Integrability of the planar $\mathcal{N} = 4$ gauge theory and the Hubbard model

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Banff, February 16, 2006

Overview

- Integrability of the $\mathcal{N} = 4$ SYM: the three loop result
- All loop integrability and the BDS ansatz
- The Hubbard model
- Conclusions and perspectives

Integrability in AdS/CFT: sigma models vs. spin chains

String side: non-linear sigma model on the coset space [Metsaev, Tseytlin 98] $PSL(2,2|4)/SO(4,1) \times SO(5) \Rightarrow$ non-local conserved charges [Bena, Polchinski, Roiban^{03}]

– rotating string solutions at large $J \rightarrow$ classical integrable model (Neumann system) [Frolov, Tseytlin, Arutyunov, Russo$^{02-03}$]

$$\frac{E(\lambda)}{J} = \varepsilon_0\left(\frac{\lambda}{J^2}\right) + \frac{1}{J}\,\varepsilon_1\left(\frac{\lambda}{J^2}\right) + \frac{1}{J^2}\,\varepsilon_2\left(\frac{\lambda}{J^2}\right) + \dots$$

- full solution for the classical sigma model $\longleftrightarrow$ algebraic curve
[Kazakov, Marshakov, Minahan, Zarembo; Kazakov, Zarembo; Schäfer-Namecki; Kazakov, Beisert, Sakai, 04]

- quantizing the string sigma model [Kazakov et al.; Zarembo, Klose 06]

Integrability in AdS/CFT: sigma models vs. spin chains

perturbative $\mathcal{N} = 4$

- so(6) sector at one loop is integrable [Minahan, Zarembo 02]
- psl(2,2|4) sector at one loop is integrable [Beisert, Staudacher 03]
- su(2|3) sector is integrable up to three loop order [Beisert, Kristjansen, Staudacher; Beisert 03]

e.g. su(2) sector: $\uparrow \; \equiv \; Z = \Phi_1 + i\Phi_2 \; , \quad \downarrow \; \equiv \Phi = \Phi_3 + i\Phi_4 \; ,$

$$\uparrow\uparrow\downarrow\downarrow\uparrow\uparrow\downarrow\uparrow\uparrow\uparrow\downarrow\downarrow\uparrow\uparrow\downarrow$$

$$D = L + \lambda \sum_{i=1}^{L} 2(1 - P_{i,i+1}) + \lambda^2 \sum_{i=1}^{L} (8P_{i,i+1} - 2P_{i,i+2} - 6) + \ldots$$

Bethe Ansatz

Integrability in AdS/CFT: sigma models vs. spin chains

comparison of the Bethe Ansatz and string results:

$$\frac{\Delta - L}{L} = \frac{\lambda}{L^2}\left(a_0 + \frac{a_1}{L} + \ldots\right) + \left(\frac{\lambda}{L^2}\right)^2 \left(b_0 + \frac{b_1}{L} + \ldots\right) + \ldots \quad \lambda \ll 1$$

$$\frac{E(\lambda)}{J} = \varepsilon_0\left(\frac{\lambda}{J^2}\right) + \frac{1}{J}\,\varepsilon_1\left(\frac{\lambda}{J^2}\right) + \frac{1}{J^2}\,\varepsilon_2\left(\frac{\lambda}{J^2}\right) + \ldots \quad \lambda/J^2 \ll 1\,, \quad J \to \infty$$

BMN scaling

Bethe Ansatz solution to three loop order [Serban, Staudacher 04]

$\longleftrightarrow$ solution of the classical sigma model [KMMZ 04]

Discrepancy at three loop order ! [Callan et al. 03]

- order of limits?
- non-analytic corrections to strings [Beisert, Tseytlin 05] $\sim \left(\lambda/J^2\right)^{5/2}$
- non-perturbative mixing of the sectors [Minahan; Alday, Arutyunov, Frolov 05]

All loop integrability: the BDS conjecture

[Beisert, Dippel, Staudacher 04]

There is a unique spin chain obeying:

- diagrammatic constraint
- integrability up to five loops $\longrightarrow$ candidate for the dilatation operator
- BMN scaling

all loop Bethe ansatz for L infinite :

$$e^{ip_k L} = \prod_{j \neq k}^{M} \frac{u_k - u_j + i}{u_k - u_j - i}\,, \quad k = 1, \ldots, M\,,$$

$$u(p) = \frac{1}{2} \cot \frac{p}{2} \sqrt{1 + 8g^2 \sin^2 \frac{p}{2}}\,,$$

$$E(g) = -\frac{M}{g^2} + \frac{1}{g^2} \sum_{k=1}^{M} \sqrt{1 + 8g^2 \sin^2 \frac{p_k}{2}}$$

