Superstring as integrable sigma model?

V. Kazakov (ENS, Paris)

Banff 12/02/06

(to appear, in collaboration with N.Gromov, K.Sakai and P.Vieira)

Integrability in AdS/CFT

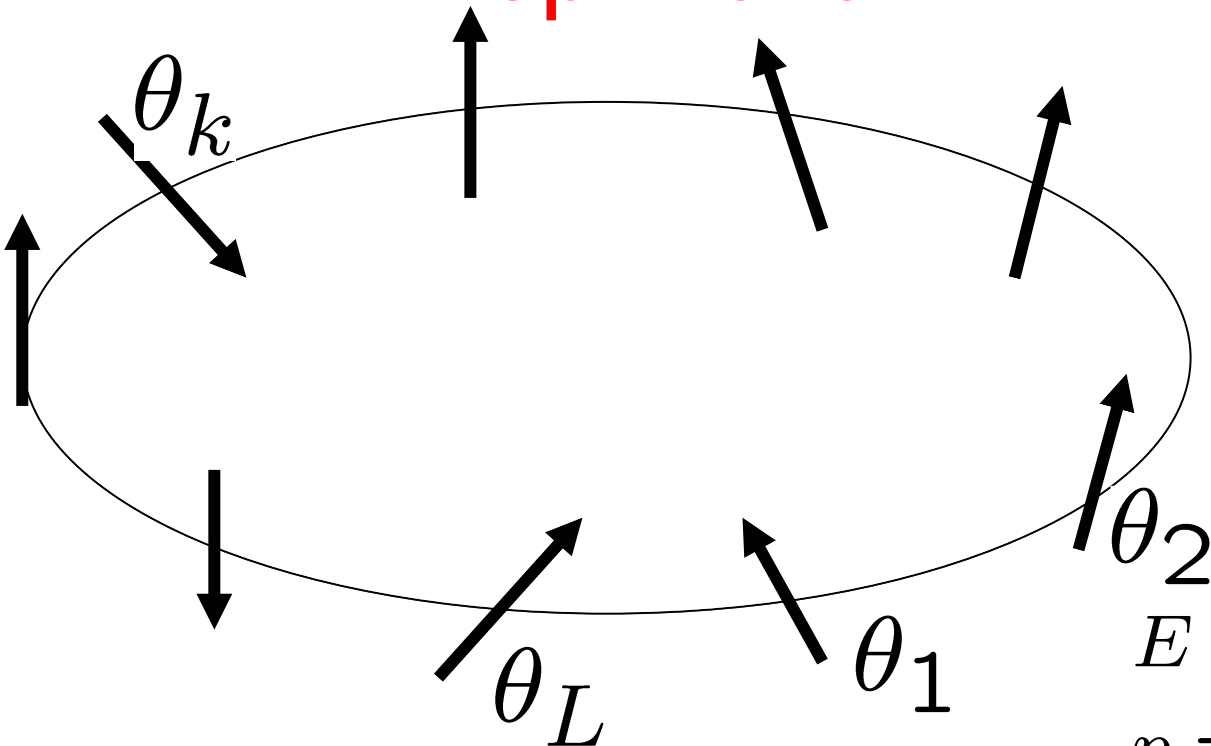
- Superstring is probably an integrable sigma model, with factorized physical S-matrix.
- Planar N=4 SYM is integrable in 3 (and may be all) loops: dilatation operator is a hamiltonian of integrable non-local spin chain.
  [Minahan,Zarembo'02],[Beisert,Kristjansen,Staudacher'03]
- Proposal: integrable model describing both sides of AdS/CFT duality is a

Dynamical (inhomogeneous) spin chain.

(greatly inspired by [Mann,Polchinski'05])

- We conjecture such a model and its Bethe equations (up to an unknown function)
- Correct classical limit (algebraic curve) reproduced for the full scalar SO(6) sector
- Demonstration for SO(4) sigma model.

