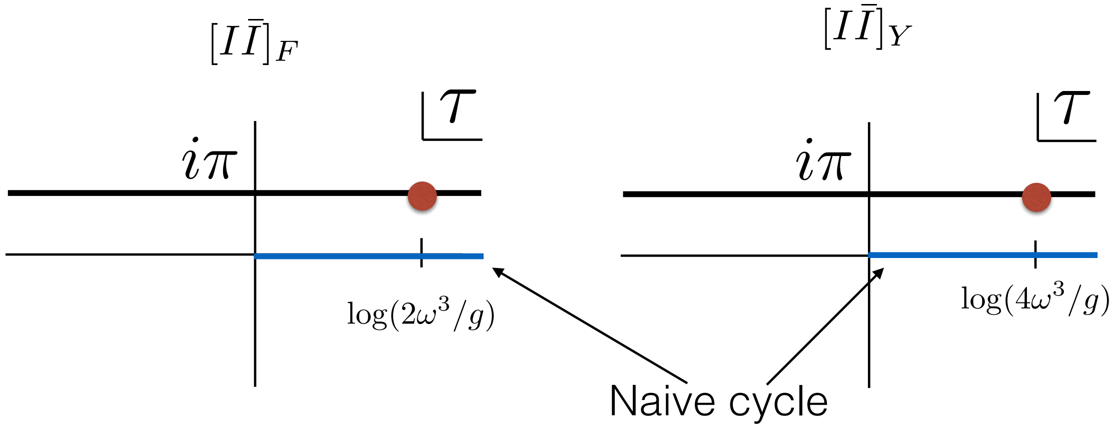
The curious case of multi-instantons and the necessity of Lefshetz thimbles

Erich Poppitz (Toronto)

with Alireza Behtash, Tin Sulejmanpasic, Mithat Unsal (NC State)

“Thimble and buttons”

in “Fashion District,” Toronto



Instanton—anti-instanton thimble

in N=2 SUSY QM, 1507.04063

Motivation: ... really, more than half of my slides

Instantons play a role in many physical problems.
In QFT, whenever semiclassics "works",
key to understanding important physics, e.g.:

N=1 SUSY theories: nonperturbative superpotentials.

N=2 SUSY theories: Seiberg-Witten curves.

Phenomenological models of chiral symmetry breaking in QCD.

...

Mass gap, confinement & center stability:

QCD(adj)/SYM & deformed Yang-Mills theory on $R^{1,2} \times S^1_L$, at small L

already at weak coupling, a major difficulty:
"How to define & calculate multi-instanton contributions?"

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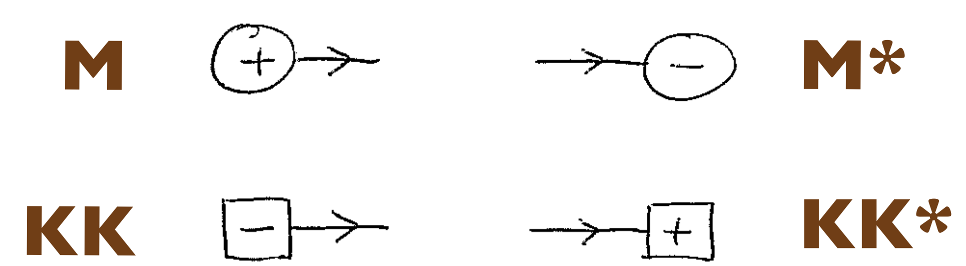

for SU(2)

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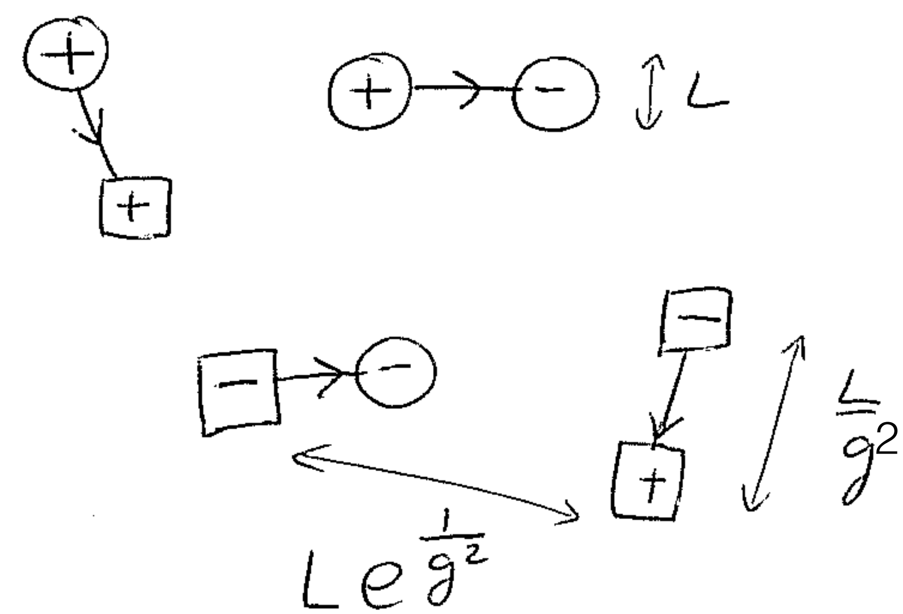

