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Lecture 3.

D-branes extremal and nonextremal, and where they go bad.

Concepts introduced:

- BPS branes, and the no-force condition
- D-branes as probes
- nonextremal D-branes
- Gregory-Laflamme instability
- the Black Hole Correspondence Principle
- where Schwarzschild black holes go bad
- where BPS D-branes go bad
- singularities and what to do about them

Here we study BPS Dp -brane spacetimes, with symmetry $SO(1, p) \times SO(9 - p)$. In string frame, solutions are

$$\begin{aligned} dS^2 &= H_p(r)^{-\frac{1}{2}} \left(-dt^2 + dx_{\parallel}^2 \right) + H_p(r)^{+\frac{1}{2}} dx_{\perp}^2 \\ e^{\Phi} &= H_p(r)^{\frac{1}{4}(3-p)} \\ C_{01\dots p} &= g_s^{-1} H_p(r)^{-1} \end{aligned} \quad (1)$$

Function $H_p(r)$ is harmonic; it satisfies $\partial_{\perp}^2 H_p(r) = 0$,

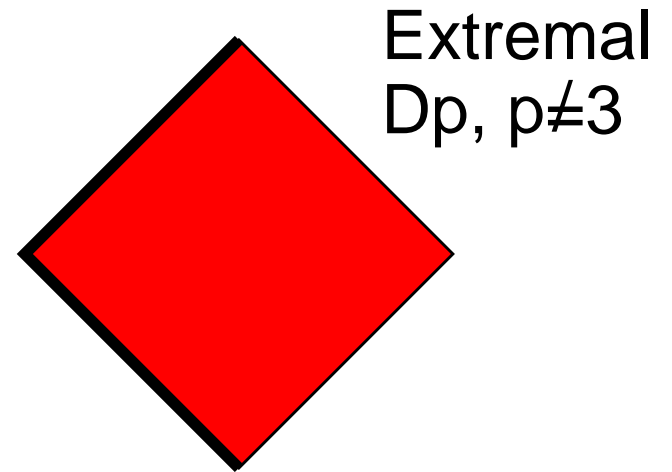
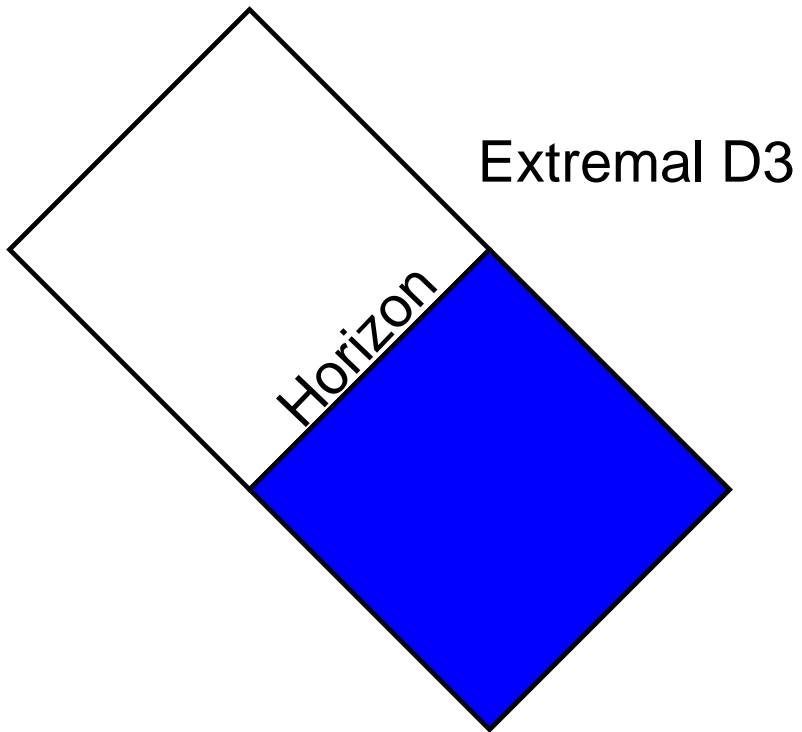
$$H_p = 1 + \frac{c_p g_s N_p \ell_s^{7-p}}{r^{7-p}} \quad c_p \equiv (2\sqrt{\pi})^{(5-p)} \Gamma \left[\frac{1}{2}(7 - p) \right] \quad (2)$$

Function H_p would still be harmonic if 1 were missing. Asymptotically flat part of geometry would be absent for this solution (see later in Lecture).