Particles on a ring as dynamical spin chain



Length= $\mathcal{L}$

$$E = m \ \cosh\theta$$
$$p = m \ \sinh\theta$$

- Large density, classical, conformal limit in the asymptotically free theory:

$$\mu = \mathcal{L}m \to 0, \qquad M \sim L \to \infty, \qquad \xi = \frac{\theta}{M} \sim 1,$$

σ-model on $S^3 x R_1$

[Frolov,Tseytlin'02]

$$S = \frac{\sqrt{\lambda}}{4\pi}\int d\sigma d\tau \sum_{a=0}^{4}(\partial_\mu X_a)^2\,, \qquad X_1^2+\ldots+X_4^2 = 1$$

- Gauge for AdS time: $X_0(\sigma,\tau) = E\ \tau$
- Equivalent to SU(2)xSU(2) principal chiral field:

$$S = -\frac{\sqrt{\lambda}}{8\pi}\int d\sigma d\tau\ \mathrm{Tr}\ j_a^2$$

$$j_a = g^{-1}\partial_a g, \qquad g = \begin{pmatrix} X_1 + iX_2 & X_3 + iX_4 \\ -X_3 + iX_4 & X_1 - iX_2 \end{pmatrix} \in SU(2)$$

- Virasoro conditions: $\mathrm{tr} j_\pm^2(\sigma,\tau) = \frac{E}{\sqrt{\lambda}}$
- Classical integrability: $\left[\mathcal{L}_+(x), \mathcal{L}_-(x)\right] = 0$

$$\mathcal{L}_\pm = \partial_\pm + \frac{j_\pm}{x \mp 1}$$ - Lax operators

- Classical multi-soliton "finite gap" solutions for all string motions.

S-matrix for SU(2)xSU(2) chiral field

- Equivalent to σ-model on S^3, a subsector of superstring

- S-matrix: $$\widehat{S}(\theta) = S_L(\theta) \times S_R(\theta) \Rightarrow$$ 

$$\widehat{S}_{L,R}(\theta) = S_0(\theta) \left(P^{+}_{L,R} + \frac{\theta + i}{\theta - i} P^{-}_{L,R} \right)$$

$$S_0(\theta) = \frac{\Gamma\left(-\frac{\theta}{2i}\right)\Gamma\left(\frac{1}{2}+\frac{\theta}{2i}\right)}{\Gamma\left(\frac{\theta}{2i}\right)\Gamma\left(\frac{1}{2}-\frac{\theta}{2i}\right)} \quad \to \pm \exp\left(-\frac{i}{\theta}\right), \quad \theta \to \infty$$

[(Zamolodchikov)x2 '79], [Wiegmann'84]

- Periodicity:

$$e^{-i\mu \sinh \pi\theta_\alpha} = \prod_{\beta=\alpha+1}^{L} \hat{S}\left(\theta_\alpha - \theta_\beta\right) \prod_{\gamma=1}^{\alpha-1} \hat{S}\left(\theta_\alpha - \theta_\gamma\right)$$

- Bethe equations (diagonalization of periodicity cond.):

$$\begin{aligned}
e^{-i\mu \sinh \pi\theta_\alpha} &= \prod_{\beta\neq\alpha}^{L} S_0^2\left(\theta_\alpha - \theta_\beta\right) \prod_{j}^{J_u} \frac{\theta_\alpha - u_j + i/2}{\theta_\alpha - u_j - i/2} \prod_{k}^{J_v} \frac{\theta_\alpha - v_k + i/2}{\theta_\alpha - v_k - i/2}, \\
1 &= \prod_{\beta}^{J_u} \frac{u_j - \theta_\beta - i/2}{u_j - \theta_\beta + i/2} \prod_{i\neq j}^{J_u} \frac{u_j - u_i + i}{u_j - u_i - i}, \\
1 &= \prod_{\beta}^{J_v} \frac{v_k - \theta_\beta - i/2}{v_k - \theta_\beta + i/2} \prod_{l\neq k}^{J_v} \frac{v_k - v_l + i}{v_k - v_l - i},
\end{aligned}$$