(talk at Quarks2012/SUSY2013 on work with Schafer/Unsal)

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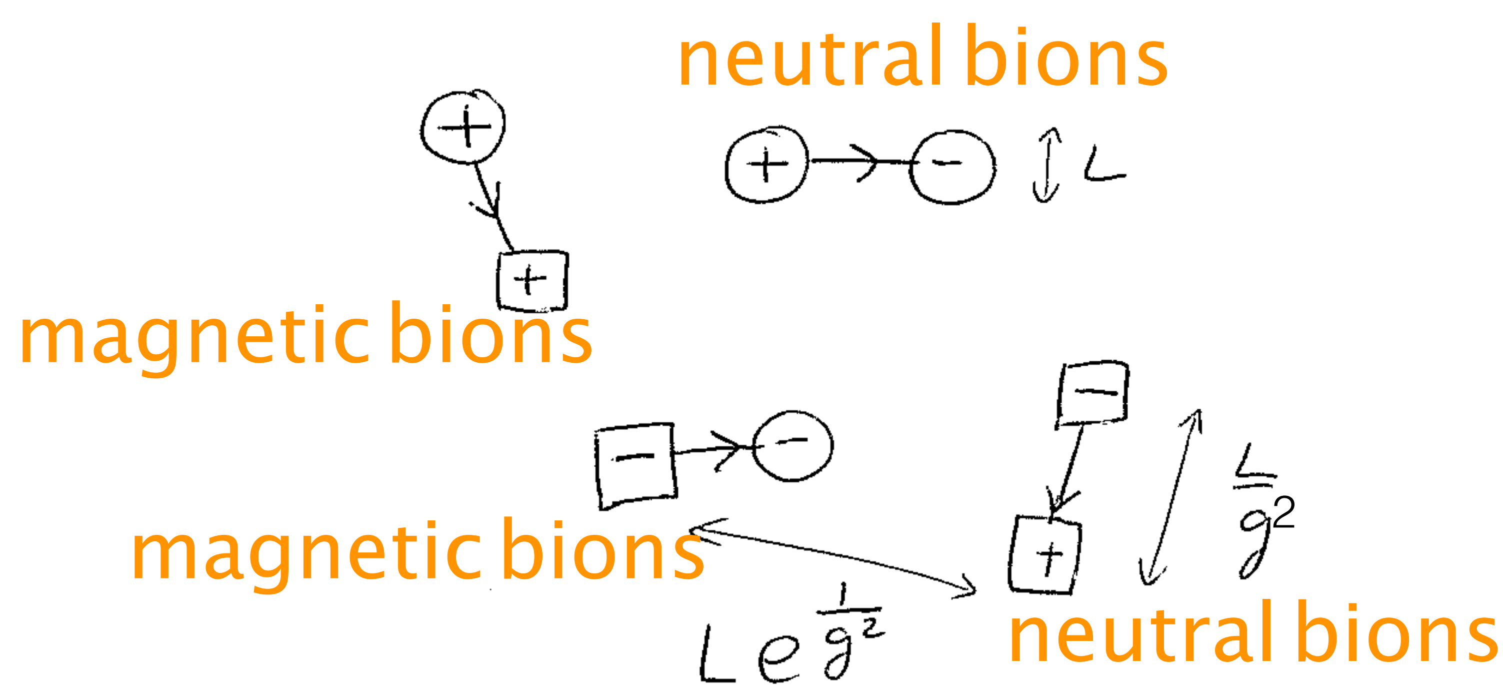

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"neutral bions" are particularly bizarre: they are MM* "molecules"

2 x M action — **deviation of holonomy from center**

MM* 't Hooft vertex ~ $\mathcal{A} e^{-2S_0} e^{\pm 2b}$

$$\mathcal{A} \sim \int_0^\infty dr e^{-\left(-2\times\frac{4\pi L}{g_4^2 r}+(4n_f-2)\log r\right)}, \quad n_f = 1$$

