On the mixed 0-form/1-form anomaly:

pouring the new wine into old bottles

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with **Andrew Cox** *and* **F. David Wandler**

2106.11442 + work (in progress) since …

- **importance of symmetries in QFT**
- **'t Hooft anomalies since1980's: e.g. Seiberg dualities, preons**
- **new developments !**

this talk

higher-form/discrete: *Gaiotto, Kapustin, Komargodski, Seiberg 2014-…*

"[GKKS+]"

mixed 0-form/1-form anomaly "new" vs "old"…

this talk: theories with 1-form center symmetry + 0-form discrete

1. pure 4d YM, any G, $\theta = 0$ or π

parity: at $\theta = 0,\pi$ only

$$\hat{P}_\pi = \hat{V}_{2\pi}\hat{P}_0$$

2. 4d N=1 SYM, any G, *or G(adj) with* n_f *massless Weyl, or…*

discrete chiral symmetry: $Z_{2c_2 n_f}$

for $G = SU(N): U(1) \to Z_{2n_f N}$

$$\hat{X}_{\mathbb{Z}^{(0)}_{2n_f N}} = e^{i\frac{2\pi}{2n_f N}\hat{Q}_5} = e^{i\frac{2\pi}{2n_f N}\int d^3x \hat{j}^0_f}\ \hat{V}^{-1}_{2\pi}$$

conserved non-gauge invariant U(1) charge

$$\hat{Q}_5 = \int d^3x \hat{J}^0_5 = \int d^3x \hat{j}^0_f - 2n_f N \int d^3x \hat{K}^0$$

this talk: theories with 1-form center symmetry + 0-form discrete

1. pure 4d YM, any G, $\theta = 0$ or π

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mixed ’t Hooft anomaly of 1-form center symmetry/0-form discrete

usually seen in Euclidean path integral after gauging center… *“[GKKS+]”*

this talk: mixed anomaly in Hilbert space on torus

why?

- desire to understand a new observation from different view points

Hamiltonian was particularly useful in 2d, see work w/ Anber 1807, 1811:
2d charge-N Schwinger model w/ similar anomaly: exact degeneracies
of all states in Hilbert space on circle (later called 'universes' …)

- lattice is usually a torus

- this study reveals connections to "old" work, perhaps somewhat unknown/unappreciated

- last but not least: allows for simple understanding of anomaly, e.g. w/out "$\mathscr{P}(B^{(2)})$"

main result: **anomaly has immediate consequences for spectrum of $\hat{H}$: exact degeneracies of "electric flux" states in appropriately twisted Hilbert space at any size torus**

N.B.: different from the 'topological order' (e.g. Z_2 in superconductors), where torus degeneracy only in "topological scaling limit," neglecting tunnelling