Classical limit: $\mu \to 0, \quad L \sim J_u \sim J_v \to \infty$

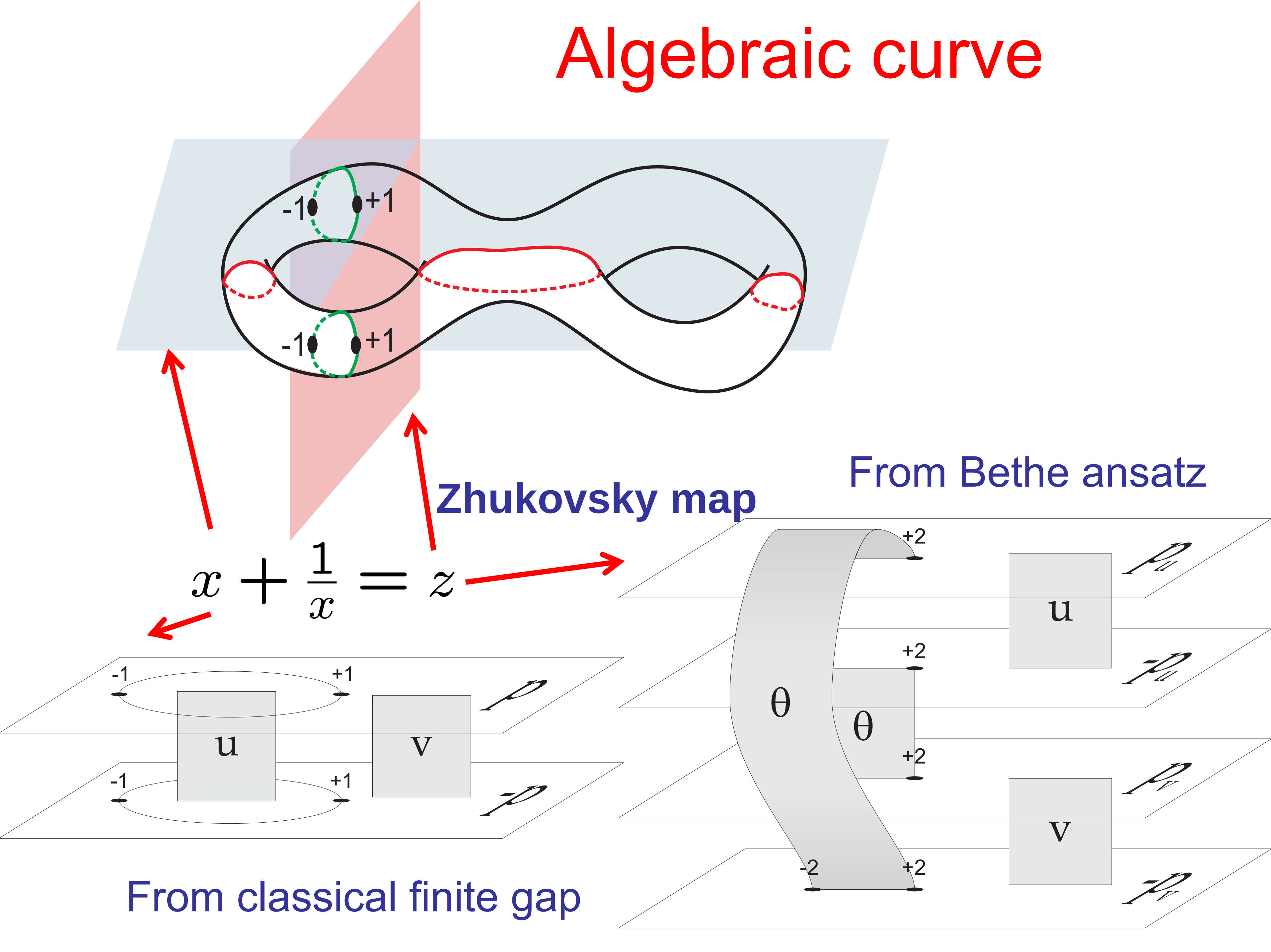
- 2D Coulomb charges with coordinates θk in potential $\mu \cosh \theta$
- Width of θ-distribution ~ $M \equiv -\frac{\log \mu}{2\pi} = \frac{\sqrt{\lambda}}{2\pi}$
- Rescale: $u = M\,x, \quad v = M\,y, \quad \theta = M\xi, \quad M \sim L$
- Potential becomes a square box
  with width of ξ-distribution ~ $t = -\frac{1}{\pi M}\log \mu \simeq \frac{2M}{L}.$
- Resolvents and quasi-momenta:

$$G_\theta(x) \equiv \sum_{\beta=1}^{L} \frac{1}{xM - \theta_\beta}\,, \quad G_u(x) \equiv \sum_{i=1}^{J_u} \frac{1}{xM - u_i}\,, \quad G_v(x) \equiv \sum_{l=1}^{J_v} \frac{1}{xM - v_l}$$

$$p_1 = -p_2 = G_u - \tfrac{1}{2}G_\theta \qquad p_3 = -p_4 = G_v - \tfrac{1}{2}G_\theta$$

