Computational complexity of the landscape, open string flux vacua, D-brane ground states, multicentered black holes, S-duality, DT/GW correspondence, and the OSV conjecture

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Banff, February 14, 2006

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Fun with fluxes

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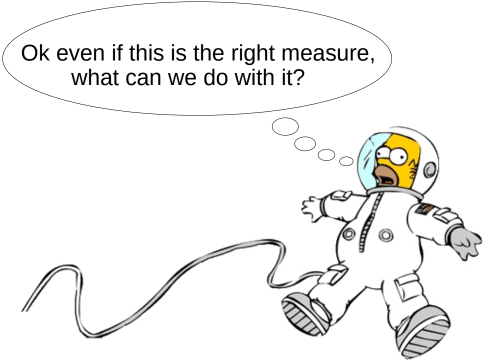
Outline

Computational complexity of the landscape

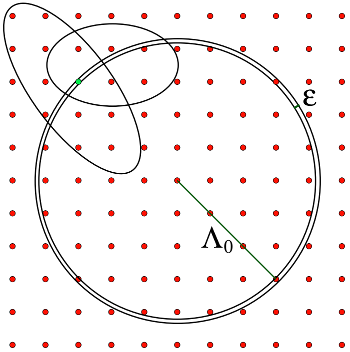
The landscape of open string flux vacua and OSV at large g_{top}

A general derivation of OSV


Ok even if this is the right measure,
what can we do with it?

Computational complexity of the landscape

Basic landscape problem: matching data



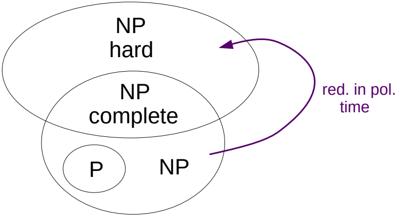
E.g. cosmological constant in Bousso-Polchinski model:

$$\Lambda(N) = -\Lambda_0 + g_{ij} N^i N^j$$

with flux $N \in \mathbb{Z}^K$. Example question: $\exists N : 0 < \Lambda(N) < \epsilon$?

Can be extended to more complicated models, other parameters, ...

Basic complexity classes



- P = yes/no problems solvable in polynomial time (e.g. is $n_1 \times n_2 = n_3$?, primality)
- NP = problems for which a candidate solution can be *verified* in polynomial time (e.g. subset sum: given finite set of integers, is there subset summing up to zero?)
- NP-hard = loosely: problem at least as hard as *any* NP problem, i.e. *any* NP problem can be reduced to it in polynomial time.
- NP-complete = NP $\cap$ NP-hard (e.g. subset-sum, 3-SAT, traveling salesman, $n \times n$ Sudoku, ...)

So: if *one* NP-complete problem turns out to be in P, then NP = P.
Widely believed: NP $\neq$ P, but no proof to date (Clay prize problem).
Therefore: expect no P algorithms for NP-complete problems.

Complexity of BP

Clear: BP $\in$ NP

Bad news: BP is NP-complete

Proof: by mapping version of subset sum to it.

Intuition: exponentially many local minima for local relaxation $\Delta N_i = \pm\delta_{ki}$, already for $g_{ij} \equiv g_i \delta_{ij}$:

$$|\Delta\Lambda| = g_k |1 \pm 2N^k| > g_k.$$

$\Rightarrow$ any $|\Lambda| < \min_k g_k/2$ is local minimum, but if $\epsilon \ll \min_k g_k$, still very far from target range.

Simulated annealing: add thermal noise to get out of local minima and gradually cool.

$\rightsquigarrow$ converges to Boltzmann distribution, so will always find target range, but only guaranteed in time exponential in $K + \log\epsilon$.

Prospects for solving NP-hard problems

- Parallel processing? (P) ✗
- Classical polynomial time probabilistic algorithms? (BPP) ✗
- Polynomial time quantum computing? (BQP) ✗
- Other known physical models of computation? ✗

Sharp selection principles based on optimization

Example: HH measure selects smallest positive Λ with overwhelming probability. $\Rightarrow$ No need to match data, just find and predict.