$$\Delta = L + g^2 E(g)\,, \qquad g^2 \equiv \frac{\lambda}{8\pi^2}$$

all loop psl(2,2|4): [Beisert, Staudacher; Beisert 05]

$$
\begin{aligned}
\mathbf{H}_8 = & +\tfrac{1479}{8}\{\} + \big(-\tfrac{1043}{4} - 12\alpha + 4\beta_1\big)\{1\} + \big(-19 + 8\alpha - 2\beta_1 - 4\beta_2\big)\{1,3\} \\
& + \big(5 + 2\alpha + 4\beta_2 + 4\beta_3\big)\{1,4\} + \tfrac{1}{8}\{1,5\} + \big(11\alpha - 4\beta_1 + 2\beta_3\big)\big(\{1,2\} + \{2,1\}\big) \\
& - \tfrac{1}{4}\{1,3,5\} + \big(\tfrac{251}{4} - 5\alpha + 2\beta_1 - 2\beta_3\big)\big(\{1,3,2\} + \{2,1,3\}\big) \\
& + \big(-3 - \alpha - 2\beta_3\big)\big(\{1,2,4\} + \{1,3,4\} + \{1,4,3\} + \{2,1,4\}\big) \\
& - \tfrac{1}{8}\big(\{1,2,5\} + \{1,4,5\} + \{1,5,4\} + \{2,1,5\}\big) \\
& + \big(\tfrac{41}{4} - 6\alpha + 2\beta_1 - 4\beta_3\big)\big(\{1,2,3\} + \{3,2,1\}\big) + \big(-\tfrac{107}{2} + 4\alpha - 2\beta_1\big)\{2,1,3,2\} \\
& + \big(\tfrac{1}{4} + \beta_2\big)\big(\{1,3,2,5\} + \{1,3,5,4\} + \{1,4,3,5\} + \{2,1,3,5\}\big) \\
& + \big(\tfrac{183}{4} - 6\alpha + 2\beta_1 - 2\beta_2\big)\big(\{1,3,2,4\} + \{2,1,4,3\}\big) \\
& + \big(-\tfrac{3}{4} - 2\beta_2\big)\big(\{1,2,5,4\} + \{2,1,4,5\}\big) + \big(1 + 2\beta_2\big)\big(\{1,2,4,5\} + \{2,1,5,4\}\big) \\
& + \big(-\tfrac{51}{2} + \tfrac{5}{2}\alpha - \beta_1 + \beta_2 + 3\beta_3\big)\big(\{1,2,4,3\} + \{1,4,3,2\} + \{2,1,3,4\} + \{3,2,1,4\}\big) \\
& - \beta_2\big(\{1,2,3,5\} + \{1,3,4,5\} + \{1,5,4,3\} + \{3,2,1,5\}\big) \\
& + \big(\tfrac{35}{4} + \alpha + 2\beta_3\big)\big(\{1,2,3,4\} + \{4,3,2,1\}\big) \\
& + \big(-\tfrac{7}{8} - \alpha + 2\beta_3\big)\big(\{1,4,3,2,5\} + \{2,1,3,5,4\}\big) \\
& + \big(\tfrac{1}{2} + \alpha\big)\big(\{1,3,2,5,4\} + \{2,1,4,3,5\}\big) \\
& + \big(\tfrac{5}{8} + \tfrac{1}{2}\alpha - \beta_3\big)\big(\{1,3,2,4,3\} + \{2,1,3,2,4\} + \{2,1,4,3,2\} + \{3,2,1,4,3\}\big) \\
& + \big(\tfrac{1}{4} - 2\beta_3\big)\big(\{1,2,5,4,3\} + \{3,2,1,4,5\}\big) \\
& + \big(\tfrac{1}{4} + \tfrac{1}{2}\alpha + \beta_3\big)\big(\{1,2,4,3,5\} + \{1,3,2,4,5\} + \{2,1,5,4,3\} + \{3,2,1,5,4\}\big) \\
& + \big(-\tfrac{1}{2}\alpha - \beta_3\big)\big(\{1,2,3,5,4\} + \{1,5,4,3,2\} + \{2,1,3,4,5\} + \{4,3,2,1,5\}\big) \\
& - \tfrac{7}{8}\big(\{1,2,3,4,5\} + \{5,4,3,2,1\}\big)
\end{aligned}
$$

$$
\{m,n,p\} \equiv \sum_i P_{i+m,i+m+1} P_{i+n,i+n+1} P_{i+p,i+p+1}
$$

BDS ansatz from the Hubbard model at half filling

[Rej, Serban, Staudacher 05]

energy of the AF state [RSS; Zarembo 05] :

$$E_{\mathrm{AF}}(g) = \frac{4L}{\sqrt{2}g} \int_0^\infty \frac{dt}{t} \frac{J_0(\sqrt{2}gt)\, J_1(\sqrt{2}gt)}{1+e^t}$$

Lieb, Wu 1968!