this talk: mixed anomaly in Hilbert space on torus

OUTLINE

1. **mixed 0-form/1-form anomaly in torus (T^3) Hilbert space**

   "old bottles": *1980's+... T^3 "femtouniverse" w/ twists*

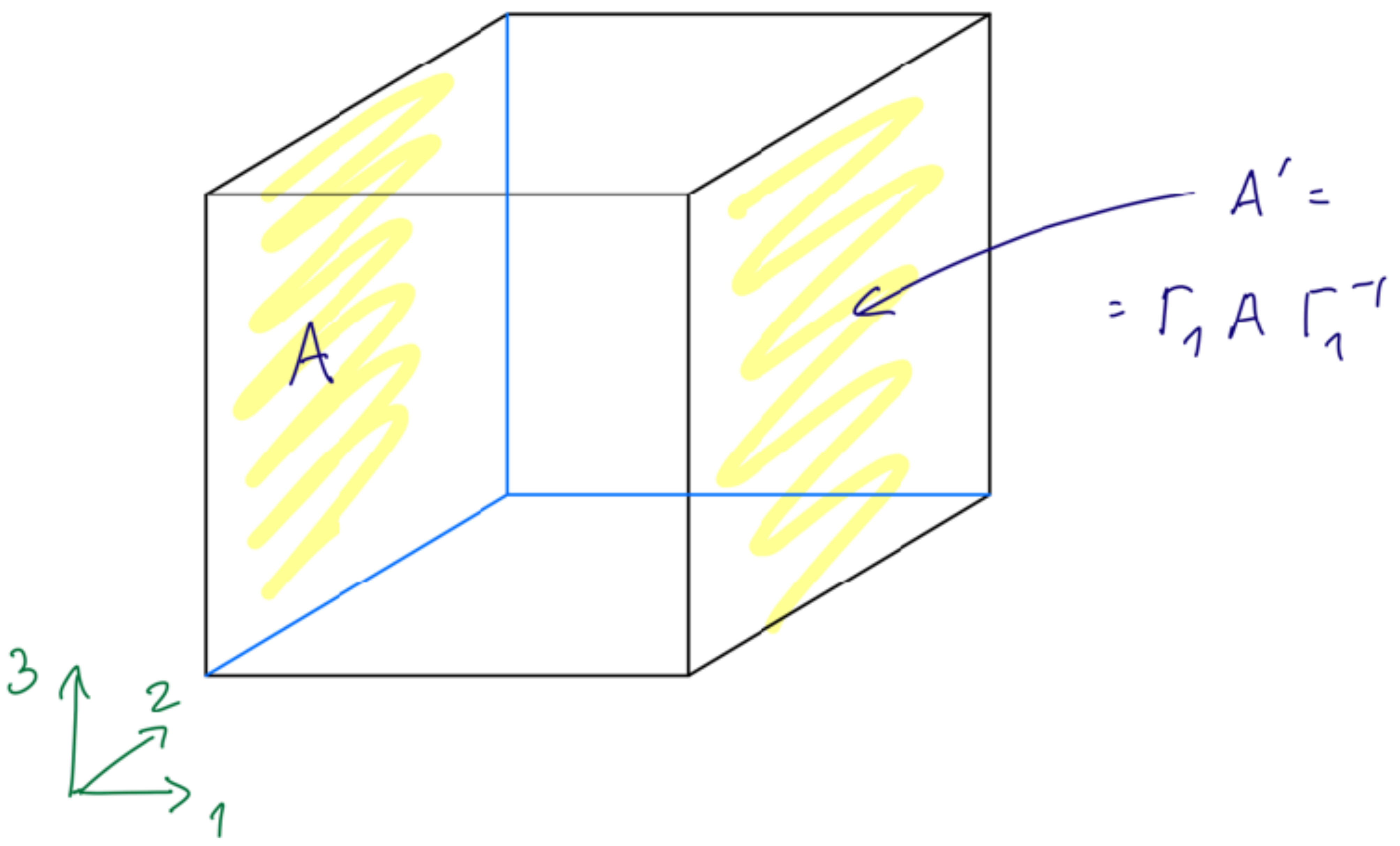
   "new wine": *anomaly interpretation*

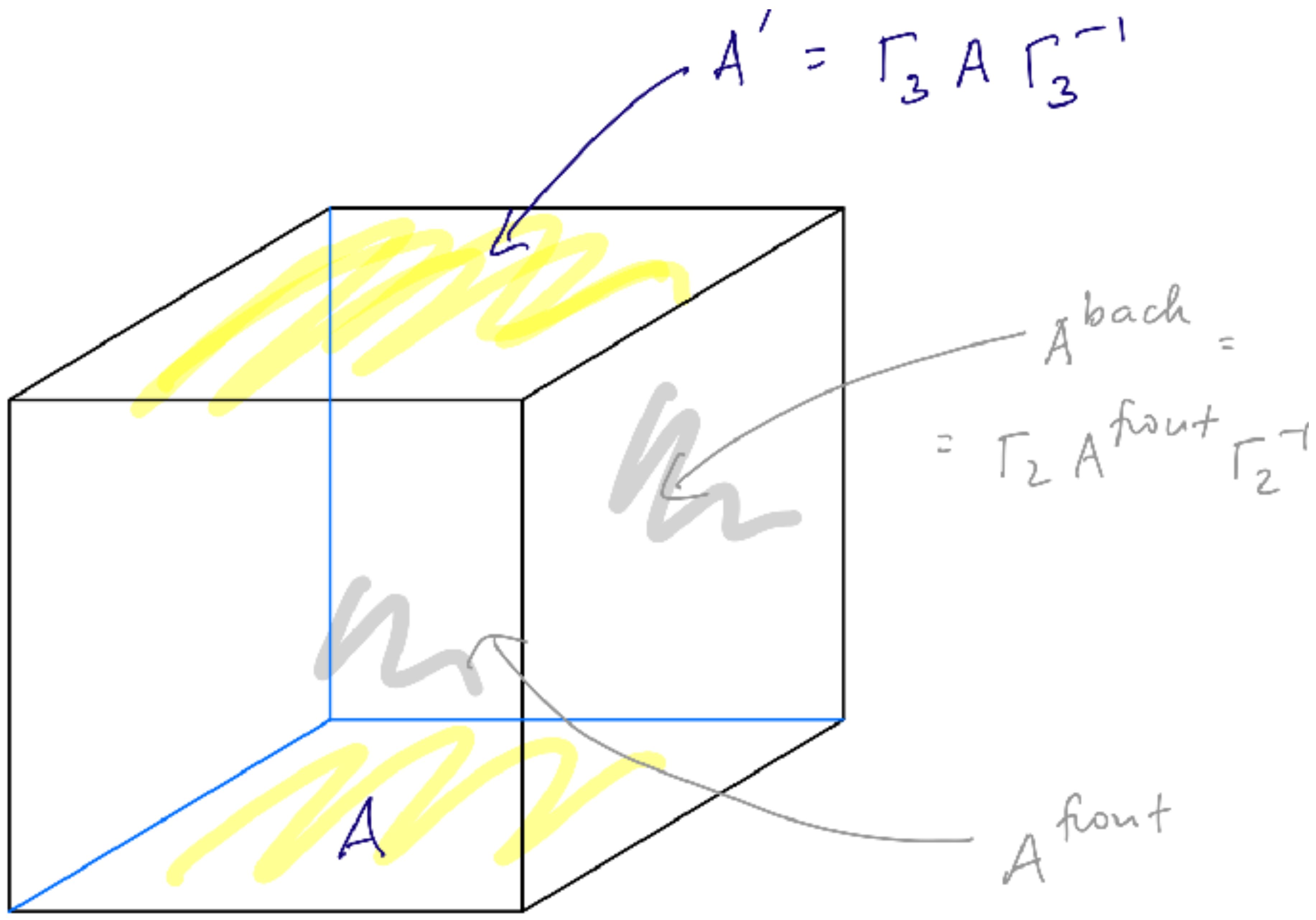
2. **consequence for spectrum of $\hat{H}$: exact degeneracies of "electric flux" states at any size T^3**

3. **implications for infinite volume phases, semiclassics, and connection to Euclidean discussions**

Will skip derivations - see Cox, Wandler, EP, 2106.11442, nicely explained! - ***and focus on discussing the implications (mostly on example of SU(N) at $\theta = \pi$)***