Problem: finding minimal $\Lambda(N)$ in BP is even harder than NP-complete! (is in DP, i.e. conjunction of NP and co-NP)

Caveats and indirect approaches

- NP-completeness is asymptotic, worst case notion. Particular instances may turn out easy. Cryptographic codes do get broken.
- String theory may have much more (as yet hidden) structure and underlying simplicity than current landscape models suggest. $\rightsquigarrow$ extra motivation to find this.
- As in statistical mechanics, one could hope to compute probabilities on low energy parameter space without need for exact construction of corresponding microstates.
- Already without dynamics, number distribution estimates together with experimental input could lead to virtual exclusion of certain future measurable properties. [Douglas et al]
- As about 20,000 Google hits note: We are humans, not computers!

String vacuum factory, A.D. 2024

For the time being: other applications of techniques developed for analyzing the landscape?

The landscape of open string flux vacua and OSV at large g_{top}

OSV for D4

Consider a D4-brane wrapped on a divisor $P = p^A D_A$ and define

$$\mathcal{Z}_{osv}(\phi^0, \Phi^A) = \sum_{q_0, q_A} \Omega(q_0, q_A)\, e^{-2\pi\phi^0 q_0 - 2\pi\Phi^A q_A}$$

where $\Omega(q_0, q_A)$ is index of BPS states with D0-charge q_0 and D2-charges q_A.

[Ooguri-Strominger-Vafa] conjectured:

$$\mathcal{Z}_{osv}(\phi^0, \Phi^A) \sim Z_{top}(\lambda, t)\, \overline{Z_{top}(\lambda, t)}$$

with substitutions:

$$\lambda \to \frac{2\pi}{\phi^0}, \qquad t^A \to \frac{-ip^A/2 + \Phi^A}{\phi^0}$$

Relation to flux vacua

BPS states realized as single smooth D4 wrapped on P with $U(1)$ flux F turned on and N pointlike (anti-)D0 branes bound to it:

$$q_A = D_A \cdot F, \qquad q_0 = -N + F^2/2 + \chi(P)/24$$

where $\chi(P) = P^3 + c_2 \cdot P$ = Euler characteristic P.

Susy condition [MMMS]:

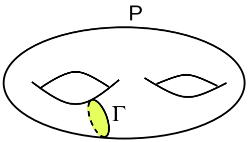
$$F^{2,0} = 0$$

$\rightsquigarrow$ puts constraints on divisor deformation moduli, freezing to isolated points for sufficiently generic F [MGM, S et al] $\rightsquigarrow$ open string flux vacua

Because in general $H^2(P) \gg H^2(X)$, there are many different (F, N) giving same (q_0, q_A). Each sector gives moduli space $\mathcal{M}_{P,F,N}$ of divisors deformations + D0-positions, and

$$\Omega(q_0, q_A) = \sum_{F,N \Leftrightarrow q_0, q_A} \chi(\mathcal{M}_{P,F,N})$$

Superpotential and $\mathcal{N} = 1$ special geometry structure



$F^{(2,0)} = 0 \Leftrightarrow W_F' = 0$ with

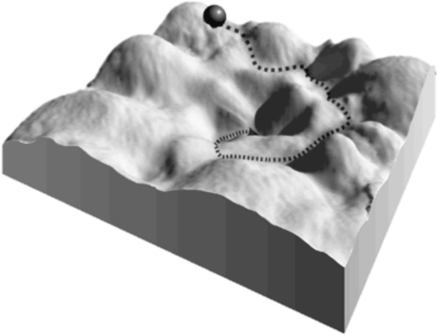
$$W_F(z) \equiv \int_{\Gamma(z)} \Omega$$

with $\partial\Gamma(z) \subset P(z)$ Poincaré dual to F.

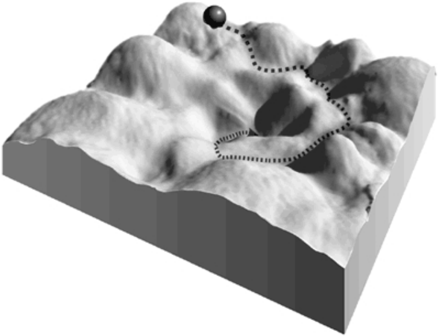
Deformation moduli space $\mathcal{M}_P$ has $\mathcal{N} = 1$ special geometry structure [Lerche-Mayr-Warner].

$\Rightarrow$ Problem of counting open string flux vacua formally almost identical to counting closed string flux vacua.

