservation of angular momentum
- ✓ Gauge transformations ➔ conservation of charge

e.g. $\psi \rightarrow e^{i\alpha}\psi$

Here ψ is the electron wavefunction.

Gauge transformation is "global" if α = const, "local" if $\alpha = \alpha(x)$

Our fundamental theories (e.g. those forming the Standard Model) are all based on the principle of local gauge invariance.

A brief pause

What are we doing ? Investigating fundamental particles and their interactions, which we probe experimentally through investigations of particle scattering processes and particle decays. Bound states also yield relevant information but we will skip these for the most part. What issues do we need to consider ?

- Relativistic kinematics
- Dynamics, including “internal” symmetries (quantum numbers) which differ for the three fundamental forces. That is, conservation laws associated with these symmetries.
- Spin is also conserved. We will need to deal with this as well.

  We will deal with conservation of spin (or more generally, conservation of angular momentum) for the remainder of this lecture and the next. The tools we develop during this discussion will be useful for other things as well.

Nuclear β Decay

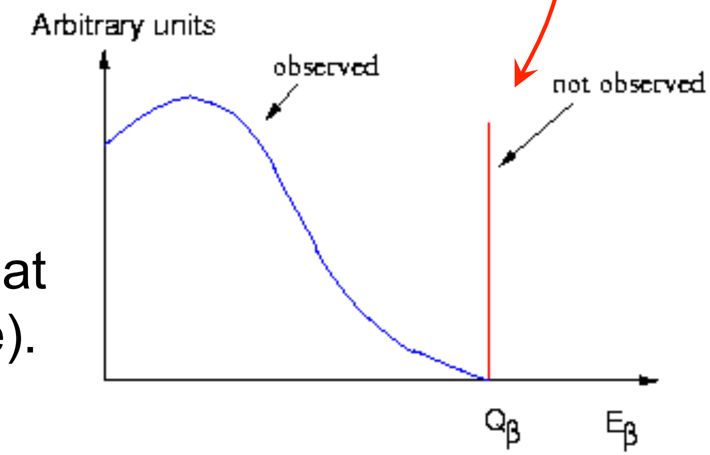
The early days of nuclear β decay are described in Griffiths §1.5

This process was initially interpreted as $n \to p + e$. If this were the case, we would expect to see mono-energetic electrons in observations of β decay processes.

See problem 3.19: for the process $A \to B + C$ $\quad E_B = \dfrac{M_A^2 + M_B^2 - M_C^2}{2M_A}$

[We did this in the last lecture]

Instead a spectrum is observed that has this value as its (high-energy) endpoint



This dilemma lead Pauli to postulate the existence of a third particle in this decay that was undetected (and perhaps undetectable).

(it was either this or give up on conservation of energy).

Fermi called this particle the neutrino.

Another view of this…..

Can also look at this issue in terms of the particle spins: $n \to p + e$

In terms of spins of the particles involved this is : $\frac{1}{2} \to \frac{1}{2} + \frac{1}{2}$

As you may know (or as we shall soon see) there is no way for a system of two spin-1/2 particles have a total spin of 1/2, so this process is forbidden by conservation of angular momentum.

[if this is not clear to you, convince yourself after we have discussed the rules for adding angular momentum vectors]

The addition of a third spin-1/2 particle to the final state (the neutrino) also solves this problem. [you should convince yourself of this as well, at some point]

Angular momentum in Quantum Mechanics

To evaluate the effects of spin and orbital angular momentum on particle collisions and decays, need to understand how to calculate the total angular momentum of a system of particles, *e.g.* how to add angular momentum vectors.

Brief review (see a Quantum Mechanics text if this is entirely unfamiliar).

Spin angular momentum is quantized in half-integer units of $\hbar$.

Orbital angular momentum is quantized in integer units of $\hbar$.

QM forbids us from simultaneously specifying more than one component of $\vec{L}$.

The best we can do is to specify the magnitude $L^2 = \vec{L} \bullet \vec{L} = \ell(\ell+1)\hbar^2$ and one component (conventionally chosen to be L_z):

$$L_z = m_\ell \hbar \qquad m_\ell = -\ell, -\ell+1, \ldots\ldots, \ell-1, \ell \qquad \textit{e.g.}\ (2\ell+1) \text{ possible values}$$

Here I am referring to the possible values that can be returned by a measurement of L^2 and/or L_z.

Intrinsic Spin

For fundamental particles, spin is an INTRINSIC property.