Double horizon of Dp -brane geometry occurs at $r = 0$ in isotropic coords. In every case except D3-branes, singularity at $r = 0$ as well. Hence, for Dp -branes with $p \neq 3$, singularity is null. (Only D6 naked.)

D3-brane maximal analytic extension nonsingular. Near-horizon D3-brane spacetime is $AdS_5 \times S^5$.

BPS D_p -brane Penrose diagrams:



No-force condition for BPS branes

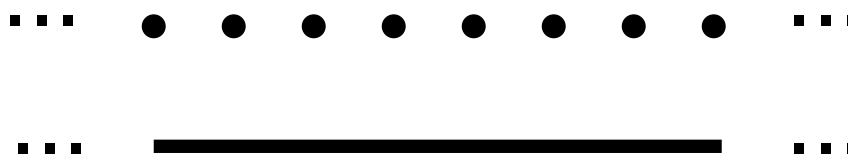
In fact, BPS multi-centre solutions are also allowed because the equation for H_p , $\partial_{\perp}^2 H_p = 0$, is linear (this is exact, not an approximation!)

$$H_{\bar{p}} = 1 + c_p g_s N_p \ell_s^{7-p} \sum_i \frac{1}{|x_{\perp} - x_{\perp i}|^{7-p}}; \quad (3)$$

Physical reason this works is that parallel BPS branes of same kind are in static equilibrium: repulsive gauge forces cancel against attractive gravitational and dilatonic forces. *Neutral* equilibrium.

Consider making an array of D_p branes.

Take limit \longrightarrow linear
density of branes.



Linear density of p -branes per unit length in string units becomes number of $(p+1)$ -branes, in T-dual picture: $N_{p+1} = N_p / (R/\ell_s)$.

Worldvolume directions are already isometry directions, so in reducing along a worldvolume direction of a D_p -brane we get $N_{p-1} = N_p$.

Dp-branes as probes

Consider what happens when probe Dp-brane spacetime, using another Dp-brane. Treat probe as “test” brane, i.e. ignore backreaction. Very good approximation provided that N , the number of branes sourcing spacetime, is large.

Action of a *flat* probe brane in a SUGRA background has two pieces,

$$S_{\text{probe}} = S_{\text{DBI}} + S_{\text{WZ}} \quad (4)$$

which are, to lowest order in derivatives,

$$\begin{aligned} S_{\text{DBI}} &= -\frac{1}{g_s(2\pi)^p l_s^{p+1}} \int d^{p+1}\sigma e^{-\Phi} \sqrt{-\det \mathbb{P} \left(G_{\alpha\beta} + [2\pi F_{\alpha\beta} + B_{\alpha\beta}] \right)} \\ S_{\text{WZ}} &= -\frac{1}{(2\pi)^p l_s^{p+1}} \int \mathbb{P} \exp (2\pi F_2 + B_2) \wedge \bigoplus_n C_n \end{aligned} \quad (5)$$

where σ are worldvolume coordinates and \mathbb{P} denotes pullback to world-volume of bulk fields. [More details in TASI notes.]

