Fewer questions this time please

Black Hole Thermodynamics

Schwarzschild (mass-only) black hole:

- Classically, just sits there for eternity.
- *Semiclassically* - for classical gravity + quantum matter, emits Hawking radiation.

Many ways (see e.g. TASI notes) to calculate *Hawking temperature* of Schwarzschild black hole:

$$T_{\mathsf{H}} \sim \frac{1}{(G_d M)^{1/(d-3)}} \tag{1}$$

T_H is physical temperature felt by an observer at infinity. Temperature blueshifts as approach horizon; Hawking radiated particles have temperature T_P at proper distance l_P from horizon.

Additional calculable physics: BH radiates with a thermal spectrum; gravitational backscattering on way out from horizon causes wavelength-dependent filtering, and gives *greybody factors*.

Runaway evaporation

Since $T_H \uparrow$ as $M \downarrow$, specific heat is negative.
Physical effect: runaway evaporation of black hole at low mass!

BH lifetime? BH radiates approx. like blackbody, so luminosity is

$$-\frac{dM}{dt} \sim (\text{Area})\ {T_{\mathrm{H}}}^{d} \sim (G_d M)^{-\frac{2}{(d-3)}} \tag{2}$$

Therefore,

$$\frac{\Delta t}{\ell_{P,d}} \sim \left(\ell_{P,d} M\right)^{\frac{(d-1)}{(d-3)}}$$

Endpoint of Hawking radiation = ??
(peek:) Hot string state - see later this Lecture.

Mass of $d = 4$ BH with lifetime $\sim$ age of Universe ($\sim$ 14 Gyr): $\sim 10^{12}$kg. Schwarzschild radius about a femtometre. c.f. Earth-mass black hole Schwarzschild radius of ping-pong ball.

Entropy of black holes

No-hair theorems indicate that we know *very* little about a BH by looking from outside. Only quantum numbers conserved because of a gauge symmetry survive. This suggests that a black hole will possess a degeneracy of states, and hence an entropy, as a function of its conserved quantum numbers:

$$S(M, J, Q) \tag{3}$$

In late 1960's and early 1970's, laws of classical black hole mechanics were discovered. Striking resemblance to laws of thermodynamics.
Zeroth black hole law says that surface gravity $\widehat{\kappa}$ is constant over the horizon of a stationary black hole.
First law is

$$dM = \widehat{\kappa}\frac{dA}{8\pi} + \omega_{\mathsf{H}} dJ + \Phi_{\mathsf{e}} dQ \tag{4}$$

where ω_{H} is angular velocity at horizon and Φ_{e} electrostatic potential.
Second law says that horizon area A must be nondecreasing in any classical process. (Singularity theorems: horizons don't bifurcate.)
Third law says that it is impossible to achieve $\widehat{\kappa}$=0 via a physical process such as emission of photons. (Does *not* say $S(T{=}0) = 0$!)

After doing many Gedankenexperiments, Bekenstein proposed that entropy of black hole should be proportional to area of event horizon. Hawking's semiclassical calculation of black hole temperature

$$T_H = \frac{\hbar \widehat{\kappa}}{2\pi} \tag{5}$$

made entropy-area identification precise by fixing the coefficient. (In semiclassical approximation, spacetime is treated classically, while matter fields interacting with it are treated quantum-mechanically.)

BH entropy (BH stands for Bekenstein-Hawking or Black Hole) is in any spacetime dimension d

$$S_{\mathsf{BH}} = \frac{A_d}{4\hbar G_d} \tag{6}$$

where A_d is area of event horizon, and G_d is Newton constant (dimensions of (length)$^{d-2}$) This is a *universal* result for any black hole, applicable to any theory with Einstein gravity as its classical action.

Enormous entropy: for Earth-mass BH, $r_H \sim 1\text{cm}$ and $S_{\mathsf{BH}} \sim 10^{66}$.

Rumour is that Hawking wants this engraved on his headstone...
à la Boltzmann...

Bekenstein would (presumably) be furious -
but the historical parallel could be quite precise!