Coulomb attraction — **fermion-zero mode exchange attraction**

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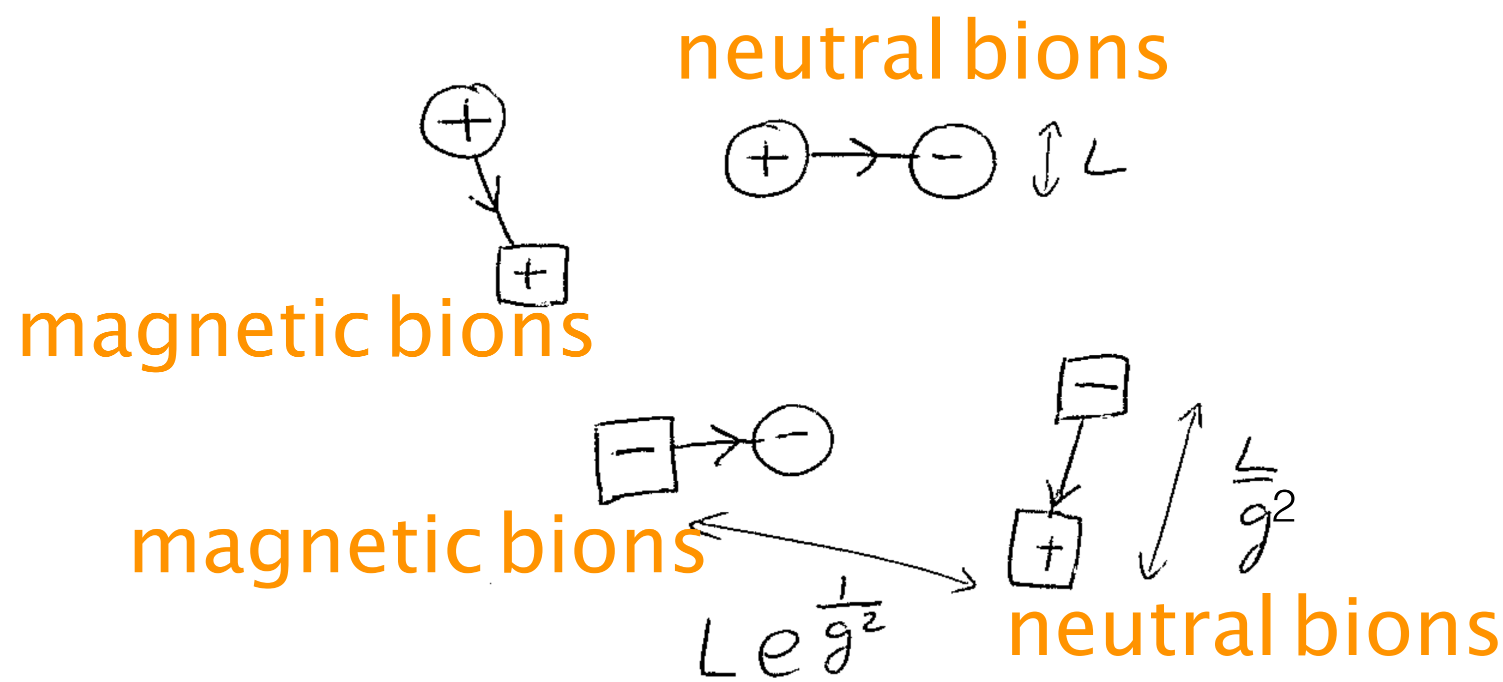

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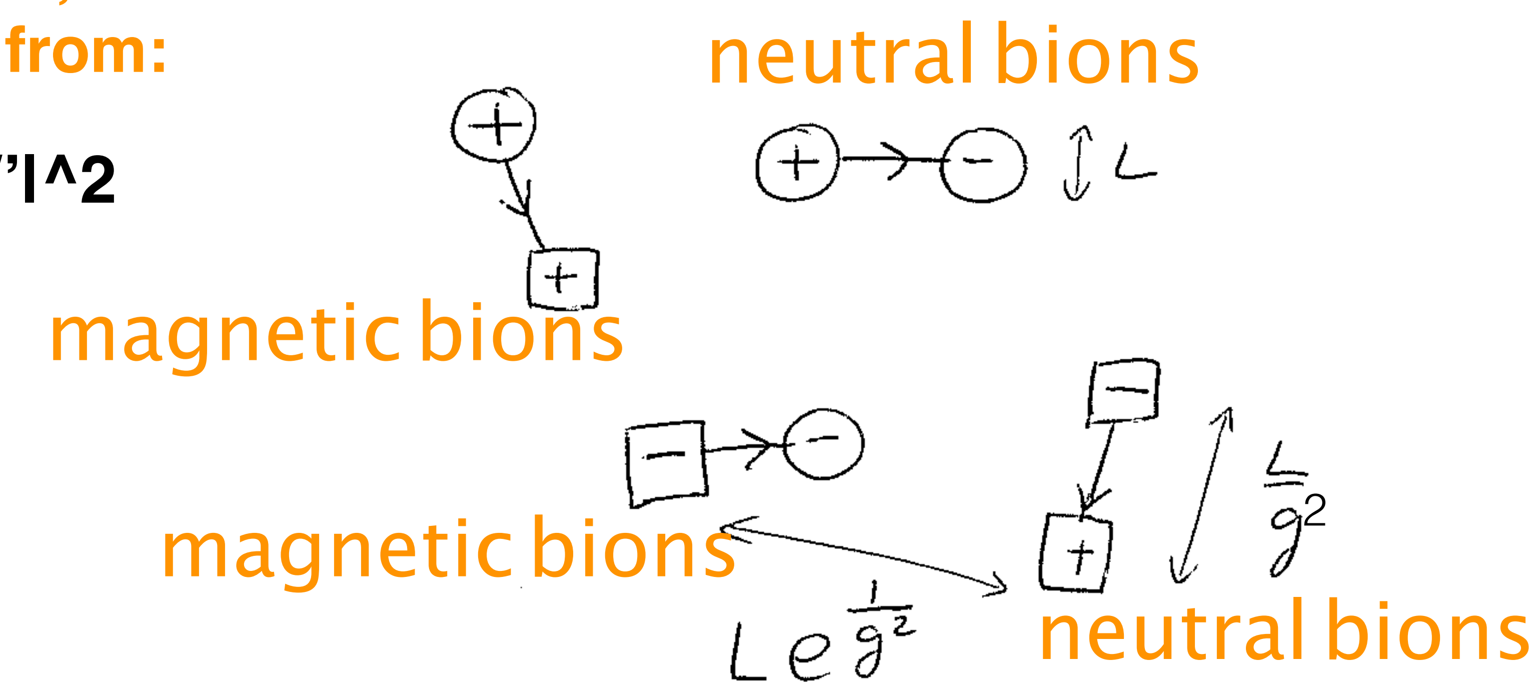
Despite the attractive-only interactions,
a “stable molecule” exists! We know from:

1. supersymmetry, exact W -> V=lW’l^2

2. analytic continuation:
   MM* “live” at complex separation



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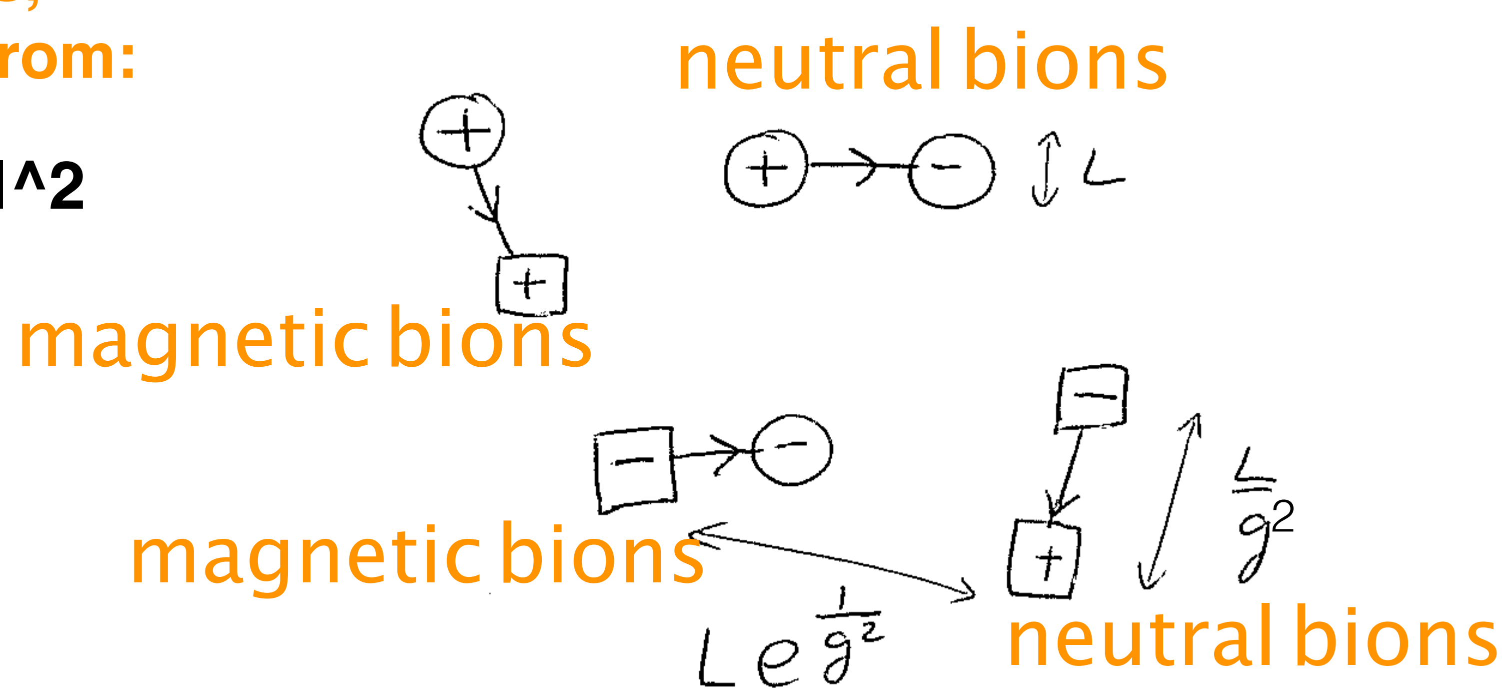

MM* in some sense
"classical" (live in Euclidean)
- no time and no quantum fluctuations
to stabilize, not, e.g. positronium!