We had for SUGRA background fields

$$\begin{aligned} dS^2 &= H_p^{-\frac{1}{2}} \left(-dt^2 + dx_{\parallel}^2 \right) + H_p^{+\frac{1}{2}} dx_{\perp}^2 \\ e^{\Phi} &= H_p^{\frac{1}{4}(3-p)} \\ C_{01\dots p} &= g_s^{-1} H_p^{-1} \end{aligned}$$

Fix worldvolume reparametrisation invariance: set static gauge

$$X^{\alpha} = \sigma^{\hat{\alpha}} \quad \alpha = 0, \dots, p \quad (6)$$

Take $9 - p$ transverse scalar fields X^i to be functions of time only,

$$X^i = X^i(t) \quad i = p + 1 \dots 9 \quad (7)$$

Denote transverse velocities as v^i ,

$$v^i \equiv \frac{dX^i}{dt} \quad (8)$$

Now we can compute pullback of metric to brane.

$$\begin{aligned} \mathbb{P}(G_{00}) &= (\partial_0 X^{\alpha})(\partial_0 X^{\beta})G_{\alpha\beta} + (\partial_0 X^i)(\partial_0 X^i)G_{ij} \\ &= G_{00} + G_{ij}v^i v^j = -H_p^{-\frac{1}{2}} + H_p^{+\frac{1}{2}} \vec{v}^2; \\ \mathbb{P}(G_{\alpha\beta}) &= H_p^{-\frac{1}{2}} \end{aligned} \quad (9)$$

Next ingredient we need is determinant of metric. To start, notice that

$$-\det \mathbb{P}(G_{\alpha\beta})(\vec{v} = \vec{0}) = H_p^{-\frac{1}{2}(p+1)} \quad (10)$$

so that

$$\sqrt{-\det \mathbb{P}(G_{\alpha\beta})} = H_p^{-\frac{1}{4}(p+1)} \sqrt{1 - \vec{v}^2 H_p} \quad (11)$$

Putting this together with expression for dilaton and R-R field, obtain

$$S_{\text{DBI}} + S_{\text{WZ}} = \frac{1}{(2\pi)^{p+1} g_s \ell_s^{p+1}} \int d^{p+1} \sigma \left[-H_p^{-1} \sqrt{1 - \vec{v}^2 H_p} + H_p^{-1} \right] \quad (12)$$

From this action, learn that position-dependent part of static potential vanishes, as it must for a supersymmetric system.

In addition, can expand out this action in powers of transverse velocity. To lowest order,

$$S_{\text{probe}} = \frac{1}{(2\pi)^{p+1} g_s \ell_s^{p+1}} \int d^{p+1} \sigma \left[\frac{1}{2} \vec{v}^2 + \mathcal{O}(\vec{v}^4) \right] \quad (13)$$

Metric on moduli space, which is coefficient of $v^i v^j$, is flat. This is in fact a consequence of having sixteen supercharges preserved by static system.

Nonextremal branes

In string frame and with a Schwarzschild-type radial coordinate r , SUGRA fields of D_p -branes can be written as

$$dS^2 = D_p(r)^{-\frac{1}{2}} \left(-K(r)dt^2 + dx_{\parallel}^2 \right) + D_p(r)^{\frac{1}{2}} \left(\frac{dr^2}{K(r)} + r^2 d\Omega_{8-p}^2 \right) \quad (14)$$

where

$$D_p(r) = 1 + \zeta c_p g_s N \left(\frac{\ell_s}{r} \right)^{7-p} \quad K(r) = 1 - \left(\frac{r_H}{r} \right)^{7-p} \quad (15)$$

Other fields are

$$e^{\Phi} = D_p(r)^{(3-p)/4} \quad (16)$$

$$C_{01\dots p} = \frac{1}{\zeta g_s D_p(r)} \quad (17)$$

Change in harmonic function due to nonextremality is codified* in parameter $\zeta \in [1, 0]$:

$$\zeta = \sqrt{1 + \left[\frac{r_H^{7-p}}{2c_p g_s N \ell_s^{7-p}} \right]^2} - \left[\frac{r_H^{7-p}}{2c_p g_s N \ell_s^{7-p}} \right] \quad (18)$$

*Don't worry about R-R field as $\zeta \rightarrow 0$; constant is not gauge-invariant

If make this spacetime by boosting with parameter β , relation is

$$\zeta = \tanh \beta \quad (19)$$

Two horizons, at $r = r_H, r = 0$. Causal structure? Inner horizon is singular, so Penrose diagram in (t, r) plane is that of Schwarzschild.