framework: T^3 Hilbert space: $A_0 = 0$: $\Psi[A]$ with A obeying





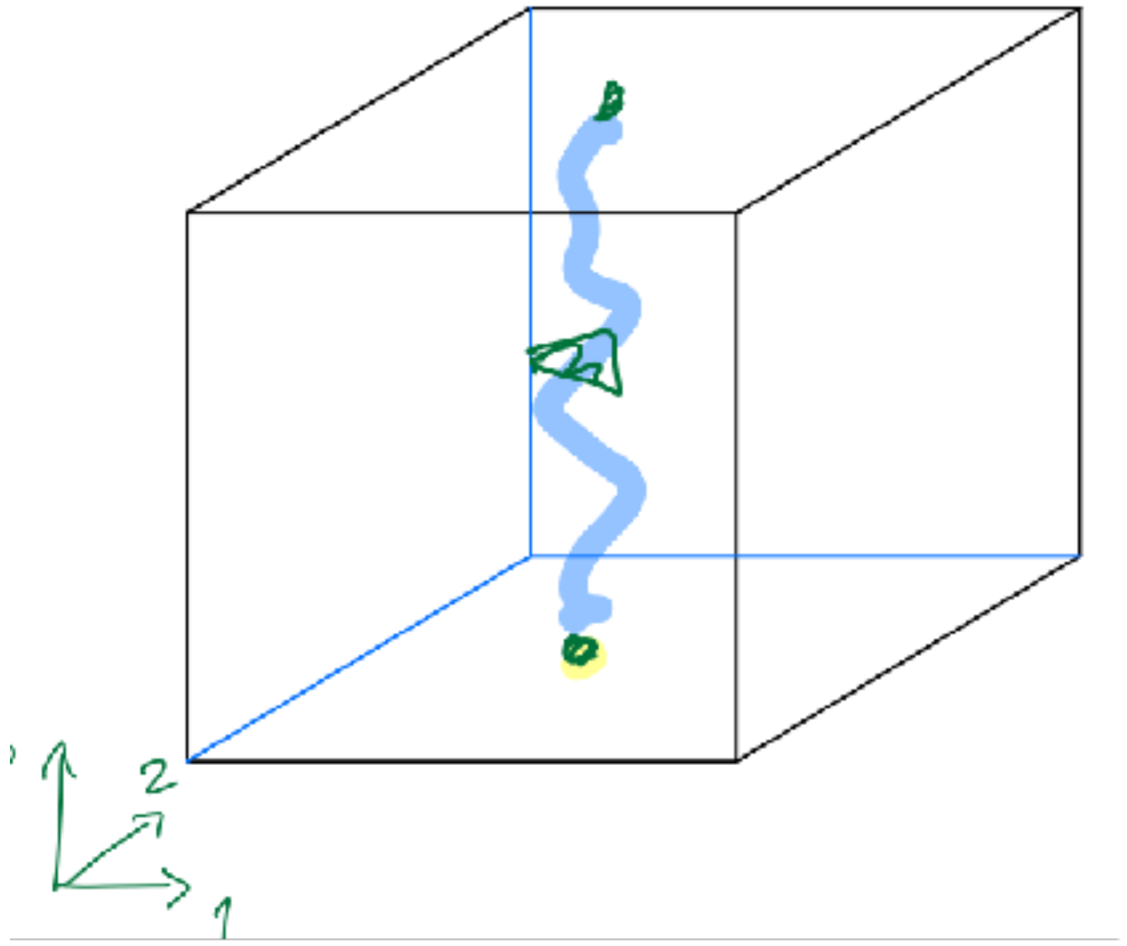
$$\Gamma_i \Gamma_j = e^{i\frac{2\pi}{N} n_{ij}} \Gamma_j \Gamma_i$$

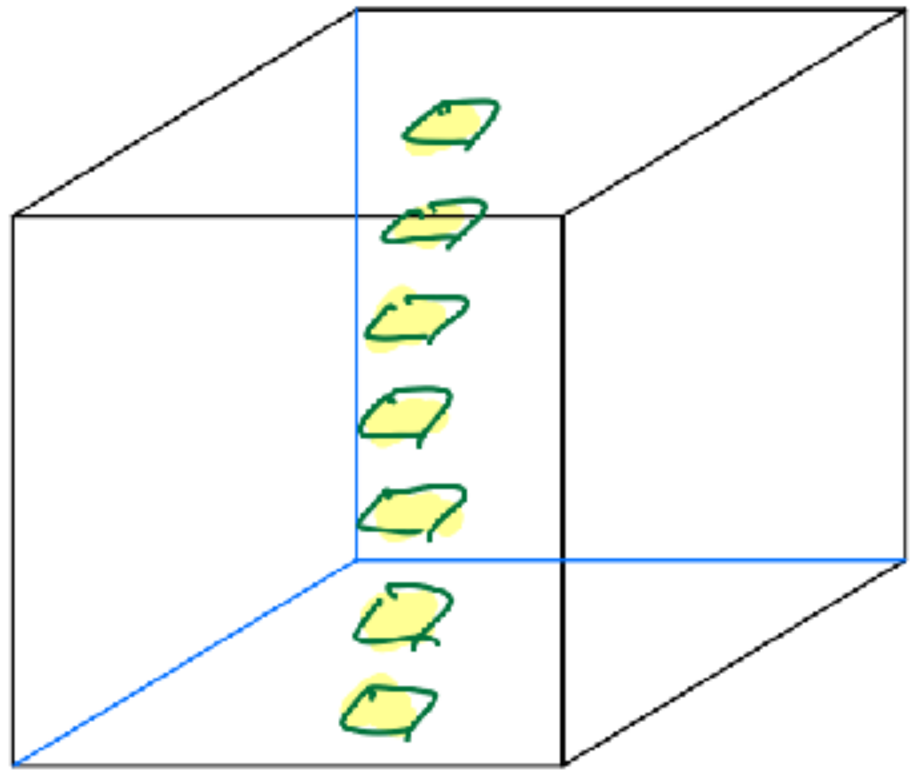
$$= e^{i\frac{2\pi}{N}\epsilon_{ijk} m_k} \Gamma_j \Gamma_i$$

"constant-twist (Γ_i) gauge"

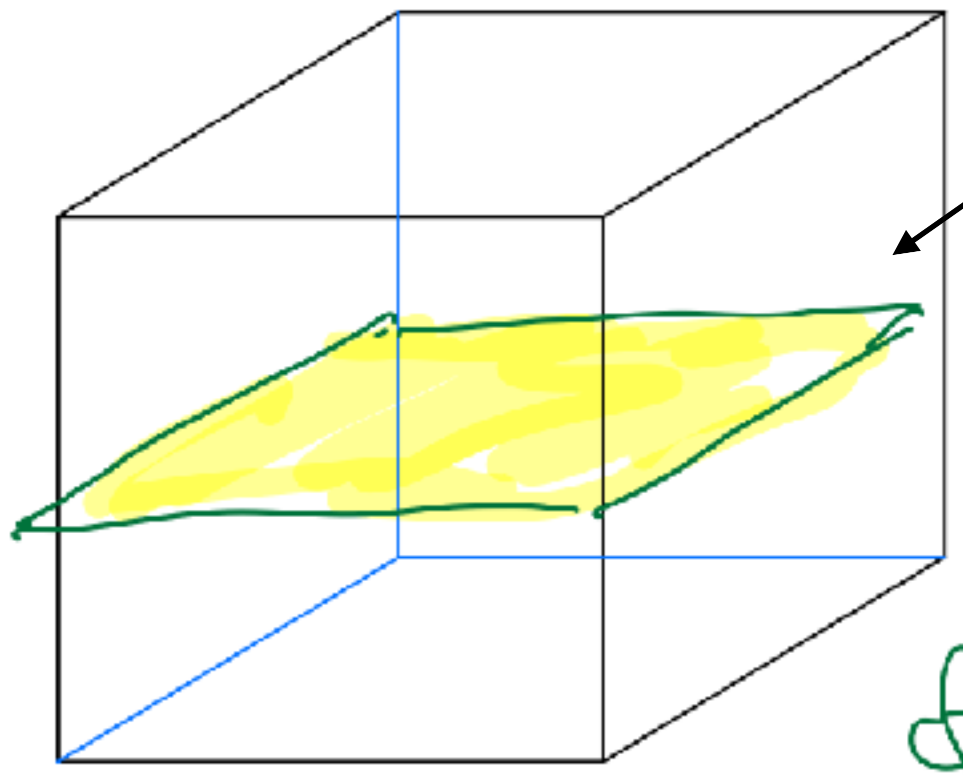
framework: T^3 Hilbert space: $A_0 = 0$: $\Psi[A]$ with A obeying