Classical Bethe eqs.

$$\begin{array}{ll} x \in C_u : & p_1{}^{+} - p_2{}^{-} = 2\not{G}_u - G_\theta = 2\pi n_u \\ x \in C_\theta : & p_2{}^{+} - p_3{}^{-} = -G_u - G_v + \not{G}_\theta = 2\pi m \\ x \in C_v : & p_3{}^{+} - p_4{}^{-} = 2\not{G}_v - G_\theta = 2\pi n_v \\ x \in C_\theta : & p_4{}^{+} - p_1{}^{-} = -G_u - G_v + \not{G}_\theta = 2\pi m \end{array}$$



Algebraic curve
-1
+1
-1
+1
Zhukovsky map
$x + \frac{1}{x} = z$
From Bethe ansatz
+2
u
p_u
+2
$-p_u$
θ
θ
+2
v
p_v
-2
+2
$-p_v$
-1
+1
u
v
p
-1
+1
$-p$
From classical finite gap

Map from Bethe ansatz to finite gap solution

- Zhukovsky map: $z = x + \frac{1}{x}, \qquad x_\pm = \frac{1}{2}\left(z \pm \sqrt{z^2-4}\right)$

- Define: $$\begin{cases} G(x_+) \equiv G_v(z) - \dfrac{G_\theta(z)}{2} + \dfrac{E}{\sqrt{\lambda}}\dfrac{2\pi}{\sqrt{z^2-4}} \\ G(x_-) \equiv -G_u(z) + \dfrac{G_\theta(z)}{2} - \dfrac{E}{\sqrt{\lambda}}\dfrac{2\pi}{\sqrt{z^2-4}} + 2\pi m \end{cases}$$

- We reproduced from the full quantum theory the finite gap eq. of classical sigma model of **[V.K.,Marshakov,Minahan,Zarembo'04]:**

[Minahan'05]

$$\not{G}(x) = \frac{E}{\sqrt{\lambda}}\,\frac{x}{x^2-1} + \pi n_{u,v}, \qquad x \in C_{u,v}.$$

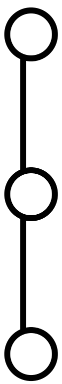
level matching for filling fractions

$$P = \frac{G(0)}{2\pi} = \frac{\mu}{2\pi}\sum_\alpha \sinh(\pi\theta_\alpha) = \sum_a n_a S_a^u + \sum_b n_b S_b^v = mL$$

$E = \frac{\mu}{2\pi}\sum_\alpha \cosh(\pi\theta_\alpha)$ - energy (AdS time generator)

Left and right global charges: $J_R = \frac{L}{2} - J_u, \qquad J_L = \frac{L}{2} - J_v$