ADM mass per unit p -volume and charge are

$$\begin{aligned} \tau &\equiv \frac{M_p}{(2\pi)^p V_p} = \frac{(r_H/\ell_s)^{7-p}}{c_p g_s^2 (2\pi)^p \ell_s^{p+1}} \left[\cosh^2 \beta + \frac{1}{(7-p)} \right] \\ N &= \frac{(r_H/\ell_s)^{7-p}}{c_p g_s} [\cosh \beta \sinh \beta] \end{aligned} \quad (20)$$

BPS bound and cosmic censorship say $\tau \geq |N/g_s|$.

Hawking temperature and Bekenstein-Hawking entropy are

$$\begin{aligned} T_H &= \frac{(7-p)}{4\pi r_H \cosh \beta} \\ S_{\text{BH}} &= \frac{\Omega_{8-p} r_H^{8-p} \cosh \beta}{4G_{10-p}} \end{aligned} \quad (21)$$

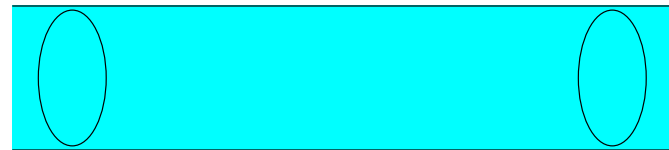
Extremal solution has double horizon, and zero Bekenstein-Hawking entropy S_{BH} . Hawking temperature T_H of extremal brane is also zero.

Gregory-Laflamme instability

Important instability of nonextremal p -branes was discovered by GF. Simplest example of this phenomenon occurs for neutral objects.

Consider first a neutral black string in $d+1$ dimensions.

Translation-invariant solution is $(d \text{ dimensional Schwarzschild}) \times \mathbb{R}$.



Horizon radius \hat{r}_H , and mass per unit length $2\pi R$

$$\frac{M_{\text{string}}}{R} \sim \frac{\hat{r}_H^{d-3}}{G_d} \quad S \sim \frac{\hat{r}_H^{d-2}}{G_d} \quad (22)$$

Question: is this thing stable?

Do perturbation theory around the translation-invariant background.

Discover tachyonic mode!

Thus, this black string is unstable.

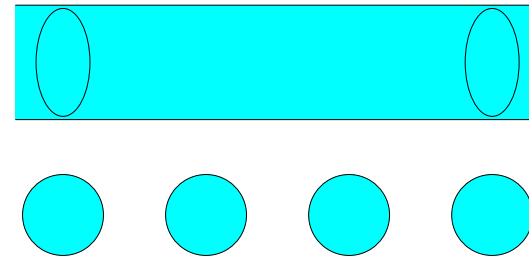
Later on, people found a potential endpoint for evolution of instability.

Consider also an array of $d+1$ -dimensional black holes, spaced by a distance $2\pi R$. [Array has to be infinite in order to get a static solution.] Roughly, this solution 'per unit $2\pi R$ ' is $d+1$ -dimensional black hole.

Or compactify this all on circle of radius R ; compare localised black hole (roughly BH of $d+1$) with translation-invariant black string.

Array: horizon radius r_H , and

$$\frac{M_{\text{array}}}{R} \sim \frac{r_H^{d-2}}{G_{d+1}} \quad (23)$$



Relation between R, r_H, \hat{r}_H chosen to give same mass per unit length as translation-invariant black string: $M_{\text{array}} = M_{\text{string}}$ so that

$$r_H^{d-2} \sim \hat{r}_H^{d-3} R \quad (24)$$

Work in microcanonical ensemble, – appropriate for fixed energy (mass) of system. Consider entropy. Physics point: entropy \propto area, so array of black holes has different entropy than black string – spheres scale differently than cylinders.