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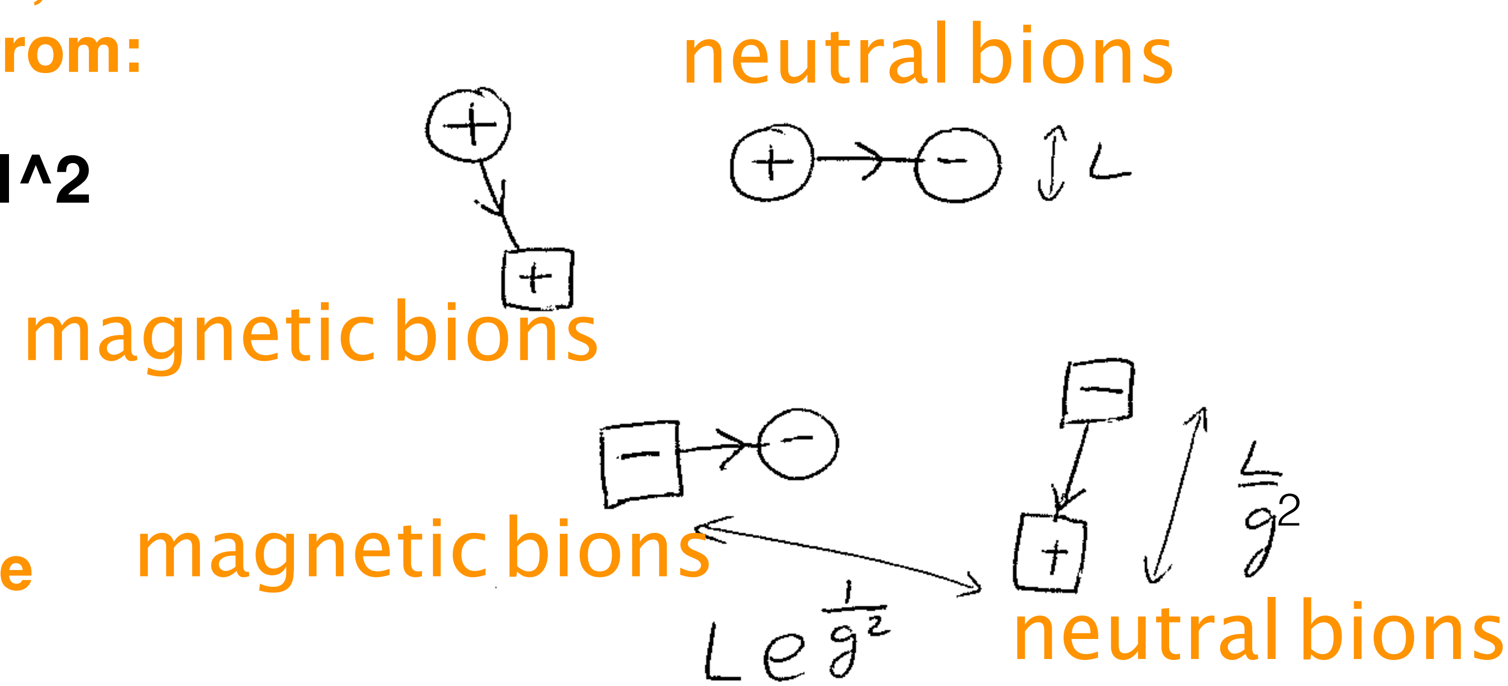

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In semiclassics, any "lump" of positive fugacity lowers vacuum energy (e.g. double well). In SYM, there are "lumps" of both positive and negative fugacity, with equal and opposite contributions to E_vac.

Complexification crucial. Hypothesis that MM* lie on a different "Lefshetz thimble" from the perturbative vacuum - distinguished by a phase ("HTA")…?

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Ex. 1: SYM, mass gap….

Ex. 2: "Resurgent" cancellations: imaginary parts due to Borel resummation of perturbation theory vs imaginary parts of I-I*

high orders of perturbation theory double-well QM, non Borel-summable:

$$E^0_{pert} = -\frac{3}{\pi}\sum_k (3g)^k\, k!$$

ambiguity of Borel sum of pert. series:

$$\Delta E^0[\text{Borel sum}] = -\frac{3}{\pi}\left(\pm\frac{i\pi}{3g}\, e^{-\frac{1}{3g}}\right)$$

II* contribution:
requires analytic continuation
Bogomolnyi, Zinn-Justin

$$E^0_{I\bar{I}} = \left(\pm i\pi + \log\frac{g}{2}\right)\frac{e^{-\frac{1}{3g}}}{g\,\pi}$$

Motivation:

Complexification seems crucial. **Hypothesis/dream/ is that MM* lie on a different "Lefshetz thimble" from the perturbative vacuum and are distinguished from it by a phase associated with the thimble… "like" in 1dim integrals:**

$$I(\hbar) = \int_{-\infty}^{\infty} dx\, e^{-\frac{1}{\hbar}f(x)} \underbrace{\longrightarrow}_{\text{steepest descent method}} \sum_{\sigma} n_\sigma \int_{\mathcal{J}_\sigma} dz\, e^{-\frac{1}{\hbar}f(z)}$$

Decompose the original real path of integration into steepest descent paths, or "thimbles", going through different saddles (recall phase is constant on each such contour)

(I think) we are far from understanding of what "DEFINING THE PATH INTEGRAL ON LEFSHETZ THIMBLES" **means.**

All I will do is to show you a simple, yet not completely trivial, example supporting the need of complexification…

Subject/summary of talk:

four real supercharges

N=2 SUSY QM = 4d WZ model reduced to 2d

$$g\mathcal{L}_E = |\dot{z}(t)|^2 + |W'(z)|^2 + \begin{pmatrix} \bar{\chi}_1 & \chi_2 \end{pmatrix} \left(-\partial_t + \begin{pmatrix} 0 & \overline{W''(z)} \\ W''(z) & 0 \end{pmatrix} \right) \begin{pmatrix} \chi_1 \\ \bar{\chi}_2 \end{pmatrix}$$

$$W(z) = \prod_{i=1}^{k+1} (z - z_i) \qquad\qquad |I_W| = k$$

Witten index=number of critical points of W(z)

E_vac=0, as opposed to N=1 SUSY QM: well known.