Black hole degeneracy of states

In general, we can invert Boltzmann's formula: $\rho(E) = e^{S(E)}$. We also know the Bekenstein-Hawking formula for the black hole entropy,

$$S \sim \frac{r_H^{d-2}}{\ell_{P,d}^{d-2}} \tag{7}$$

We also have the relation between mass and horizon radius:

$$r_H^{d-3} \sim G_d M \sim \ell_{P,d}^{d-2} M \tag{8}$$

Putting these together, we have for the entropy

$$S \sim \left(\ell_{P,d} M\right)^{\frac{(d-2)}{(d-3)}} \tag{9}$$

Therefore, the black hole degeneracy of states scales as:

$$\rho_{\text{BH}}(M) \sim \exp\left[\left(\ell_{P,d} M\right)^{\frac{(d-2)}{(d-3)}}\right] \tag{10}$$

This is to be compared to the string degeneracy

$$\rho_{\text{string}} \sim \exp\left(\ell_s m\right) \qquad \text{Q.E.D.} \tag{11}$$

BH entropy is 1/4 area of event horizon in Planck units. So... S_{BH} scales like *area* rather than volume! Violates our QFT intuition about extensivity of thermodynamic entropy. Central idea behind holography.

There are several versions of holography...
Elevation to *principle* occurred with 't Hooft and then Susskind.

Idea: since entropy scales like area rather than volume, fundamental degrees of freedom describing quantum BH are characterised by a QFT with one fewer space dimensions and with Planck-scale UV cutoff.

Hawking radiation appears to violate second law – until realize it has entropy too that should be counted. $\longrightarrow$ Bekenstein bound.

(*) *Bousso bound* is *covariant* bound. Respected whenever semiclassical approximation applied *self-consistently*.
Think of Bousso bound as semiclassical proxy for fundamental law. Or a way of guiding definition of laws of physics when we can't even define an S-matrix (asymptopia doesn't exist or doesn't cooperate).

When Black Holes go bad: the Correspondence Principle

SUGRA describes low-energy approximation to string theory.
String theory has *two* expansion parameters which encode corrections to the lowest-order action, namely

- "sigma-model" loop-counting parameter α'
  (Since $\alpha' \equiv {\ell_{\text{s}}}^2$ is dimensionful, need to fold in measure of spacetime curvature to get a dimensionless parameter, e.g. $\ell_s^2 \mathcal{R}$.);
- string loop-counting parameter g_{s}.
  (For string loop corrections actually need $g_{\text{s}} e^{\Phi}$.).

Basic idea behind Correspondence Principle: stringy/braney degrees of freedom take over when SUGRA goes bad.

For *neutral black holes* corrections to flat metric scale like

$$ {r_H}^{d-3} = \frac{16\pi G_d M}{(d-2)\Omega_{d-2}} \sim {g_{\text{s}}}^2 {\ell_{\text{s}}}^{d-2} M \qquad (12) $$

Note: if fix mass M and radius r in units of ℓ_{s}, then metric becomes flat as $g_{\text{s}} \to 0$.

Black hole breakdown

SUGRA black hole solution breaks down in sense of the Correspondence Principle when curvature invariants at *horizon* are $\mathcal{O}(\ell_s)$.
Physical reason: horizon (not singularity) signals existence of BH.
Using horizon also gives rise to sensible answers which fit together in a coherent fashion under duality maps.

Curvature invariant nonzero for neutral black hole: $R^{\mu\nu\lambda\sigma}R_{\mu\nu\lambda\sigma} \sim r_H^{-4}$ so breakdown of SUGRA occurs when

$$r_H \sim \ell_{\mathsf{s}} \tag{13}$$

Thermodynamic temperature and entropy of black hole scale as

$$T_{\mathsf{H}} \sim \frac{1}{r_H} \qquad S_{\mathsf{BH}} \sim \frac{{r_H}^{d-2}}{G_d} \tag{14}$$

so Hawking temperature at correspondence point is $T_{\mathsf{H}} \sim 1/\ell_{\mathsf{s}}$
(this is the Hagedorn temperature!)

Q: What replaces SUGRA when SUGRA fails?
A: *Follow the quantum numbers!*

String Legos

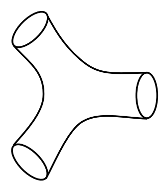
Perturbative: fundamental (super)strings

Basic length scale $\boxed{\ell_s}$. Light at weak coupling.

Groundstate of open string $\supset$ spin-1 gauge field .
Groundstate of closed string $\supset$ spin-2 graviton .