For array, approximately

$$S \sim \frac{r_H^{d-1}}{G_{d+1}} \quad (25)$$

So need to find which configuration has biggest entropy:

$$\frac{S_{\text{array}}}{S_{\text{string}}} \sim \frac{r_H^{d-1}}{G_{d+1}} \frac{G_d}{\hat{r}_H^{d-2}} \quad (26)$$

Now use $G_{d+1} \sim G_d \times R$, to find

$$\frac{S_{\text{array}}}{S_{\text{string}}} \sim \frac{r_H^{d-1}}{R \hat{r}_H^{d-2}} \quad (27)$$

and using mass equality relation $r_H^{d-2} \sim \hat{r}_H^{d-3} R$ gives

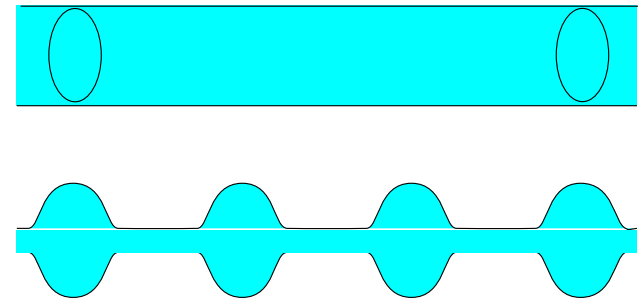
$$\frac{S_{\text{array}}}{S_{\text{string}}} \sim \left(\frac{R}{\hat{r}_H} \right)^{1/(d-1)} \sim \left(\frac{R}{r_H} \right)^{1/(d-2)} \quad (28)$$

So array has biggest entropy for large R , in particular for $R \rightarrow \infty$. Thus uncompactified translation-invariant black string always unstable. Then, argument goes, in $d + 1$ dim it becomes black hole array, which is then unstable (via similar mechanism as Jeans instability) to becoming one ginormous black hole. Black string stable for small R .

But what actually happens for large R ??

Turning the translation-invariant black string into the array would require uncovering singularity! Nakedness! Horowitz-Maeda proved that cannot happen in finite affine parameter – using arguments very similar to original *singularity theorems* of Penrose. Weaker argument that cannot happen in infinite affine parameter...

Conjectured that endpoint is instead a lumpy black string!
Interplay with breakdown of no-hair theorems in higher- d .



Gubser-Mitra conjecture (in AdS context), also Reall (AF): spacetimes have classical tachyonic instability iff have thermodynamic instability.

Interesting to ask: is the transition 'first-order' or 'second-order'?
(Compact case: sizes $\gg \ell_P$ so effectively ∞ system \Rightarrow OK.)

Question of endpoint being investigated with state-of-art numerics by Choptuik's group at UBC. But see also Barak Kol hep-th/0206220.

Original GF paper was entitled

“Black Strings and p -Branes Are Unstable” .

Horrors! Are all uncompactified SUGRA p -brane spacetimes of string theory rubbish?? What about BPS branes?

Nope. Several ways to see this. Two are:

1. Tachyonic mode disappears in extremal case; length scale of instability goes to infinity as nonextremality parameter goes to zero.
2. BPS branes are protected by Bogomolnyi bound. What could a BPS brane break up into? Dp -brane has conserved charge. E.g. if uncompactified BPS D1-brane wanted to break up into array of D0-branes, it would be out of luck because D0's and D1's do not occur in same theory. If D1 were wrapped on a circle, there would be a regime ($R < \ell_s$) in which we should more properly describe it in the T-dual theory, i.e. as a D0. In this case configuration is still stable, of course.

We did not have time to discuss branes of low codimension (give rise to IR problems – logarithmic and linear potentials). But domain walls separating different vacua of a theory will be stable even if they are neutral, because it would cost an infinite amount of energy for them to break up. So no Gregory-Laflamme for them.