example: $m_3 = n_{12} = 1$ $\Gamma_1\Gamma_2 = e^{i\frac{2\pi}{N}}\Gamma_2\Gamma_1, \Gamma_3 = 1$





$\square : \mathrm{tr}\, U_p \to e^{i\frac{2\pi}{N}}\, \mathrm{tr}\, U_p$



any noncontractible 2-plane

$\oint_{12} B^{(2)} = \frac{2\pi}{N}$ *(mod 1)*

winding Dirac surface of "'t Hooft loop" = *lattice: unit center vortex in 0-3 plane* = *turned on static topological 2-form Z_N gauge background in 1-2 plane* [Kapustin, Seiberg '14]

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quantizing in fixed $\overrightarrow{m}$ background
= in spatial 2-form flux gauging $Z_N^{(1)}$

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framework: T^3 Hilbert space: $A_0 = 0$: $\Psi[A]$ with A obeying

"old bottles"

't Hooft '81; Luscher '82; van Baal '84; Gonzalez-Arroyo; Korthals Altes +…

calculate $\vec{e}$, $\vec{m}$-flux states energies E(L) on T^3, study confinement…

Witten '82, '00: use for tr$(-1)^F$

- center-symmetry: $\hat{T}_{l,\, l=1,2,3}$ act on winding loops $\hat{T}_l \hat{W}_k \hat{T}_l^{-1} = e^{i\frac{2\pi}{N}\delta_{kl}}\, \hat{W}_k$
- $\hat{T}_l$ commute with Hamiltonian, generate 1-form $Z_N^{(1)}$; $\hat{T}_l$ eigenvalues $e^{i\frac{2\pi}{N}e_l} \in Z_N$

- all eigenvectors of $\hat{H}$ also labeled by Z_N "electric flux" $\vec{e}$

(states with different $\vec{e}$ - action of winding Wilson loops)

boundary conditions on T^3

eigenvalues of $\hat{T}_l$, generating 1-form Z_N

$\vec{m}$ (mod N) …

discrete magnetic flux

$\vec{e}$ (mod N) …

discrete electric flux

framework: T^3 Hilbert space: $A_0 = 0$: $\Psi[A]$ with A obeying

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$$\hat{T}_l | \psi_{\vec{e}} \rangle = | \psi_{\vec{e}} \rangle e^{\frac{2\pi i}{N}(e_l - \frac{\theta}{2\pi} m_l)}$$

't Hooft: center-symmetry generator "along" $\vec{m}$ has fractional $T^3 \to G$ winding #

boundary conditions on T^3

$\vec{m}$ (mod N) …

discrete magnetic flux

eigenvalues of $\hat{T}_l$, generating 1-form Z_N

$\vec{e}$ (mod N) …

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Witten '82, '00: use for tr$(-1)^F$

't Hooft: center-symmetry generator

"along" $\vec{m}$ has fractional $T^3 \to G$ winding #

$$\hat{T}_l \; \hat{V}_{2\pi} = e^{i2\pi \frac{m_l}{N}} \; \hat{V}_{2\pi} \; \hat{T}_l$$

[van Baal Ph.D. thesis Ch.3 '84, unpublished]

SU(N) YM at $\theta = \pi$

$$\hat{T}_j \; \hat{P}_\pi = e^{\frac{2\pi i}{N} m_j} \hat{P}_\pi \; \hat{T}_j^\dagger$$

"new wine": anomaly! [Cox, Wandler, EP '21]

SU(N) massless QCD(adj)

$$\hat{T}_j \; \hat{X}_{\mathbb{Z}^{(0)}_{2n_f N}} = e^{-i\frac{2\pi}{N} m_j} \; \hat{X}_{\mathbb{Z}^{(0)}_{2n_f N}} \; \hat{T}_j .$$

cf. charge-N 2d Schwinger [Anber, EP '18]

$SU(N)$ **YM, take e.g.** $\vec{m} = (0,0,1)$ at $\theta = \pi$ vs . $\theta = 0$

$[\hat{T}_3, \hat{H}_{\theta=\pi}] = 0 \; , \quad [\hat{P}_\pi, \hat{H}_{\theta=\pi}] = 0 \; , \quad \hat{T}_3 \hat{P}_\pi = e^{i\frac{2\pi}{N}} \hat{P}_\pi \hat{T}_3^\dagger \longleftrightarrow \hat{T}_3 \hat{P}_0 = \hat{P}_0 \hat{T}_3^\dagger$

deformed

dihedral D_N

(2N elements)

$\hat{P}_\pi : |E, e_3\rangle \to |E, 1 - e_3 \;(\mathrm{mod} N)\rangle$ for all E

$SU(N)$ **YM, take e.g.** $\vec{m} = (0,0,1)$ at $\theta = \pi$

$$\hat{P}_\pi : |E, e_3\rangle \to |E, 1 - e_3 \ (\mathrm{mod} N)\rangle$$ for all E

algebra: e_3 *and* $1 - e_3$ *(mod N) states degenerate at* $\theta = \pi$ *at any size torus*

odd N: $e_3 = \dfrac{N+1}{2}$ *mapped to itself at* $\theta = \pi$ *, "global inconsistency"*