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Goal: **Understand E_vac = 0 from next-order semiclassics.**

Upshot: **It’s not completely trivial. {Relation to motivation: complexification!}**

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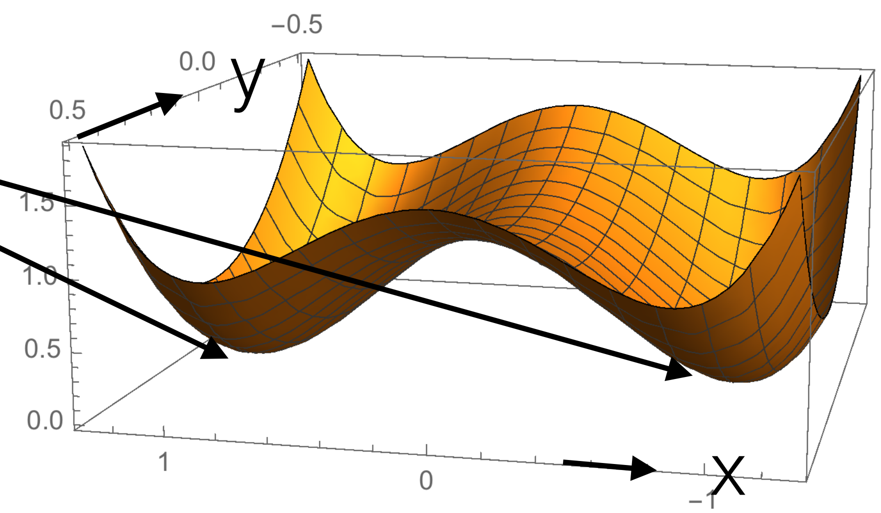
take "a" real (plot for a=1)

potential w/ two minima



BPS (anti)instantons

$$\dot{z} = \pm\overline{W'}$$

$I \rightleftarrows$

$I^* \leftrightarrows$

I,I*: tunnelling between minima; two fermion zero modes each

(with opposite "chirality" from 4d p.o.v.)

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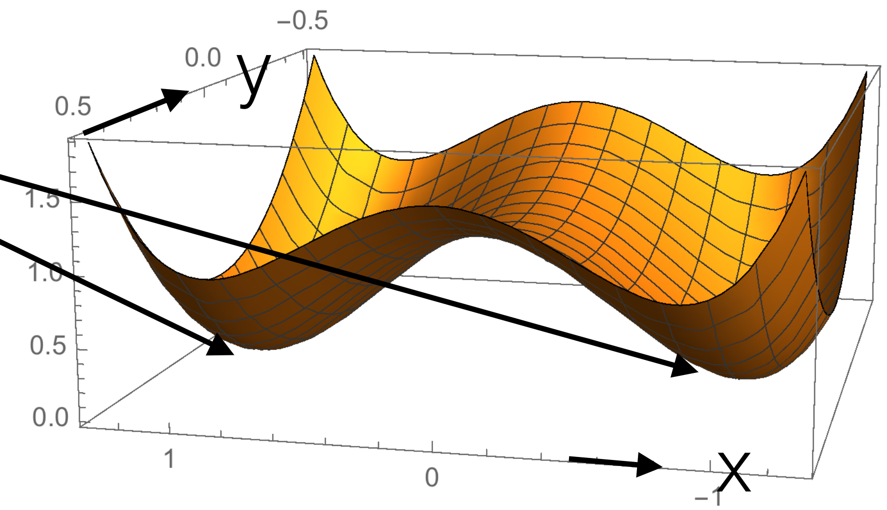

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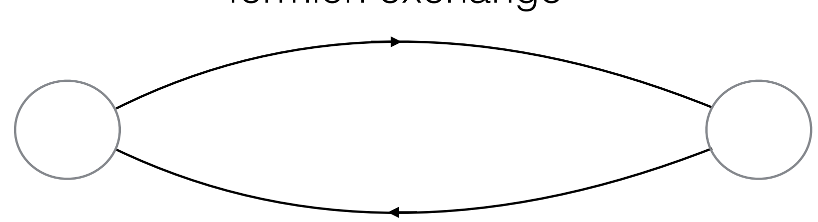

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To rephrase: Why I-I* 'events' do not contribute to E_vac?