When SUGRA goes bad: the Black Hole Correspondence Principle

SUGRA actions we met yesterday describe low-energy approximations to string theory. Appropriate for situations where corrections are small. String theory has *two* expansion parameters which encode corrections to the lowest-order action, namely sigma-model loop-counting parameter α' and string loop-counting parameter g_s . Since $\alpha' \equiv \ell_s^2$ is dimensionful, need to fold in measure of spacetime curvature to get a dimensionless parameter, e.g. $\ell_s^2 \mathcal{R}$. For string loop corrections need $g_s e^\Phi$.

Basic idea behind Correspondence Principle: stringy/braney degrees of freedom take over when SUGRA goes bad.

Neutral black holes

$$dS_d^2 = - \left[1 - \left(\frac{r_H}{r} \right)^{d-3} \right] dt^2 + \left[1 - \left(\frac{r_H}{r} \right)^{d-3} \right]^{-1} dr^2 + r^2 d\Omega_{d-2}^2 \quad (29)$$

where

$$r_H^{d-3} = \frac{16\pi G_d M}{(d-2)\Omega_{d-2}} \sim g_s^2 \ell_s^{d-2} M \quad (30)$$

Note: if fix mass M and radius r in units of ℓ_s , then metric becomes flat as $g_s \rightarrow 0$.

SUGRA black hole solution breaks down in sense of the Correspondence Principle when curvature invariants at *horizon* are $\mathcal{O}(\ell_s)$. Physical reason: horizon (not singularity) signals existence of black hole. Using horizon also gives rise to sensible answers which fit together in a coherent fashion under duality maps.

Curvature invariant nonzero for neutral black hole: $R^{\mu\nu\lambda\sigma}R_{\mu\nu\lambda\sigma} \sim r_H^{-4}$ so breakdown of SUGRA occurs when

$$r_H \sim \ell_s \quad (31)$$

Thermodynamic temperature and entropy of black hole scale as

$$T_H = \frac{(d-3)}{4\pi r_H} \quad S_{\text{BH}} = \frac{\Omega_{d-2} r_H^{d-2}}{4G_d} \quad (32)$$

so Hawking temperature at correspondence point is $T_H \sim 1/\ell_s$.

Q: What replaces SUGRA when SUGRA fails?

A: *Follow the quantum numbers!*

Simplest string theory object which carries only mass quantum number is closed fundamental string. Other motivations: Occam's razor, and lack of involvement of g_s in correspondence point.

Expectations? Black holes and stringy/braney states typically do *not* have identical entropy for all values of parameters; rather, transition between black hole and string degrees of freedom occurs at Correspondence Point. Existence of a correspondence point for every system studied is a highly nontrivial fact about string theory and the degrees of freedom that represent systems in it in different regions in parameter space.

To finish neutral BH example, need statistical entropy of closed string states due to large degeneracy at high mass. Standard result in perturbative string theory; see GSW or Polchinski book. Assume that $g_s \ll 1$ so can use *free* spectrum computation; this assumption will be justified *a posteriori*.

Using relation between oscillator number N and mass m , $\ell_s^2 m^2 \sim N$ have at leading order for large m the degeneracy

$$d_m \sim e^{m\ell_s} \tag{33}$$

Quantity $1/\ell_s$ is Hagedorn temperature. Above Hagedorn temperature, canonical ensemble is in fact no longer well-defined. This happens because partition function diverges,

$$Z = \int_0^\infty dm e^{m\ell_s} e^{-m/T} \rightarrow \infty \quad \text{above } T = 1/\ell_s \quad (34)$$