1-d Hubbard model: itinerant fermions with onsite repulsion

$$H = \frac{1}{\sqrt{2}\,g} \sum_{i=1}^{L} \sum_{\sigma=\uparrow,\downarrow} \left(e^{i\phi}\, c^\dagger_{i,\sigma} c_{i+1,\sigma} + e^{-i\phi}\, c^\dagger_{i+1,\sigma} c_{i,\sigma} \right) - \frac{1}{g^2} \sum_{i=1}^{L} c^\dagger_{i,\uparrow} c_{i,\uparrow} c^\dagger_{i,\downarrow} c_{i,\downarrow} \,,$$

$t = 1 \quad \Leftrightarrow \quad U = \sqrt{2}/g$

- solved by (nested) Bethe Ansatz

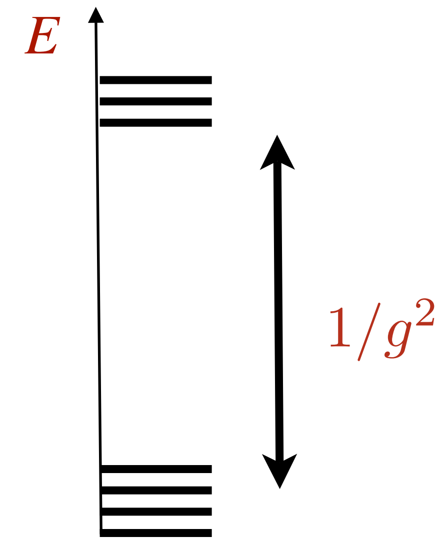
⟶ Heisenberg model at half filling and $g = 0$

ground state: ferromagnetic state | ↑↑↑↑ ... ↑>

Hubbard model at half filling

projection to a spin Hamiltonian
("strong coupling" or $g \to 0$)

[Klein, Seitz 73; Takahashi 77]

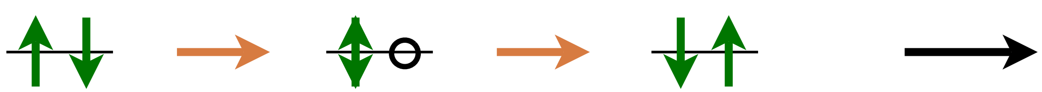


singly-occupied states

- at $g = 0$ the onsite part dominates

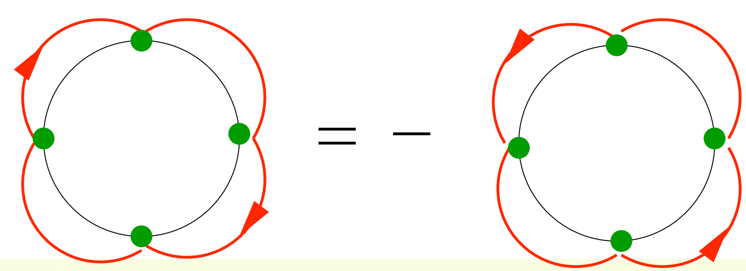
  states: $| \uparrow\downarrow\uparrow\uparrow \ldots \uparrow >$

- fluctuations

g^2  spin permutation

g^4 (unwanted) four spin term [Takahashi 77]

twisted boundary conditions
for odd chains: $t_{1L} = - t_{1L}^*$



$\longleftrightarrow$ Aharonov-Bohm flux $\Phi = \dfrac{\pi(L+1)}{2}$

Projection of the Hubbard model on the spin space:

$$h = \sum_{i=1}^{L} (h_2 + g^2 h_4 + g^4 h_6 + \dots) \,,$$

$$\begin{aligned}
h_2 &= \frac{1}{2}(1 - \vec{\sigma}_i \vec{\sigma}_{i+1}) \,, \\
h_4 &= -(1 - \vec{\sigma}_i \vec{\sigma}_{i+1}) + \frac{1}{4}(1 - \vec{\sigma}_i \vec{\sigma}_{i+2}) \,, \\
h_6 &= \frac{15}{4}(1 - \vec{\sigma}_i \vec{\sigma}_{i+1}) - \frac{3}{2}(1 - \vec{\sigma}_i \vec{\sigma}_{i+2}) + \frac{1}{4}(1 - \vec{\sigma}_i \vec{\sigma}_{i+3}) \\
&\quad - \frac{1}{8}(1 - \vec{\sigma}_i \vec{\sigma}_{i+3})(1 - \vec{\sigma}_{i+1} \vec{\sigma}_{i+2}) \\
&\quad + \frac{1}{8}(1 - \vec{\sigma}_i \vec{\sigma}_{i+2})(1 - \vec{\sigma}_{i+1} \vec{\sigma}_{i+3}) \,.
\end{aligned}$$

$\longleftrightarrow$ dilatation operator in the su(2) sector!