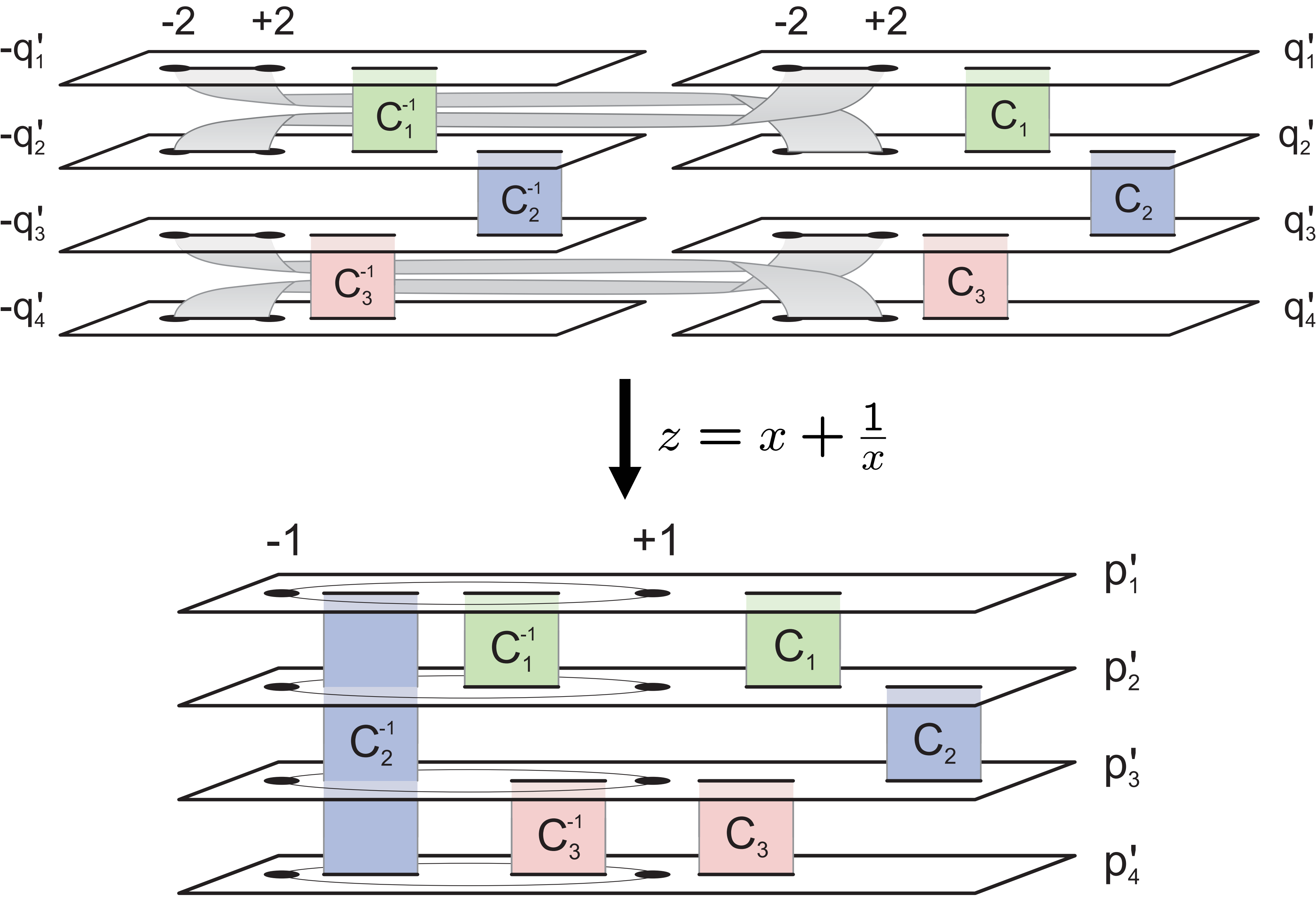
SO(6) σ-model



- Bethe eqs. follow the Dynkin diagram pattern.

$S_0(\theta)$ is known [(Zamolodchikov)x2 ’79].

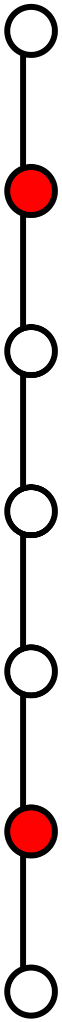
- Classical alg. curve coincides, after Zhukovski map, with the finite gap solution of [Beisert,Sakai,V.K’04]



-2
+2
-2
+2
$-q'_1$
$-q'_2$
$-q'_3$
$-q'_4$
q'_1
q'_2
q'_3
q'_4
C_1^{-1}
C_2^{-1}
C_3^{-1}
C_1
C_2
C_3
$z = x + \frac{1}{x}$
-1
+1
p'_1
p'_2
p'_3
p'_4
C_1^{-1}
C_1
C_2^{-1}
C_2
C_3^{-1}
C_3

Full Metsaev-Tseytlin superstring as a dynamical chain?