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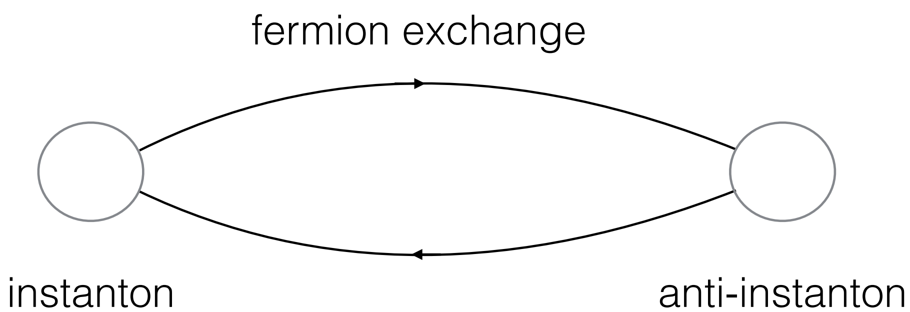

$$I_1 = \int_{\mathcal{J}_1} d\tau \; e^{\frac{4\omega^3}{g} e^{-\omega\tau} - 2\omega\tau}$$

naively, integral from "0" to infinity

quasi-zeromode

classical I-I* attraction

fermion "exchange" (attraction; "exchange of two massive particles")

Two issues:

- at small I-I* separation - all the above is nonsense
- gives *negative* E_vac if exponentiated

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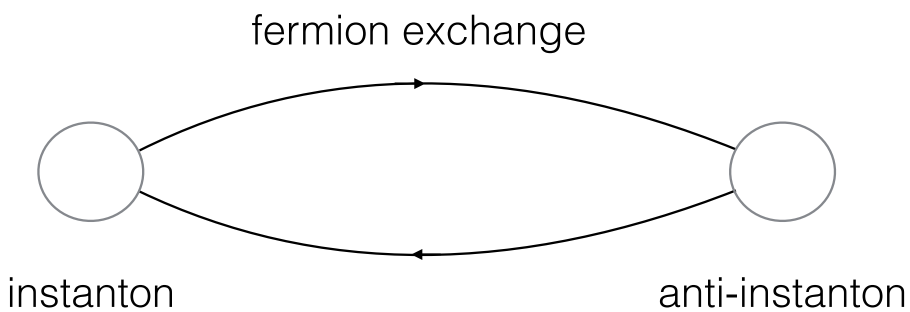

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y-scalar exchange (one massive particle)

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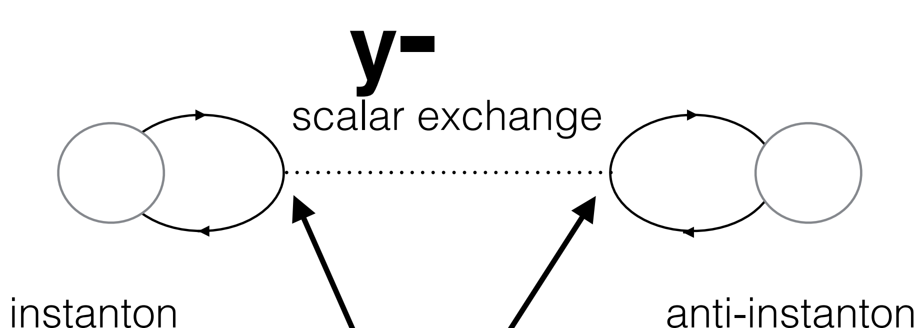

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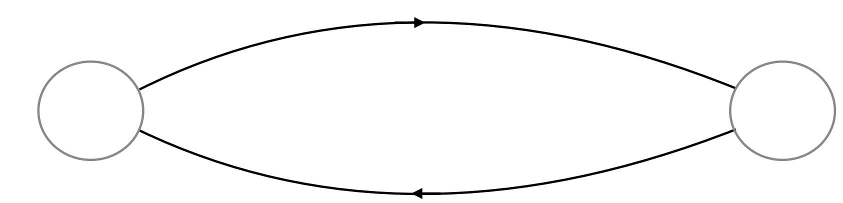

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entire story rests on relative factor - somewhat hard calculation

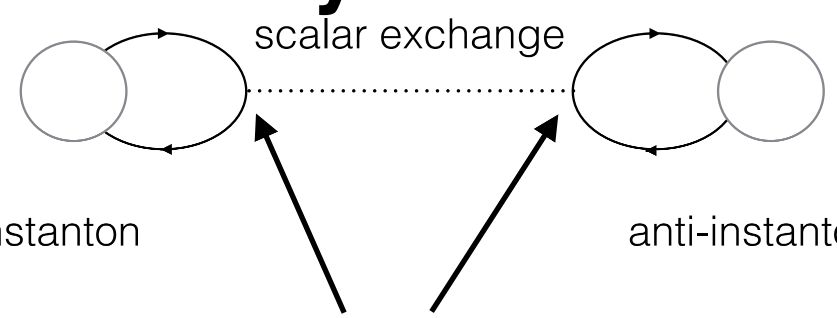
combining the two (omitting the nonzero mode I and I* determinants)

$$E_0 \propto -e^{-2S_0} \left(4\omega^3 I_1 + g I_2 \right)$$

notice: **different orders in g!**

both come with same sign: how to cancel?