- Fundamental fermions are all spin 1/2 (quarks and leptons)
- Force carriers (gauge bosons) are all spin 1
- Higgs boson is spin 0

Composite particles (atoms, mesons, baryons) have a total spin *J* that has contributions from the spins of the fundamental constituents and from any relative orbital angular momentum: $\vec{J} = \vec{L} + \vec{S}$. Here $\vec{S}$ comes from the combining the intrinsic spins and $\vec{L}$ represents the total orbital angular momentum

[e.g. 1/2 ± 1/2]

Mesons are $q\bar{q}$ states with total (intrinsic) spin 0 or 1. Orbital angular momentum can be ℓ = 0,1,2,….. (in integer steps) so the total spin $\vec{J}$ can only have integer values (as we shall see). Mesons are therefore bosons:

i.e. a measurement of $J^2 = j(j+1)\hbar^2$ can yield only give yield integer values for *j*.

Corresponding argument for baryons shows that they are always fermions

[convince yourself]

Addition of Angular Momentum Vectors

Recall the rules for the addition of angular momentum vectors:

$$\vec{J} = \vec{J}_1 + \vec{J}_2$$

Here the symbol $\vec{J}$ is used to represent an *arbitrary* angular momentum vector (*i.e*. spin, orbital, or some combination thereof):

z components add, but magnitudes do not: $J_z = J_{1z} + J_{2z}$

Magnitude is $J^2 = j(j+1)\hbar^2$ where j runs from $|j_1 - j_2|$ to $|j_1 + j_2|$ in integer steps:

As an example:

$q\bar{q}$ L=0 ($\ell = 0$) ➔ spin 0: π, K, η, η', D (different quark contents)

spin 1: $\rho, K^*, \omega, \varphi, J/\psi$

For states with L > 0, we have three angular momentum vectors to add (two spins plus one orbital angular momentum). The procedure is to add two and then add the third. The order does not matter, but it normally makes the most sense to add the two spins and then to add the orbital angular momentum:

$q\bar{q}$ states with L>0 ➔ mesons with spin(s):

$$j = \ell + 1$$

$$j = \ell \quad \text{(two ways: s = 0,1)}$$

$$j = \ell - 1$$

Rules for the addition of angular momenta are well established and tabulated:

$$|\, j_1 m_1 >|\, j_2 m_2 > = \sum_{j=|j_1 - j_2|}^{j_1+j_2} \boxed{C^{j\ j_1\ j_2}_{m\ m_1\ m_2}}\, |\, jm > \qquad m = m_1 + m_2$$

Clebsch Gordan coefficients

The square of the CG coefficient gives the probability of getting a state of total spin j from a system consisting of two angular momentum states:

$$\left| j_1, m_1 \right\rangle \left| j_2, m_2 \right\rangle$$

Clesch Gordan Coefficients from PDG Listings

35. CLEBSCH-GORDAN COEFFICIENTS, SPHERICAL HARMONICS, AND d FUNCTIONS

Note: A square-root sign is to be understood over *every* coefficient, *e.g.*, for $-8/15$ read $-\sqrt{8/15}$.

Notation:

		J	J	...
		M	M	...
m_1	m_2	Coefficients		
m_1	m_2			
.	.			
.	.			

$1/2 \times 1/2$

m_1	m_2	$J=1$, $M=+1$
+1/2	+1/2	1

m_1	m_2	$J=1$, $M=0$	$J=0$, $M=0$
+1/2	−1/2	1/2	1/2
−1/2	+1/2	1/2	−1/2

m_1	m_2	$J=1$, $M=-1$
−1/2	−1/2	1

$1 \times 1/2$

m_1	m_2	$J=3/2$, $M=+3/2$
+1	+1/2	1

m_1	m_2	$J=3/2$, $M=+1/2$	$J=1/2$, $M=+1/2$
+1	−1/2	1/3	2/3
0	+1/2	2/3	−1/3

m_1	m_2	$J=3/2$, $M=-1/2$	$J=1/2$, $M=-1/2$
0	−1/2	2/3	1/3
−1	+1/2	1/3	−2/3

m_1	m_2	$J=3/2$, $M=-3/2$
−1	−1/2	1

2×1

m_1	m_2	$J=3$, $M=+3$
+2	+1	1

m_1	m_2	$J=3$, $M=+2$	$J=2$, $M=+2$
+2	0	1/3	2/3
+1	+1	2/3	−1/3

m_1	m_2	$J=3$, $M=+1$	$J=2$, $M=+1$	$J=1$, $M=+1$
+2	−1	1/15	1/3	3/5
+1	0	8/15	1/6	−3/10
0	+1	2/5	−1/2	1/10

m_1	m_2	$J=3$, $M=0$	$J=2$, $M=0$	$J=1$, $M=0$
+1	−1	1/5	1/2	3/10
0	0	3/5	0	−2/5
−1	+1	1/5	−1/2	3/10

m_1	m_2	$J=3$, $M=-1$	$J=2$, $M=-1$	$J=1$, $M=-1$
0	−1	2/5	1/2	1/10
−1	0	8/15	−1/6	−3/10
−2	+1	1/15	−1/3	3/5