Closed string landscape



Open string landscape



Same form $\Rightarrow$ same techniques applicable.

Evaluation $\mathcal{Z}_{osv}$ at small ϕ^0

In continuum approximation for sum over F ($\Leftrightarrow$ large $|q_0|$ approx. $\Leftrightarrow$ small $|\phi^0|$ approx.): $\mathcal{Z}_{osv}$ can be evaluated as Gaussian boson-fermion integral with Q-symmetry, giving:

$$\mathcal{Z}_{osv} \approx \hat{\chi}(\mathcal{M}_P)\,(\phi^0)^{1-b_1}\,e^{-\frac{2\pi}{\phi^0}\left(\frac{\chi(P)}{24}-\frac{\Phi^2}{2}\right)}.$$

with "differential geometric Euler characteristic"

$$\hat{\chi}(\mathcal{M}_P) \equiv \frac{1}{\pi^n}\int_{\mathcal{M}_P} \det R,$$

and R curvature form of natural Hodge metric on $\mathcal{M}_P$.

singular $\Rightarrow$ not at all obvious that

$$\hat{\chi} = \chi_{top} = (\frac{1}{6}P^3 + \frac{1}{12}c_2 \cdot P)/|\mathrm{Aut}|,$$

but comparison results [Shih-Yin] for T^6 and $T^2 \times K3$ indicate it is!

Comparison to OSV conjecture

Up to prefactor refinement (which was not specified in conjecture), matches exactly in $\phi^0 \to 0$ approximation, for any compact Calabi-Yau, and any (very ample) divisor P!

Note:

- Only polynomial part of F_{top} survives when $\phi^0 \to 0$.
- Agreement somewhat surprising, given $\lambda \sim 1/\phi^0 \to \infty$ and topological string series a priori only asymptotic $\lambda \to 0$ expansion.

Main conclusion:

You can't escape the landscape!

How to understand osv more generally?

A general derivation of OSV

Physical interpretation and regularization of $\mathcal{Z}_{osv}$

Suitable topologically twisted theory of D4 on $S^1 \times P$, with S^1 Euclidean time circle of circumference β localizes on BPS configurations, i.e.

$$\mathcal{Z}_{D4}(\beta, g_s, B+iJ, C_0, C_2) =$$

$$\sum_{F,N} \Omega(F,N;B+iJ)\, e^{-\frac{\beta}{g_s}|Z(F,N;B+iJ)|+2\pi i(F-B)\cdot C_2+2\pi i[-N+\frac{1}{2}(F-B)^2+\frac{\chi}{24}]C_0}$$

where $C_{2q+1} =: C_{2q} \wedge dt/\beta$. Then formally

$$\begin{aligned} \mathcal{Z}_{osv}(\phi^0,\Phi) &= \mathcal{Z}_{D4}|_{\beta=0,B=0,C_0=i\phi^0,C_2=i\Phi,J=\infty} \\ &= \sum_{F,N} \Omega(F,N)\, e^{-2\pi\Phi\cdot F-2\pi\phi^0[-N+\frac{1}{2}F^2+\frac{\chi}{24}]}. \end{aligned}$$

$\mathcal{Z}_{D4}$ has better convergence properties than $\mathcal{Z}_{osv}$ (which diverges everywhere), so this is also a good regularization.

S-duality

Now do following chain of dualities:

- T-dualize along time circle: maps the D4 into a Euclidean D3.
- S-dualize: preserves D3.
- T-dualize back to D4.

In OSV limit this maps the background into

$$\beta'/g_s' = 0, \quad C_0' = -\frac{1}{C_0}, \quad C_2' = 0, \quad B' = C_2, \quad J' = |C_0|J = \infty.$$

Under these dualities $\mathcal{Z}_{D4}$ should be invariant or transform as a modular form. This descends to the following formal equality:

$$\mathcal{Z}_{osv} = (\phi^0)^w e^{-\frac{2\pi}{\phi^0}\left(\frac{\chi(P)}{24} - \frac{\Phi^2}{2}\right)} \sum_{F,N} \Omega(F,N)\, e^{-\frac{2\pi}{\phi^0}(-N+\frac{F^2}{2})+\frac{2\pi i}{\phi^0}\Phi\cdot F}$$

Dominant contributions

So we had

$$\mathcal{Z}_{osv} = (\phi^0)^w e^{-\frac{2\pi}{\phi^0}\left(\frac{\chi(P)}{24}-\frac{\Phi^2}{2}\right)} \sum_{F,N} \Omega(F,N)\, e^{-\frac{2\pi}{\phi^0}(-N+\frac{F^2}{2})+\frac{2\pi i}{\phi^0}\Phi\cdot F}$$

We take as usual $\mathrm{Re}\,\phi^0 < 0$ (this is the case for usual black hole saddle points in inverse Fourier transform of $\mathcal{Z}_{osv}$).