(at $\theta = 0$*,* e_3 *and* $-e_3$ *(mod N) states degenerate,* $e_3 = 0$ *mapped to itself)*

even N: *no state mapped to itself,* ***all states doubly degenerate!***

THIS IS GENERAL:

any YM at $\theta = \pi$, if center of even order, $Sp(2k+1), E_7, Spin(2k)$: double degeneracy;
if center of odd order: global inconsistency (no anomaly on T^3: $Sp(2k), Spin(2k+1)$)

$SU(N)$ **YM, take e.g.** $\vec{m} = (0,0,1)$ at $\theta = \pi$ vs . $\theta = 0$

$$\hat{T}_3 \hat{P}_\pi = e^{i\frac{2\pi}{N}} \hat{P}_\pi \hat{T}_3^\dagger \xleftrightarrow{\text{deformed}} \hat{T}_3 \hat{P}_0 = \hat{P}_0 \hat{T}_3^\dagger$$

dihedral D_{2N}

COMMENT 1:

deformed algebra was seen on $R^3 \times S^1$ by GKKS [SU(2) "partially gauged"]

as well as Aitken, Cherman, Unsal '18 in semiclassical SU(N) dYM ($R^3 \times S^1$)

$\downarrow$

with no apparent 't Hooft flux ($\vec{m}$)??

…but algebra deformation seen only on "fractional" monopole ops

$SU(N)$ **YM, take e.g.** $\vec{m} = (0,0,1)$ at $\theta = \pi$ vs. $\theta = 0$

$$\hat{T}_3 \hat{P}_\pi = e^{i\frac{2\pi}{N}} \hat{P}_\pi \hat{T}_3^\dagger \quad \longleftrightarrow \quad \hat{T}_3 \hat{P}_0 = \hat{P}_0 \hat{T}_3^\dagger$$

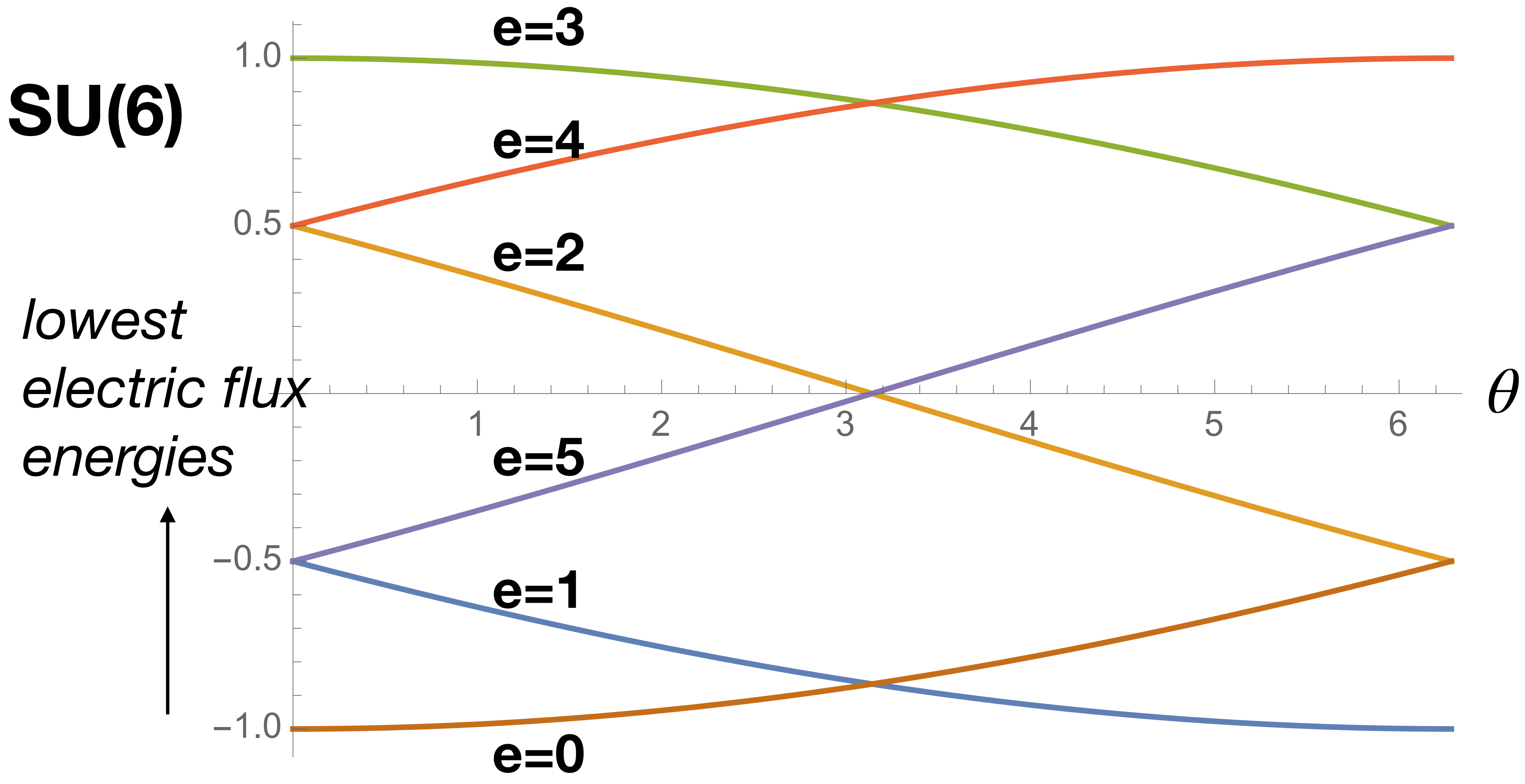
deformed *dihedral* D_{2N}

COMMENT 2:

van Baal, femtouniverse $L \ll \Lambda^{-1}$, *'84, saw this behaviour, now understood from anomaly:*

$$E(\theta, e_3) = -\frac{C e^{-\frac{8\pi^2}{g^2 N}}}{L g^4} \cos\left(\frac{2\pi}{N} e_3 - \frac{\theta}{N} m_3\right)$$

fractional instantons on $T^3 \times R$ *split perturbative degeneracy of lowest electric flux energies*



$SU(N)$ **YM, take e.g.** $\vec{m} = (0,0,1)$ at $\theta = \pi$ vs. $\theta = 0$

$$\hat{T}_3 \hat{P}_\pi = e^{i\frac{2\pi}{N}} \hat{P}_\pi \hat{T}_3^\dagger \quad \longleftrightarrow \quad \hat{T}_3 \hat{P}_0 = \hat{P}_0 \hat{T}_3^\dagger$$

deformed *dihedral* D_{2N}

COMMENT 3:

A tale of two semiclassical limits

van Baal, femtouniverse $L \ll \Lambda^{-1}$ $\longleftrightarrow$ **dYM,** $R^3 \times S^1, \Lambda L N \ll 2\pi$

Unsal, Yaffe '08 +

$$E(\theta, e_3) = -\frac{Ce^{-\frac{8\pi^2}{g^2 N}}}{Lg^4} \cos\left(\frac{2\pi}{N} e_3 - \frac{\theta}{N} m_3\right)$$

$$\rho_{vac}(k,\theta) = \frac{c}{L^4} e^{-\frac{8\pi^2}{Ng^2}} \cos\left(\frac{2\pi k}{N} - \frac{\theta}{N}\right)$$

fractional instantons on $T^3 \times R$

monopole-instanton gas $R^3 \times S^1$

accident or…?

COMMENT 3.1:

A tale of two semiclassical limits

Unsal, Yaffe '08 +

van Baal, femtouniverse $L \ll \Lambda^{-1}$ ⟷ **dYM,** $R^3 \times S^1, \Lambda L N \ll 2\pi$

dYM on $R \times T^2 \times S^1$**, with** $\vec{m}$ **through** T^2 **(along** S^1**)** [after Unsal 2020+…]

changing T^2 from $\ll \Lambda^{-1}$ to $\gg \Lambda^{-1}$, keep $L(S^1) \ll \Lambda^{-1}$

symmetries realized identically in two limits, but semiclassical objects different

femtouniverse	**dYM**
$\vert k\rangle = \hat{T}_3^k \vert 0\rangle,\ k = 0,\ldots,N-1$	$\vert k\rangle = \hat{T}_3^k \vert 0\rangle,\ k = 0,\ldots,N-1$
no magnetic flux $\langle k \vert \mathrm{tr}\hat{F}_{12}\hat{W}_3 \vert k\rangle = 0$	magnetic flux $\langle k \vert \mathrm{tr}\hat{F}_{12}\hat{W}_3 \vert k\rangle \neq 0$
$\hat{T}_3$ broken classically $\langle k \vert \mathrm{tr}\hat{W}_3 \vert k\rangle \neq 0$	$\hat{T}_3$ broken classically by flux (w/ adjoints .. Wandler EP '22
$\hat{T}_3$ restored by fractional instantons	$\hat{T}_3$ restored by "M" and "KK"

then $\vert e\rangle \sim \sum e^{i\frac{2\pi k e}{N}} \vert k\rangle$ have energies $\sim e^{-\frac{8\pi^2}{g^2 N}} \cos(\frac{2\pi e}{N} - \frac{\theta}{N})$

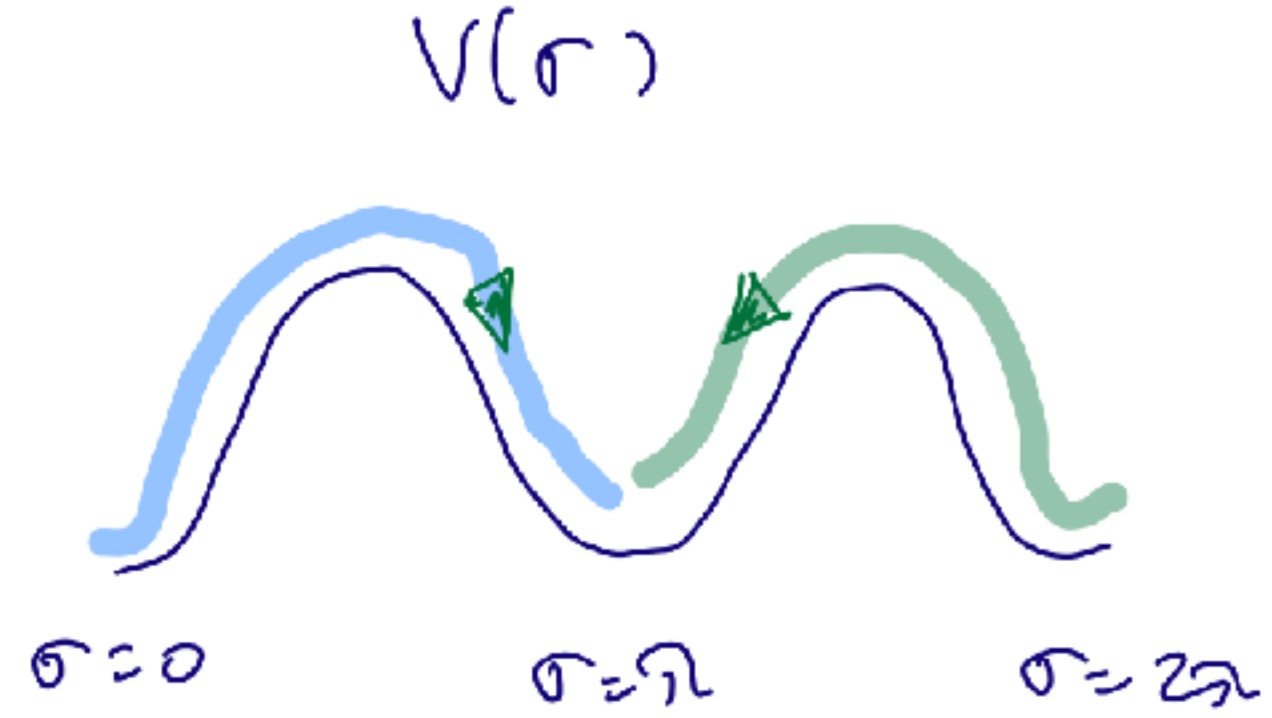
$SU(N)$ **YM, take e.g.** $\vec{m} = (0,0,1)$ at $\theta = \pi$

$$\hat{T}_3 \hat{P}_\pi = e^{i\frac{2\pi}{N}} \hat{P}_\pi \hat{T}_3^\dagger$$

COMMENT 4:

Exact degeneracy at finite volume?

dYM SU(2), $\Lambda L_{S^1} \ll 2\pi, \theta = \pi$ *Unsal '12 +…*



- two vacua (P̸)
- two distinct domain walls (lines!)