At Hagedorn temperature, excited string becomes very long and floppy; single long string dominates thermal ensemble. Boltzmann entropy of string state is

$$S_{\text{string}} = \log(d_m) \sim \frac{m}{\ell_s} \quad (35)$$

Matching masses at correspondence point for general Schwarzschild radius

$$M \sim \frac{r_H^{d-3}}{g_s^2 \ell_s^{d-2}} \sim m \quad (36)$$

yields the general entropy ratio

$$\frac{S_{\text{BH}}}{S_{\text{string}}} \sim \frac{r_H^{d-2}}{g_s^2 \ell_s^{d-2}} \frac{g_s^2 \ell_s^{d-3}}{r_H^{d-3}} \sim \frac{r_H}{\ell_s} \quad (37)$$

So indeed, crossover from black hole to string state indeed happens at $r_H \sim \ell_s$. And BH dominates for large mass, while string dominates at small mass.

Justifying previous assumption about string coupling at correspondence transition point: Since entropy at correspondence is $S \sim m\ell_s$, and $\ell_s m \sim \sqrt{N}$, we get $S \sim \sqrt{N}$. Also, $S \sim \ell_s^{d-2}/G_d \sim 1/g_s^2$. So $g_s \sim N^{-\frac{1}{4}}$ at transition. This is indeed weak coupling since N is very large.

More work has been done on physics of transition between the black hole and string state.

This river runs deeper! Conservative direction to run matching argument says: string state will collapse to a black hole when it gets heavy enough. Radical direction to run argument is other way: correspondence principle says endpoint of Hawking radiation for a Schwarzschild black hole is a hot string. Hot string then subsequently decays by emitting radiation until get bunch of massless radiation. (An interesting fact about this decay of a massive string state in perturbative string theories is that spectrum is thermal, when averaged over degenerate initial states.)

Overall, picture of decay of Schwarzschild BH in string theory is consistent with expectations that a truly unified theory should not allow loss of quantum coherence. [TASI notes: other NS-NS cases.]

For R-R charges, what happens? Here, energy above extremality ΔE can be carried by either open or closed fundamental strings – as long as they are close to the D-branes. Open and closed strings have different equations of state and different entropy. Again, assume weak string coupling; this assumption can be justified a posteriori. For open strings, free massless gas

$$\Delta E_{\text{open}} \sim N_p^2 V_p T^{p+1} \quad S_{\text{open}} \sim N_p^2 V_p T^p \quad \Rightarrow S_{\text{open}}(\Delta E_{\text{open}}) \quad (38)$$

while for closed strings, equation of state is

$$S_{\text{closed}} \sim \ell_s \Delta E_{\text{closed}} \quad (39)$$

Find: open strings dominate for near-extremal BH, closed strings dominate for nearly-neutral BH.

In terms of advances in precise computations of black hole entropy, most important examples of application of correspondence principle are systems with two or more R-R charges. This is case both for BPS and near-BPS black holes. Physics calculation shows crucial fact: for these systems, scaling works in such that there is no special correspondence point – exact comparisons can be made to weak-coupling stringy/braney calculations for black holes of *any* horizon radius. We will discuss spectacular success of these microscopic calculations in Lecture 4.

Where BPS D p -branes go bad

$$\begin{aligned}
dS^2 &= H_p(r)^{-\frac{1}{2}} \left(-dt^2 + dx_{\parallel}^2 \right) + H_p(r)^{+\frac{1}{2}} dx_{\perp}^2 \\
e^{\Phi} &= H_p(r)^{\frac{1}{4}(3-p)} \\
C_{01\dots p} &= g_s^{-1} H_p(r)^{-1}
\end{aligned} \tag{40}$$

where

$$H_p = 1 + \frac{c_p g_s N_p \ell_s^{7-p}}{r^{7-p}} \tag{41}$$

Ricci scalar is nonzero:

$$R_G = -\frac{1}{4}(p^2 - 4p - 17) (\partial_r H_p)^2 H_p^{-\frac{5}{2}} \tag{42}$$

Since the harmonic function $H_p \sim r^{p-7}$ near $r = 0$, have

$$R_G \sim r^{\frac{5}{2}(7-p)} \left(r^{p-8} \right)^2 \sim r^{\frac{1}{2}(3-p)} \tag{43}$$