The leading contribution comes from pure D4 $(N,F) = (0,0)$ because $N \geq 0, F^2 \leq 0$ on susy configurations [There is actually one "bad" positive susy F^2 mode, but this disappears in regularized version; alternatively, work at fixed q_A]

Note: in $\phi^0 \to 0$ limit this immediately (!) reproduces our previous result, provided $\Omega(0,0) = \chi(\mathcal{M}_P) = \hat{\chi}(\mathcal{M}_P)$, and $w = 1 - b_1$.

$\Rightarrow$ Black hole entropy formula at large $-q_0$, including infinite series of $1/|q_0|$ corrections, is direct consequence of S-duality!

Corrections determined by states with highest q_0, i.e. small (N,F) excitations of pure D4.

Spacetime realization of D4

Unlike high (N, F) states, pure D4 is *not* a spherically symmetric black hole, but $D6 - \overline{D6}$ two-centered bound state [D].

Pure D4 with "inert" flux pulled back from ambient X:

$$\mathrm{D6}[\mathrm{S}_1] \; \bullet \qquad\qquad \bullet \; \overline{\mathrm{D6}[\mathrm{S}_2]}$$

where $D6[S]$ = single D6-brane with flux $F = S$ turned on.

$\Rightarrow Q_{tot} = (e^{S_1} - e^{S_2})(1 + c_2/12)$, i.e.:

$$Q_{D4} = P, \quad Q_{D2} = P \cdot S, \quad Q_{D0} = \frac{1}{24}(P^3 + c_2 \cdot P) + \frac{1}{2} P \cdot S^2$$

where

$$P = S_1 - S_2, \qquad S = \frac{S_1 + S_2}{2}.$$

= charges of D4 on P with flux $F = S$ turned on. ✓

Wrapping D6 r times gives $q_0 \sim P^3/24r^2$ in large P limit, much smaller than maximal q_0. $\rightsquigarrow$ strongly suppressed in $\mathcal{Z}_{osv}$.

Spacetime realization of D4 + "small" excitations

For e.g. $q_A = 0$, one needs $N - F^2/2 > \chi(P)/24 \sim P^3/24$ to get spherically symmetric black hole solution. Hence in limit

$$P^3/\phi^0 \to -\infty$$

only surviving contributions to $\mathcal{Z}_{osv}$ look like:

but now more generally with pure D6 + flux replaced by D6-D2-D0 + flux (higher r D6 can again be shown to contribute at $q_0 \lesssim P^3/r^2 \Rightarrow$ asymptotically vanishing contribution).

BPS states of D6-D2-D0 system considered in physics by [Iqbal,Nekrasov,Okounkov,Vafa], [Dijkgraaf-Verlinde-Vafa], presumably counted by Donaldson-Thomas invariants.

Counting D6-D2-D0 BPS states

- For suitable stabilizing value of B-field, D6+D2+D0 counted by Donaldson-Thomas generating function

$$\mathcal{Z}_{DT}(q,v) = \sum_{n,\beta \in H_2(X)} N_{DT}(n,\beta) q^n v^\beta$$

  where β is homology class of D2 defects in D6 and n holomorphic Euler characteristic of D2 and D0 defects.
- Sufficiently large B-field needed to bind D0's to D6. Contribution from D0's alone ($\beta = 0$) is $\mathcal{Z}^0_{DT}$, conjectured by [Maulik-Nekrasov-Okounkov-Pandharipande] to equal $M(q)^{-\chi(X)}$, with $M(q) = \prod_n (1-q^n)^n$.
- D6-D2 (with induced D0) bound states do not need B-field, counted by reduced $\mathcal{Z}'_{DT} \equiv \mathcal{Z}_{DT}/\mathcal{Z}^0_{DT}$.