$$e_s(u) = \frac{u+is/2}{u-is/2}$$

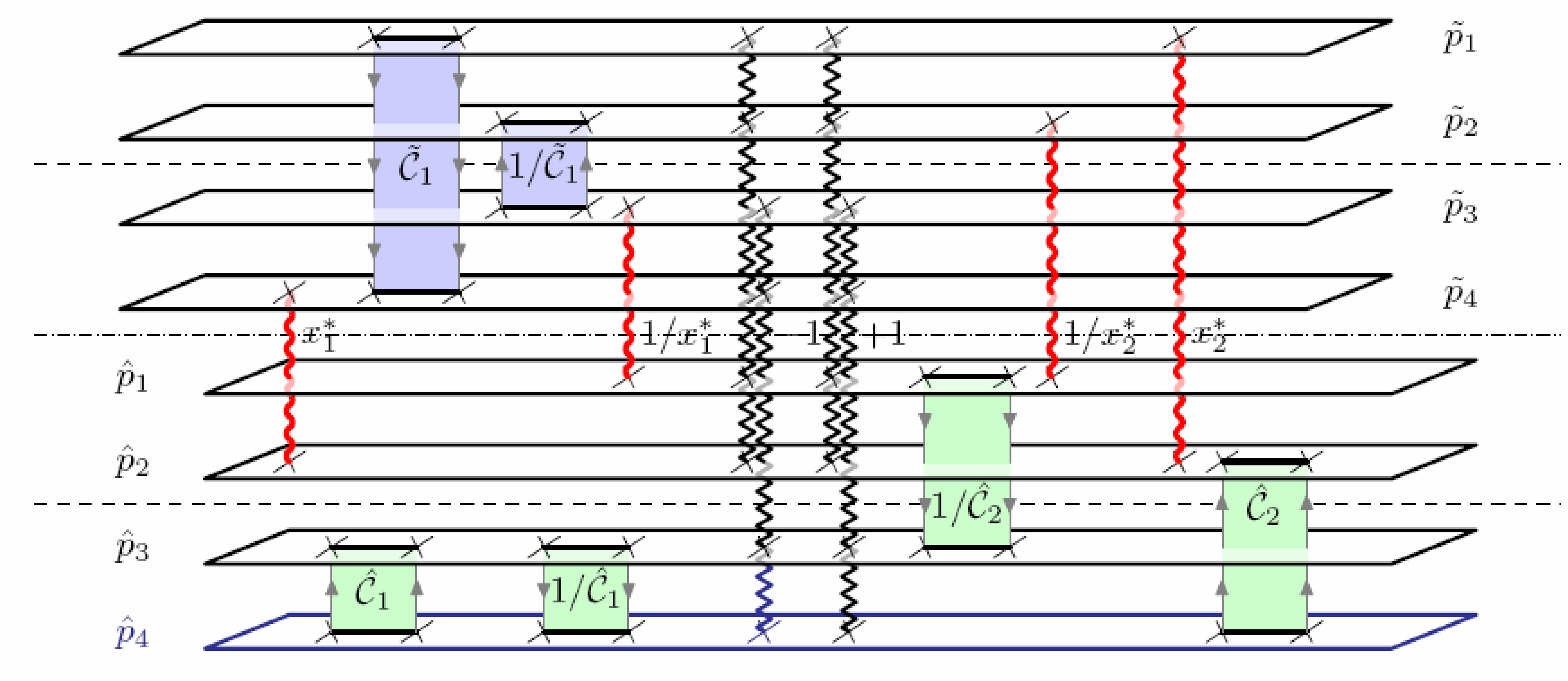


$S_0(\theta)$ Is not known, since the model is non-compact.

- If we expect conformal symmetry: $\sinh c\ \theta \to \pm e^{\pm c\ \theta}$
- Everything else fixed by PSU(2,2|4) symmetry and integrability!

Riemann surface of the curve

[Beisert,V.K.,Sakai,Zarembo'05]



- Classical integrability used [Bena,Polchinski,Roiban'02]
- Algebraic curve encodes all "action" variables;
- "Angle" variables defined by holomorphic integrals.
- Possible to restore corresponding classical string motion (see for S3xR1 sector [Dorey,Vicedo'06]).

Problems and prospects

- Define $S_0(\theta)$ and the dispersion (may be, using the conformal symmetry.
- Restore the SYM perturbation theory (1-loop of [Beisert,Staudacher'03] works! Physical particles form a lattice).
- Understand the SYM periodicity $p \to p + 2\pi$ (may be Hubbard model mechanism of [Rej,Serban,Staudacher'06] can help).
- Quantum 1/L corrections.

- Prospects: obvious!