m_1	m_2	$J=3$, $M=-2$	$J=2$, $M=-2$
−1	−1	2/3	1/3
−2	0	1/3	−2/3

m_1	m_2	$J=3$, $M=-3$
−2	−1	1

1×1

m_1	m_2	$J=2$, $M=+2$
+1	+1	1

m_1	m_2	$J=2$, $M=+1$	$J=1$, $M=+1$
+1	0	1/2	1/2
0	+1	1/2	−1/2

m_1	m_2	$J=2$, $M=0$	$J=1$, $M=0$	$J=0$, $M=0$
+1	−1	1/6	1/2	1/3
0	0	2/3	0	−1/3
−1	+1	1/6	−1/2	1/3

m_1	m_2	$J=2$, $M=-1$	$J=1$, $M=-1$
0	−1	1/2	1/2
−1	0	1/2	−1/2

m_1	m_2	$J=2$, $M=-2$
−1	−1	1

$$Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_1^1 = -\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi}$$

$$Y_2^0 = \sqrt{\frac{5}{4\pi}}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$$

$$Y_2^1 = -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{i\phi}$$

$$Y_2^2 = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta\, e^{2i\phi}$$

$3/2 \times 1$

m_1	m_2	$J=5/2$, $M=+5/2$
+3/2	+1	1

m_1	m_2	$J=5/2$, $M=+3/2$	$J=3/2$, $M=+3/2$
+3/2	0	2/5	3/5
+1/2	+1	3/5	−2/5

m_1	m_2	$J=5/2$, $M=+1/2$	$J=3/2$, $M=+1/2$	$J=1/2$, $M=+1/2$
+3/2	−1	1/10	2/5	1/2
+1/2	0	3/5	1/15	−1/3
−1/2	+1	3/10	−8/15	1/6

m_1	m_2	$J=5/2$, $M=-1/2$	$J=3/2$, $M=-1/2$	$J=1/2$, $M=-1/2$
+1/2	−1	3/10	8/15	1/6
−1/2	0	3/5	−1/15	−1/3
−3/2	+1	1/10	−2/5	1/2

m_1	m_2	$J=5/2$, $M=-3/2$	$J=3/2$, $M=-3/2$
−1/2	−1	3/5	2/5
−3/2	0	2/5	−3/5

m_1	m_2	$J=5/2$, $M=-5/2$
−3/2	−1	1

$2 \times 1/2$

m_1	m_2	$J=5/2$, $M=+5/2$
+2	+1/2	1

m_1	m_2	$J=5/2$, $M=+3/2$	$J=3/2$, $M=+3/2$
+2	−1/2	1/5	4/5
+1	+1/2	4/5	−1/5

m_1	m_2	$J=5/2$, $M=+1/2$	$J=3/2$, $M=+1/2$
+1	−1/2	2/5	3/5
0	+1/2	3/5	−2/5

m_1	m_2	$J=5/2$, $M=-1/2$	$J=3/2$, $M=-1/2$
0	−1/2	3/5	2/5
−1	+1/2	2/5	−3/5

m_1	m_2	$J=5/2$, $M=-3/2$	$J=3/2$, $M=-3/2$
−1	−1/2	4/5	1/5
−2	+1/2	1/5	−4/5

m_1	m_2	$J=5/2$, $M=-5/2$
−2	−1/2	1

$3/2 \times 1/2$

m_1	m_2	$J=2$, $M=+2$
+3/2	+1/2	1

m_1	m_2	$J=2$, $M=+1$	$J=1$, $M=+1$
+3/2	−1/2	1/4	3/4
+1/2	+1/2	3/4	−1/4

m_1	m_2	$J=2$, $M=0$	$J=1$, $M=0$
+1/2	−1/2	1/2	1/2
−1/2	+1/2	1/2	−1/2

m_1	m_2	$J=2$, $M=-1$	$J=1$, $M=-1$
−1/2	−1/2	3/4	1/4
−3/2	+1/2	1/4	−3/4

m_1	m_2	$J=2$, $M=-2$
−3/2	−1/2	1

$$Y_\ell^{-m} = (-1)^m Y_\ell^{m*}$$

$$d^{\ell}_{m,0} = \sqrt{\frac{4\pi}{2\ell+1}}\, Y_\ell^m\, e^{-im\phi}$$

$$\langle j_1 j_2 m_1 m_2 | j_1 j_2 J M\rangle = (-1)^{J-j_1-j_2}\langle j_2 j_1 m_2 m_1 | j_2 j_1 J M\rangle$$

Griffiths Example 4.3

An electron occupies the orbital state $|2,-1\rangle$ and the spin state $|1/2,1/2\rangle$. What are the possible outcomes of a measurement of J^2 and with what probabilities do they occur?