Counting $D6-\overline{D6}$ bound states

Schematically:

$$\mathcal{Z}_{tot} = L\,\mathcal{Z}^0_{DT}\,\mathcal{Z}'_{DT,1}\,\mathcal{Z}'_{DT,2}$$

Factors resp.:

- L = Landau degeneracy from $D6-\overline{D6}$ e/m spin; $L = \langle Q_1, Q_2 \rangle = P^3/6 + c_2 \cdot P/12 = \chi(\mathcal{M}_P)$, corrections for excitations unimportant in $P^3/\phi^0 \to \infty$ limit.
- It turns out that in the supergravity solutions, depending on B, mobile $D0$'s either bind to D6 *or* to anti-D6, so only one contribution from degree 0.
- remaining contributions come from all possible D6-D2 and anti-(D6-D2) bound states.

Computing $\mathcal{Z}_{osv}$

Thus, in limit $\chi(P)/\phi^0 \sim (P^3 + c_2 \cdot P)/\phi^0 \to -\infty$, keeping P/ϕ^0 possibly finite but large, after some work:

$$\begin{aligned}\mathcal{Z}_{osv} \;\approx\; & \chi(\mathcal{M}_P)\,(\phi^0)^{1-b_1}\, M(e^{2\pi/\phi^0})^{-\chi(X)}\, e^{-\frac{2\pi}{24\phi^0}\chi(P)} \\ & \times \sum_{S\in\frac{P}{2}+H^2(X)} e^{\frac{\pi}{\phi^0}(\Phi+iS)^2}\,\mathcal{Z}'_{DT}[-e^{2\pi/\phi^0}, e^{\frac{\pi}{\phi^0}P-\frac{2\pi i}{\phi^0}(\Phi+iS)}] \\ & \qquad\qquad \times \mathcal{Z}'_{DT}[-e^{-2\pi/\phi^0}, e^{\frac{\pi}{\phi^0}P+\frac{2\pi i}{\phi^0}(\Phi+iS)}]\end{aligned}$$

[recall $P = S_1 - S_2$ and $S = (S_1 + S_2)/2$].

DT-GW correspondence and OSV

[INOV] (phys.) [MNOP] (math.) conjectured relation between $\mathcal{Z}'_{DT}[q = e^{i\lambda}, v] = \mathcal{Z}'_{GW}[\lambda, v] \equiv \exp F'_{GW}[\lambda, v]$, recently clarified by [DVV], which applied to our formula for $\mathcal{Z}_{osv}$ gives

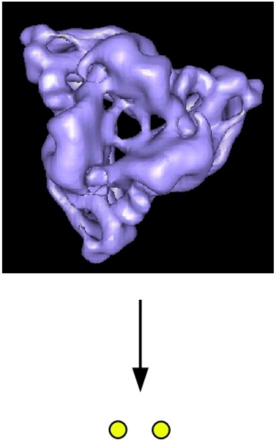
$$\begin{aligned}
\mathcal{Z}_{osv} \approx\ & \chi(\mathcal{M}_P)\,(\phi^0)^{1-b_1}\, M(e^{2\pi/\phi^0})^{-\chi(X)}\, e^{-\frac{2\pi}{24\phi^0}\chi(P)} \\
& \times \sum_{S \in \frac{P}{2}+H^2(X)} e^{\frac{\pi}{\phi^0}(\Phi+iS)^2}\, \mathcal{Z}'_{GW}[-2\pi i/\phi^0, e^{\frac{\pi}{\phi^0}P - \frac{2\pi i}{\phi^0}(\Phi+iS)}] \\
& \qquad \times \mathcal{Z}'_{GW}[2\pi i/\phi^0, e^{\frac{\pi}{\phi^0}P + \frac{2\pi i}{\phi^0}(\Phi+iS)}]
\end{aligned}$$

agrees with and refines OSV conjecture.

Note:

- for "small black holes", $P^3 = 0$, so when $\chi(P)/\phi^0 \to \infty$, $P/\phi^0 \to \infty$, so we cannot take clean limit keeping instantons. $\rightsquigarrow$ explains some "problems" with small black holes.
- Must be dual to [Gaoitto-Strominger-Yin] picture through [Dijkgraaf-Vafa-Verlinde].

A lesson for the landscape?



(S-)duality to fight computational complexity?