["double-string confinement"… Anber, Tin et al 2015]

$SU(N)$ **YM, take e.g.** $\vec{m} = (0,0,1)$ at $\theta = \pi$

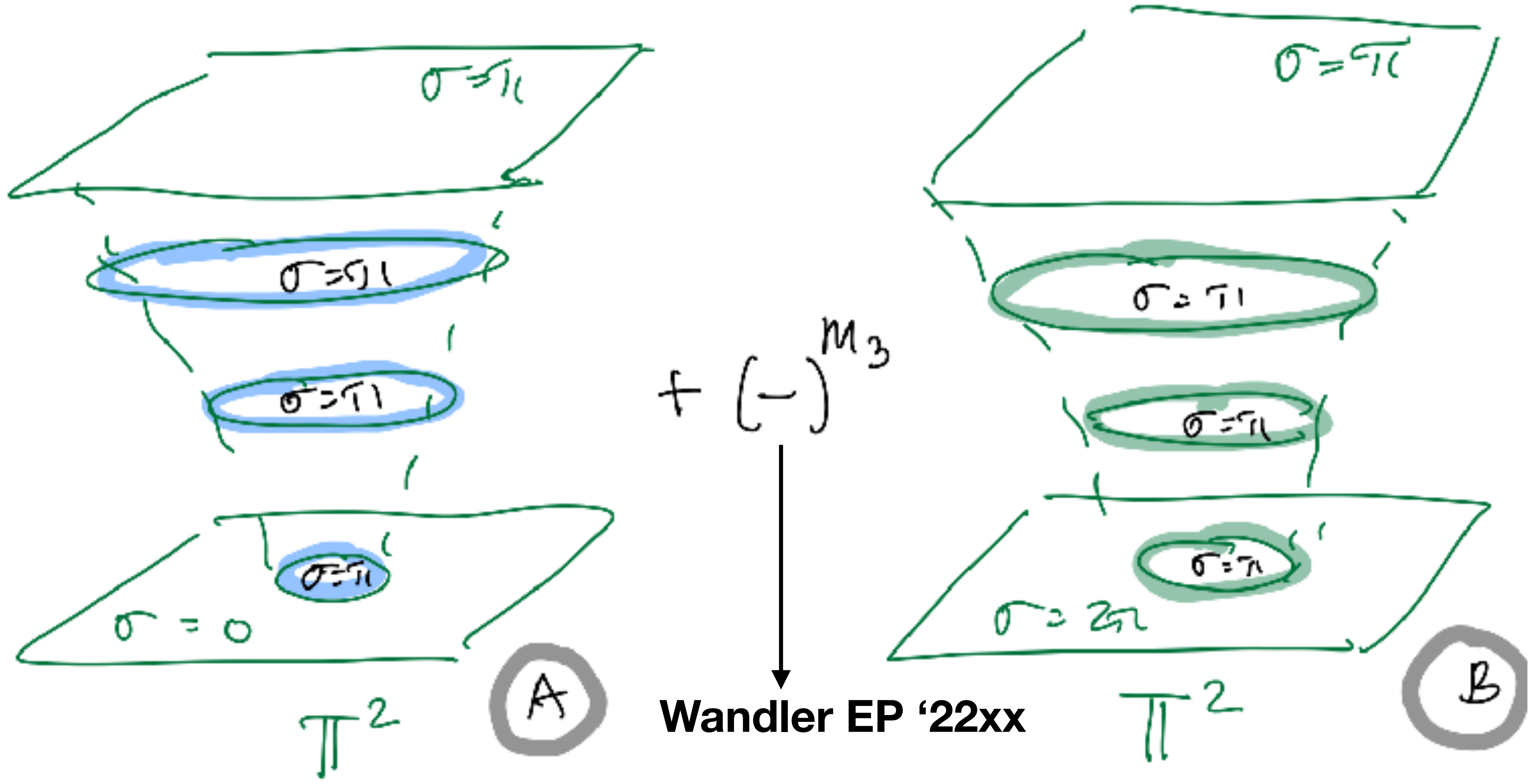
$$\hat{T}_3 \hat{P}_\pi = e^{i\frac{2\pi}{N}} \hat{P}_\pi \hat{T}_3^\dagger$$

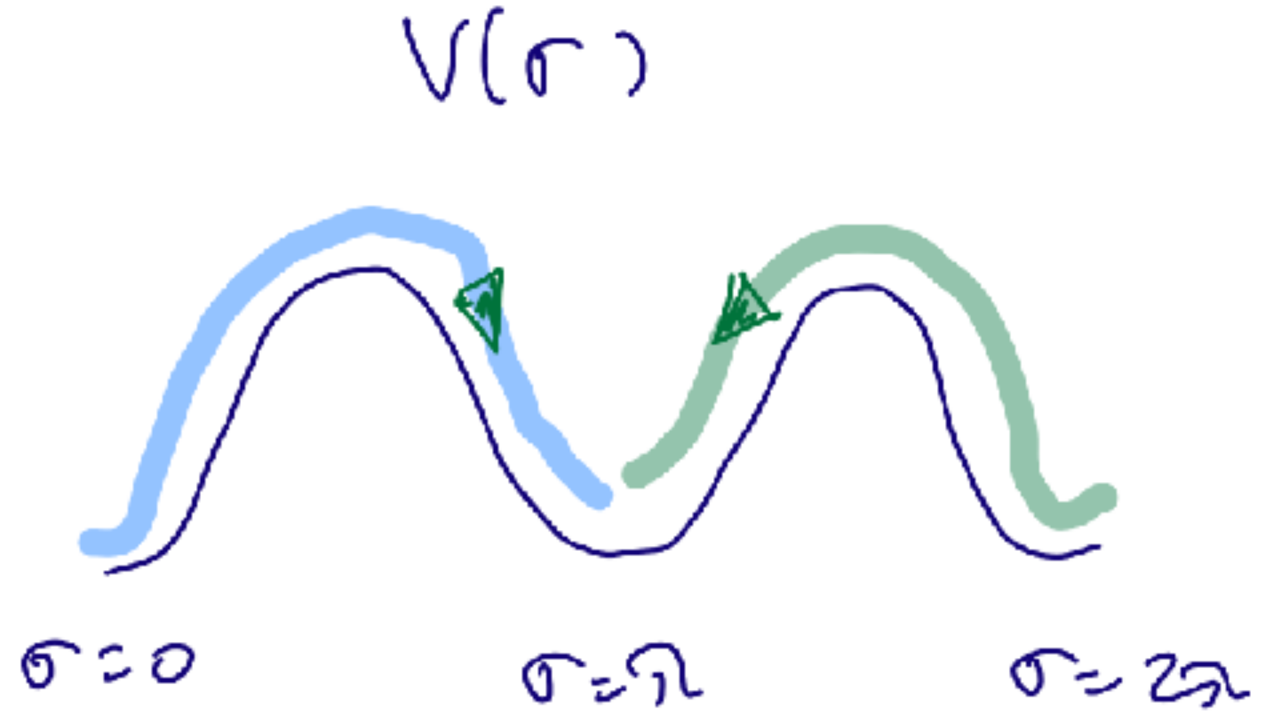
COMMENT 4:

Exact degeneracy at finite volume?

SU(2) **dYM:** $R \times T^2 \times S^1$, **large-**T^2, **small** S^1 **+** $m_3 = 1$

dYM SU(2), $\Lambda L_{S^1} \ll 2\pi, \theta = \pi$





- two vacua (∅)
- two distinct domain walls (lines!)

$= (\#)\left(1 + (-1)^{m_3}\right)$ ***-> no tunnelling even at finite volume: A+B =0 when*** $m_3 = 1$

$SU(N)$ **YM, take e.g.** $\vec{m} = (0,0,1)$ at $\theta = \pi$ vs. $\theta = 0$

$$\hat{T}_3 \hat{P}_\pi = e^{i\frac{2\pi}{N}} \hat{P}_\pi \hat{T}_3^\dagger \quad \xleftrightarrow{\text{deformed}} \quad \hat{T}_3 \hat{P}_0 = \hat{P}_0 \hat{T}_3^\dagger$$

dihedral D_{2N}

COMMENT 5:

What about "real" infinite-T^3 world ?

focus on **even N:** *no state mapped to itself, all states doubly degenerate!*

confinement: $\theta = 0$ **expect** $e_3 = 0$ **flux to have finite E at** $L \to \infty$**, others** $E_{flux} = \sigma L$

$\theta = \pi$ **e.g.** $|e_3 = 0\rangle$ ***and*** $|e_3 = 1\rangle$ ***states of finite E*** **at** $L \to \infty$

clustering: ***if center preserved, parity must be broken:***

$|0\rangle, |1\rangle$ ***parity breaking vacua*** *as in dYM,…, lattice [Kitano et al 2021]…?*

expect uniqueness of infinite volume limit,
no directional dependence due to b.c. in gapped theory

$SU(N)$ **YM, take e.g.** $\vec{m} = (0,0,1)$ at $\theta = \pi$ vs. $\theta = 0$

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deformed *dihedral* D_{2N}

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focus on **even N:** *no state mapped to itself, all states doubly degenerate!*

no confinement?

$\theta = \pi$ ***e.g.*** $|e_3 = 0\rangle$ ***and*** $|e_3 = 1\rangle$ ***states of finite E*** **at** $L \to \infty$

clustering: ***if parity preserved, center must be broken, still double degeneracy***

Explicit example: take 4d Georgi-Glashow $SU(2) \to U(1)$ with triplet Higgs with $v \gg 1/L$; long-distance free CFT, on T^3 with flux $\vec{m} = (0,0,1)$. Double degeneracy at $\theta = \pi$ for any L: $|\langle \mathrm{tr}\hat{F}_{12}\hat{\phi}\rangle| = 2\pi v/L^2$ (two signs of flux mix by 't H.-P. monopole tunnelling), $\langle \mathrm{tr}\hat{W}_3\rangle = \pm 2$ (can't tell the latter apart at infinite volume)

$SU(N)$ **YM, take e.g.** $\vec{m} = (0,0,1)$ at $\theta = \pi$ vs . $\theta = 0$

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deformed *dihedral* D_{2N}

COMMENT 6:

The Euclidean connection ?

$$Z[k_3, m_3] \equiv \mathrm{tr}_{\mathscr{H}^{phys.}_{\theta=0,m_3}} (e^{-\beta \hat{H}_{\theta=\pi}} \hat{T}_3^{k_3})$$

twist by $\hat{T}_3^{k_3}$ *- path integral configurations w/* $Q_{top.} = \dfrac{k_3 m_3}{N} + n$*, summed over* n*.*

$SU(N)$ **YM, take e.g.** $\vec{m} = (0,0,1)$ at $\theta = \pi$ vs. $\theta = 0$

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deformed *dihedral* D_{2N}

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$$Z[k_3, m_3] = Z[-k_3, m_3] \, e^{i\frac{2\pi}{N} k_3 m_3}$$

"$\mathscr{P}(B^{(2)})$"

mixed anomaly in path $\int$

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"$\mathcal{P}(B^{(2)})$"

mixed anomaly in path $\int$

solution: $Z[k_3, m_3] = e^{i\frac{\pi}{N} k_3 m_3}\, \Xi$ w/ $\Xi(k_3) = \Xi(-k_3)$

e.g. the two degenerate fluxes of van Baal's = TQFT, w/ $\Xi = e^{-\beta E_{\mathrm{vac}}}\, 2\cos\frac{\pi k m_3}{N}$

COMMENT 7:

Discrete chiral symmetry study proceeds similarly. Strongest constraints for SU(N): N-fold degeneracy of all electric flux states on T^3 at any L, here, anomaly: ``confinement -> chiral breaking”

[constraints from $Z_k^{(0)}$-gravity (Cordova, Ohmori ’19), assuming gap, are stronger for $G \neq SU(N)$… due to smaller rank centers]

SUMMARY:

Studied mixed 0-form/1-form anomaly: "new" vs "old" *- Hilbert space w/ twist*

main result:

Set up offers a relatively simple understanding of this type of anomaly.

Quantization in discrete $\vec{m}$ background implies exact degeneracies between $\vec{e}$-flux states, due to deformed symmetry algebra, at any finite size torus.

Symmetry realizations in $\vec{m}$ backgrounds imply that semiclassics in different regimes related… details may be of interest?

Degeneracies in $\vec{m}$ background may be useful for lattice?

Other symmetries and anomalies?