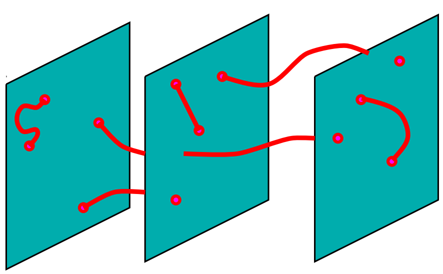
Infinite tower of excitations of increasing m^2, s [no propagating ghosts!]

Coupling constant $\boxed{g_s}$.



Nonperturbative: Dp-branes: Hypersurfaces where open strings end.
Heavy at weak coupling (tension$\sim 1/g_s$).

e.g.
N=3



[Plus others: NS5, Op, KK, W, etc.]

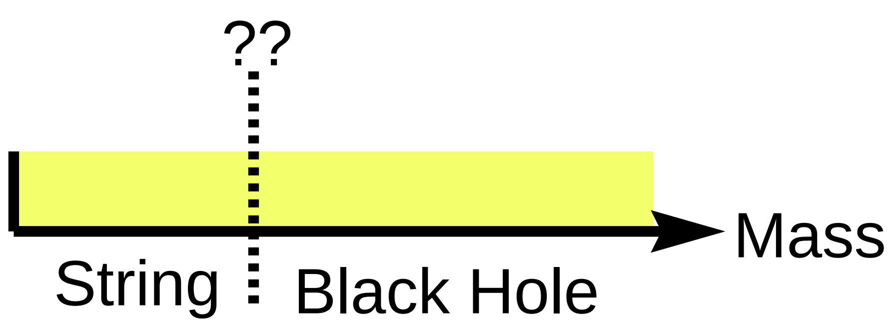
This string physics motivated ADD, RS "brane world" scenarios.

Correspondence Point

To resolve Schwarzschild decay puzzle, need string object to match onto. Simplest string theory object which carries only mass quantum number is closed fundamental string. Other motivations: Occam's razor, and lack of involvement of g_s in correspondence point.

Expectations?

Black holes and stringy/braney states typically do *not* have identical entropy for all values of parameters; rather, transition between black hole and string degrees of freedom occurs at Correspondence Point. Existence of a correspondence point for every system studied is a highly nontrivial fact about string theory and the degrees of freedom that represent systems in it in different regions in parameter space.



Assume that $g_s \ll 1$ so can use *free* spectrum computation; easily justified *a posteriori*). At leading order, recall that we had

$$\rho_{\text{string}}(m) \sim e^{m\ell_s} \tag{15}$$

Boltzmann entropy of string state is

$$S_{\text{string}} = \log(\rho_{\text{string}}) \sim \frac{m}{\ell_{\text{s}}} \tag{16}$$

Matching masses at correspondence point for general horizon radius

$$M \sim \frac{{r_H}^{d-3}}{{g_{\text{s}}}^2 {\ell_{\text{s}}}^{d-2}} \sim m \tag{17}$$

yields the general entropy ratio

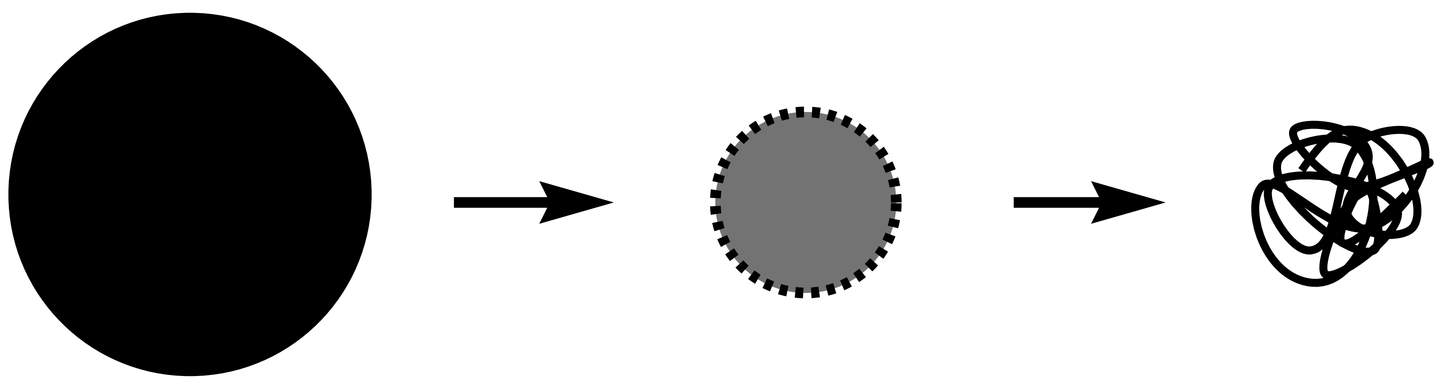
$$\frac{S_{\text{BH}}}{S_{\text{string}}} \sim \frac{{r_H}^{d-2}}{{g_{\text{s}}}^2 {\ell_{\text{s}}}^{d-2}} \frac{{g_{\text{s}}}^2 {\ell_{\text{s}}}^{d-3}}{{r_H}^{d-3}} \quad \sim \frac{r_H}{\ell_{\text{s}}} \tag{18}$$