Remember, z components add: $m=-1+1/2=-1/2$

Magnitude (in terms of *j*) is either 3/2 or 5/2 [*i.e.* 2 ± 1/2]

get these coefficients from CG tables

$$|2,-1\rangle\left|\frac{1}{2},\frac{1}{2}\right\rangle = A\left|\frac{5}{2},-\frac{1}{2}\right\rangle + B\left|\frac{3}{2},-\frac{1}{2}\right\rangle$$

$$|j_1,m_1\rangle|j_2,m_2\rangle$$

Actually, don't even need to work out the possible values of j. The tables do this for you as well. There is a separate table for each possible j_1, j_2 combination: here we need the 2 x 1/2 table

Clebsh-Gordan Coefficients for 2x1/2

for $-8/15$ read $-\sqrt{8/15}$.

Notation:

		J	J	...
		M	M	...
m_1	m_2	Coefficients		
m_1	m_2			
.	.			
.	.			
.	.			

$2\times 1/2$

m_1 m_2	5/2 +5/2
+2 +1/2	1

m_1 m_2	5/2 +3/2	3/2 +3/2
+2 −1/2	1/5	4/5
+1 +1/2	4/5	−1/5

m_1 m_2	5/2 +1/2	3/2 +1/2
+1 −1/2	2/5	3/5
0 +1/2	3/5	−2/5

m_1 m_2	5/2 −1/2	3/2 −1/2
0 −1/2	3/5	2/5
−1 +1/2	2/5	−3/5

m_1 m_2	5/2 −3/2	3/2 −3/2
−1 −1/2	4/5	1/5
−2 +1/2	1/5	−4/5

m_1 m_2	5/2 −5/2
−2 −1/2	1

$m_1, m_2 = -1, 1/2$

$$|2,-1\rangle\left|\frac{1}{2},\frac{1}{2}\right\rangle = \sqrt{\frac{2}{5}}\left|\frac{5}{2},-\frac{1}{2}\right\rangle - \sqrt{\frac{3}{5}}\left|\frac{3}{2},-\frac{1}{2}\right\rangle$$

Negative sign from -3/5 in table

Probability of measuring *j = 5/2 is 2/5*
Probability of measuring *j = 3/2 is 3/5*
sum to 1 as required

Additional Problems

4.11

In the decay $\Delta^{++} \rightarrow p\pi^{+}$ what are the possible values of the orbital angular momentum quantum number, ℓ, in the final state?

4.12

An electron in a hydrogen atom is in a state with orbital angular momentum number $\ell = 1$. If the total angular momentum quantum number is *j=3/2*, and the *z* component of total angular momentum is $\hbar/2$ what is the probability of finding the electron with $m_s = +1/2$? (next slides).

See also problems 4.13, 4.14: There are similar problems on the first assignment, changed slightly to make them different than the problems in the text.

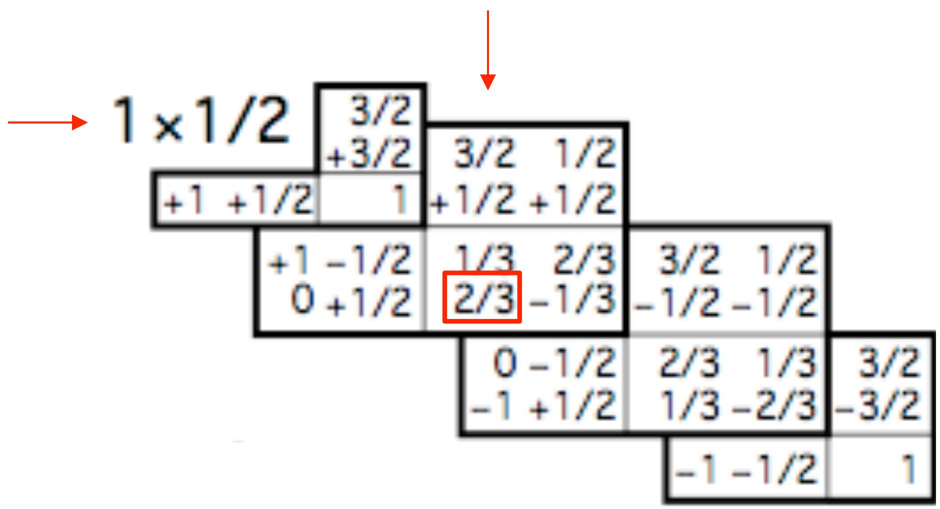
Try these. We can discuss them next time if you have difficulties.

Griffiths Problem 4.12

4.12

An electron in a hydrogen atom is in a state with orbital angular momentum number $\ell = 1$. If the total angular momentum quantum number is *j=3/2*, and the *z* component of total angular momentum is $\hbar / 2$ what is the probability of finding the electron with $m_s = +1/2$?

Need ½ x 1 Clebsch-Gordan Table (spin ½ + $\ell = 1$)



Probability is thus 2/3.