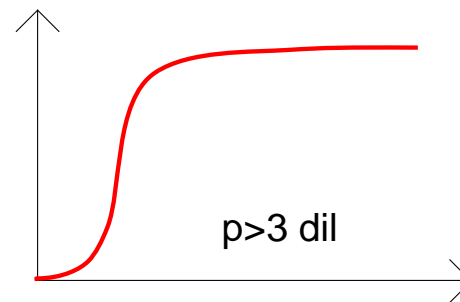
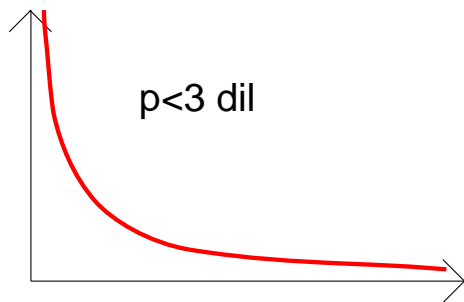
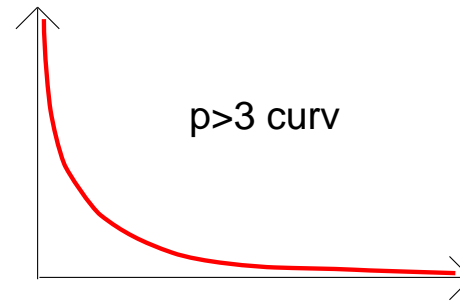
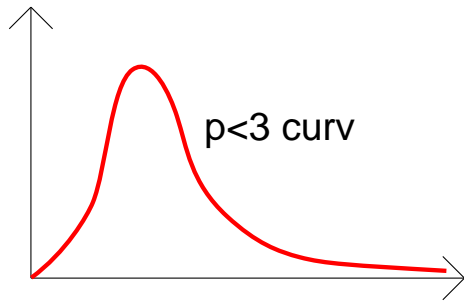
Blows up at $r = 0$ for big branes $p > 3$. Similarly for $R^{\mu\nu} R_{\mu\nu}$, $R^{\mu\nu\lambda\sigma} R_{\mu\nu\lambda\sigma}$.

Dilaton behaves differently:

$$e^{\Phi} = H_p^{\frac{1}{4}(3-p)} \sim r^{-\frac{1}{4}(7-p)(3-p)} \tag{44}$$

This blows up at $r = 0$ for small branes, i.e. for $p < 3$.

(For $p = 3$, recall from Lecture 2 that curvature and dilaton are both boring.)



Note interesting fact: if asymptotically flat part of geometry is removed, i.e. lose 1 in harmonic function H_p , then behaviour of both curvature and the derivative of dilaton becomes monotonic. This turns out to be a crucial SUGRA fact in context of Dp -brane gravity/gauge correspondences of [IMSY].

Singularities

Does string theory always fix singularities by smoothing them out?

[Horowitz-Myers]: *not always!*

Some singularities are so bad they should be thrown away altogether. Prototypical example is negative-mass Schwarzschild geometry. Since $M < 0$, horizon is absent, so singularity is naked. If singularity were smoothed out by stringy phenomena, resulting finite-sized blob would be an allowed object with overall negative mass. It would then destabilise vacuum - via pair production, for example. Upshot: negative-mass Schwarzschild geometry is a figment of classical physicist's imagination.

Important note: whether a spacetime geometry is singular depends on dimension of SUGRA theory it is embedded in. Some spacetimes singular in lower- d are nonsingular when lifted to higher- d . For understanding possible resolution of singularities in terms of basic stringy objects like D-branes, best dimension to do singularity analysis is $d=10$, which is dimension in which D-branes naturally live. Generally more confusing to try to do analysis directly in lower dimensions. Also, need to be sure that any operation you do in SUGRA also makes sense in string theory.