So indeed, crossover from black hole to string state indeed happens at $r_H \sim \ell_{\text{s}}$. And BH dominates for large mass, while string dominates at small mass.

More work has been done on physics of transition between the black hole and string state.

This river runs deeper!

Conservative direction to run matching argument says: string state will collapse to a black hole when it gets heavy enough.
Radical direction to run argument is other way: correspondence principle says endpoint of Hawking radiation for a Schwarzschild black hole is a hot string.



Hot string then subsequently decays by emitting radiation until get bunch of massless radiation. (An interesting fact about this decay of a massive string state in perturbative string theories is that spectrum is thermal, when averaged over degenerate initial states.)

Charges

String theorists have names for higher-dimensional analogues of electric charge - charge per unit length, area, volume... They are carried by *branes*. [*Bulk* charges, distinct from brane gauge fields.]

- "NS-NS" charge is carried by strings*.
- "R-R" charge is carried by D-branes

Cool fact: Entropy is exactly the same whether you compute it with branes in higher dimension, or compactified branes in lower dimension!!

For R-R charged systems, what happens?

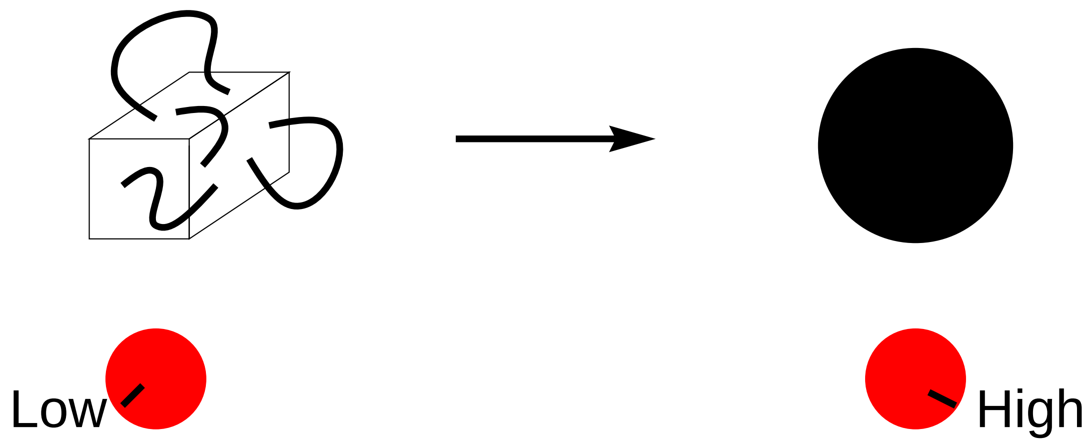
In terms of advances in precise computations of black hole entropy, most important examples of application of correspondence principle are systems with two or more R-R charges. This is case both for "BPS" and "near-BPS" black holes. Physics calculation shows crucial fact: for these systems, scaling works in such that there is no special correspondence point – exact comparisons can be made to weak-coupling stringy/braney calculations for black holes of *any* horizon radius. Let's do it now!

*And by NS5-branes

Strominger-Vafa

One of biggest results of second superstring revolution was Jan'96 counting of entropy of some very special black holes, using D-braney stringy statistical mechanics. HUGE! Finally string theory had a result that blew socks off people all around the physics community!