BDS ansatz from Lieb-Wu equations

Lieb-Wu equations (half filling):

$$e^{i\tilde{q}_n L} = \prod_{j=1}^{M} \frac{u_j - \sqrt{2}g\sin(\tilde{q}_n + \phi) - i/2}{u_j - \sqrt{2}g\sin(\tilde{q}_n + \phi) + i/2}\,, \qquad n = 1, \ldots, L$$

$$\prod_{n=1}^{L} \frac{u_k - \sqrt{2}g\sin(\tilde{q}_n + \phi) + i/2}{u_k - \sqrt{2}g\sin(\tilde{q}_n + \phi) - i/2} = \prod_{\substack{j=1\\ j\neq k}}^{M} \frac{u_k - u_j + i}{u_k - u_j - i}\,, \quad k = 1, \ldots, M$$

$$E = \frac{\sqrt{2}}{g} \sum_{n=1}^{L} \cos(\tilde{q}_n + \phi) \qquad\qquad \phi = \frac{\pi(L+1)}{2L}$$

$g \to 0$ Heisenberg Bethe ansatz

L fermions, L large $\longrightarrow$ integral equations [Lieb, Wu 68]

BDS ansatz from Lieb-Wu equations

Shiba (particle/hole) transformation:

$$\uparrow \;\longleftrightarrow\; \circ \qquad\qquad c_{i,\circ} = c^{\dagger}_{i,\uparrow}$$

$$\downarrow \;\longleftrightarrow\; \updownarrow \qquad\qquad c_{i,\updownarrow} = c_{i,\downarrow}$$

$$H(g;\phi,\phi) \longrightarrow -H(-g;\pi-\phi,\phi) - \frac{M}{g^2}$$

attractive interaction

Dual Lieb-Wu equations

$$e^{iq_nL} = \prod_{j=1}^{M} \frac{u_j - \sqrt{2}g\sin(q_n-\phi) - i/2}{u_j - \sqrt{2}g\sin(q_n-\phi) + i/2}, \qquad n = 1,\dots,2M$$

$$\prod_{n=1}^{2M} \frac{u_k - \sqrt{2}g\sin(q_n-\phi) + i/2}{u_k - \sqrt{2}g\sin(q_n-\phi) - i/2} = -\prod_{\substack{j=1\\ j\neq k}}^{M} \frac{u_k - u_j + i}{u_k - u_j - i}, \qquad k = 1,\dots,M$$

$$E = -\frac{M}{g^2} - \frac{\sqrt{2}}{g}\sum_{n=1}^{2M}\cos(q_n-\phi)$$

2M fermions, M magnons

BDS ansatz from Lieb-Wu equations

Magnons as bound states of fermions:
Lieb -Wu equations have bound-states solutions (strings) [Takahashi 72]

q-u strings: 2 fermions (q_1 and q_2) and one rapidity u

$$q_1 - \phi = \frac{\pi}{2} + \frac{p}{2} + i\,\beta\,, \qquad q_2 - \phi = \frac{\pi}{2} + \frac{p}{2} - i\,\beta$$

1st LW equation $\longrightarrow$ $u \pm i/2 = \sqrt{2}\,g\,\cos\left(\frac{p}{2} \mp i\beta\right)$ $L \to \infty$

$$\sinh\beta = \frac{1}{2\sqrt{2}\,g\,\sin\frac{p}{2}} \qquad u(p) = \frac{1}{2}\cot\frac{p}{2}\,\sqrt{1 + 8g^2\sin^2\frac{p}{2}}$$

$$E(p) = \frac{1}{g^2}\left(\sqrt{1 + 8g^2\sin^2\frac{p}{2}} - 1\right)$$

M magnons:

1st LW equation: $\longrightarrow$ $e^{ip_k L} = \prod_{j\neq k}^{M} \frac{u_k - u_j + i}{u_k - u_j - i}\,, \quad k = 1, \ldots, M$

BDS equation

Finite size corrections: $\mathcal{O}(e^{-\beta L})$

$$g \ll 1 \qquad \Rightarrow \qquad e^{-\beta L} \sim g^{2L} \qquad \text{as expected}$$

No order-of-limits problem!

- Hubbard model has a space of states much larger than the su(2) sector of the dilatation operator!

Solutions with real q ? important at finite g

- comparison with the strings solutions around the AF state qualitatively correct [Roiban, Tîrziu, Tseytlin 06]

Conclusions

- Extension of the Hubbard model to psl(2,2|4)?
- Four loop computation in the gauge theory/ direct derivation from the gauge theory?
- Comparison with the Bethe ansatz solution for the string sigma model?

[Kazakov et al. 06] [Zarembo, Klose 06]