Special technical tool:*supersymmetric nonrenormalisation theorems*.
Allows*, with enough supersymmetry, to say that
weak coupling degeneracy = strong coupling degeneracy



Red: coupling constant knob on God's stove (- female, and cooks!)

*See next page for details

For reading later: Supersymmetry (SUSY)

Supersymmetry: bosons $\leftrightarrow$ fermions. Operator Q.

Mixes nontrivially with Hamiltonian:

$$\left\{Q, Q^\dagger\right\} = H - G$$

Here, G= symmetry operator, e.g. electric charge.

Sandwich normalised physical state around this:

$$\langle \text{phys}|QQ^\dagger|\text{phys}\rangle + \langle \text{phys}|Q^\dagger Q|\text{phys}\rangle = m - g$$

$$= \left|\left|Q|\text{phys}\rangle\right|\right|^2 + \left|\left|Q^\dagger|\text{phys}\rangle\right|\right|^2 \geq 0$$

Therefore, mass bounded below: "BPS bound"

$$m \geq g$$

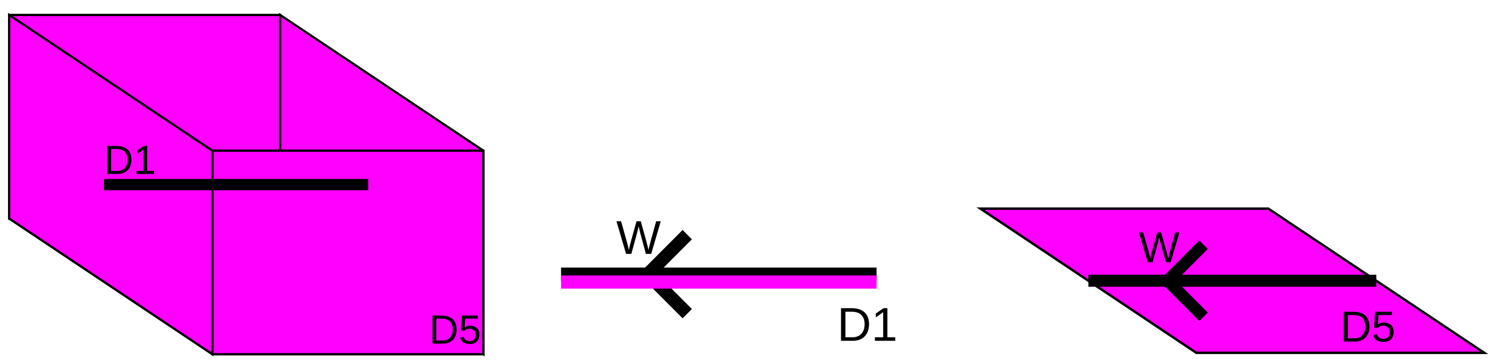
Special property of SUSY: bosons and fermions are paired, so quantum corrections to BPS bound (+ and -) cancel!

$\Rightarrow$ can reliably follow "BPS states" for any coupling.
Ability to do this crucial in discovery of dualities.

Brane cooking

Recipe for making BPS (supersymmetric) black holes is considerably simpler than recipe for making nonextremal ones. First part of recipe is how to combine different ingredients.

No-force pairs:



Therefore, $W \parallel D1 \parallel D5$ can all be in neutral equilibrium in a mutually consistent fashion!

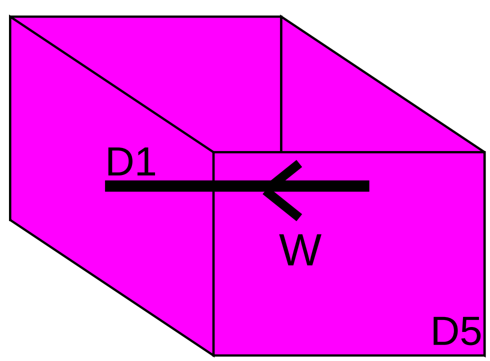
Problems with too few ingredients

BPS black holes in dimensions $d = 4 \dots 9$ may be constructed from BPS building blocks. Typically, however, they have zero horizon area and therefore non-macroscopic entropy.

E.g.1: if D1 is wrapped on circle S^1, its gravity wants to shrink the circle to zero size at the horizon.

E.g.2: if D1 joined by D5 wrapped on $T^4 \times S^1$, D1 pressure balances D5 tension and T^4 stays finite size at horizon. But S^1 shrinks even more!

E.g.3: add gravitational wave W moving along circle at c. This pushes out with enough pressure along S^1 to make it also stay finite size at horizon.



Just-right recipe

Thermodynamics? (D1,D5,W) BPS black hole has $T_{\mathsf{H}} = 0$.
For Bekenstein-Hawking entropy, find amazingly simple result

$$S_{\mathsf{BH}} = 2\pi\sqrt{N_1 N_5 N_m} \tag{19}$$

Independent of all moduli. SUGRA solution valid for large $N_{1,5,m}$.

The D-brane picture: Same quantum numbers for our ingredients, but our red knob turned very low now. Just do string/D-brane stat mech.

Beginning ingredients: D1 branes and D5 branes. What are degrees of freedom carrying momentum quantum number?

D5 branes and smeared D1 branes have a symmetry group $SO(1,1)\times SO(4)_{||}\times SO(4)_{\perp}$. This symmetry forbids (rigid) branes from carrying linear or angular momentum, so we need something else.

Obvious modes in the system to try are massless 1-1, 5-5 and 1-5 strings, which come in both bosonic and fermionic varieties.

• Momentum N_m/R carried by bosonic and fermionic strings, $1/R$ each.
• Angular momentum is carried *only* by fermionic strings, $\frac{1}{2}\hbar$ each.
Both linear and angular momenta can be built up to macroscopic levels.

Next step: identify degeneracy of states of this system.

Choose the four-volume of the torus small by comparison to circle radius, so can do computation in *two-dimensional CFT*:
For large N's, as we have here, we can use Cardy formula

$$\Omega(N_m) \sim \exp\sqrt{\frac{\pi\, c\, E\,(2\pi R)}{3}} = \exp\left(2\pi\sqrt{\frac{c}{6}\, ER}\right) \tag{20}$$

We know R, radius of circular dimension. Need c and E.
Central charge

$$c = n_{\text{bose}} + \tfrac{1}{2} n_{\text{fermi}} \tag{21}$$

How many bosons (and fermions) do we have???
Boson and fermion count in system of D1, D5 and open strings?
Can be done rigorously; here is the basic physics:
• $N_1 N_5$ 1-5 strings can move in 4 torus directions so $c = (4+2)N_1 N_5$.

Now, how about energy E? Supersymmetry dictates that in $d = 1 + 1$ $E = |P|$. In $d = 1 + 1$ things can move only to R or L. Our sign conventions make us have R-moving groundstate, and put all the action in L-movers. Momentum was $P = \pm N_m/R$, so $E = N_m/R$.

Therefore, using Cardy's formula,

$$S_{\text{micro}} = 2\pi\sqrt{N_1\, N_5\, N_m} \tag{22}$$

This agrees exactly with black hole result!!

<u>Rotation</u>

In $d = 5$ there are two independent angular momentum parameters. BPS entropy:

$$S_{\text{BH}} = 2\pi\sqrt{N_1 N_5 N_m - J^2} \tag{23}$$

Note $|J_{\text{max}}| = \sqrt{N_1 N_5 N_m}$; beyond J_{max}, closed timelike curves develop, and entropy walks off into complex plane.

Basic D-braney physics is simple: aligning $\frac{1}{2}\hbar$'s all in a row to build up macroscopic angular momentum *costs oscillator degeneracy*. Energy is reduced, and so is entropy. Get entropy agreeing with black hole calculation again.

Turning on finite temperature

SUGRA: nonextremal branes cannot be in static equilibrium with each other – they want to fall towards each other, and they do *not* satisfy simple superposition rule.

Physics for *near-BPS*: new energy adds a small number of *R-movers* as well as L-movers. Think of R-movers and L-movers as dilute gases, interacting only very infrequently. Energy and momentum are additive, and so is entropy.

Amazingly, entropy agrees with near-extremal black brane entropy. Why? - no theorem protecting degeneracy of non-BPS states. What is going on physically is that conformal symmetry possessed by the $d = 1 + 1$ theory is sufficiently restrictive, even when it is broken by finite temperature, for black hole entropy to be reproduced by field theory.

Also greybody factors can be computed. Mindbogglingly, D-brane story gives same answer!! *Multi-parameter agreement.* ↑⟶