y-



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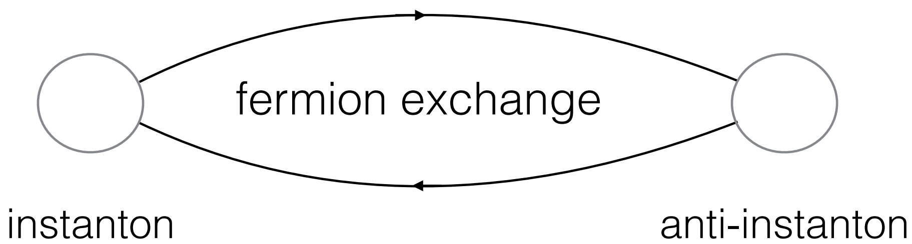

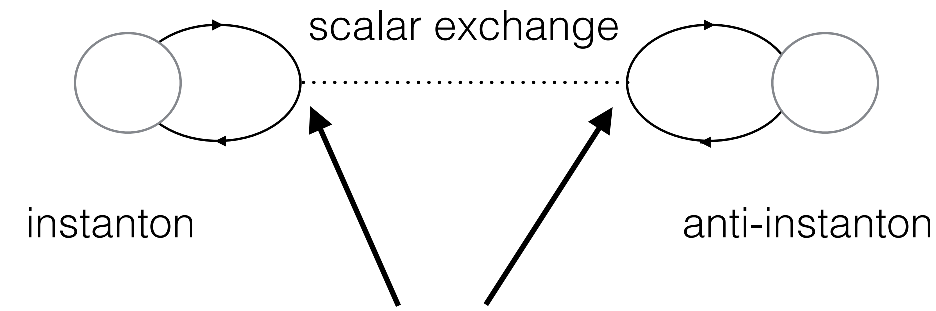

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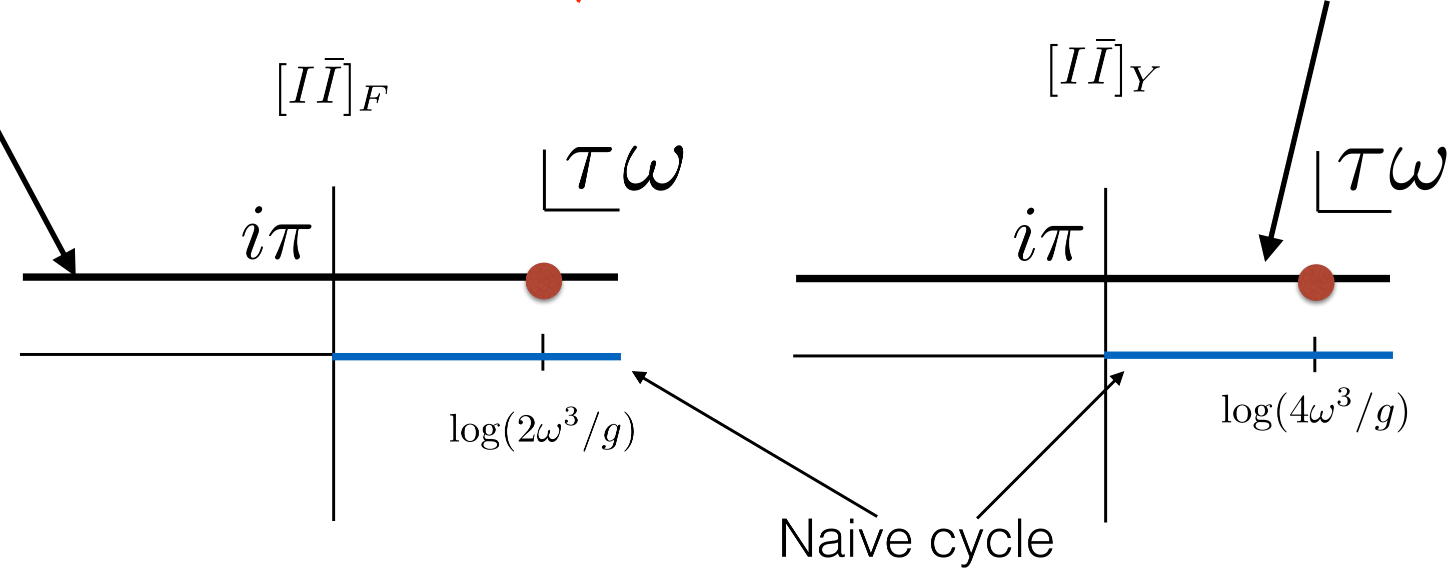

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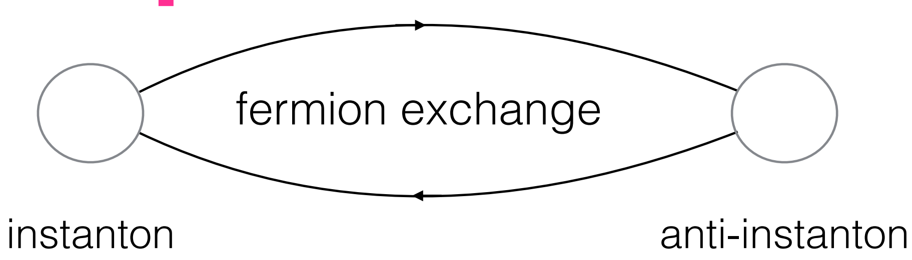

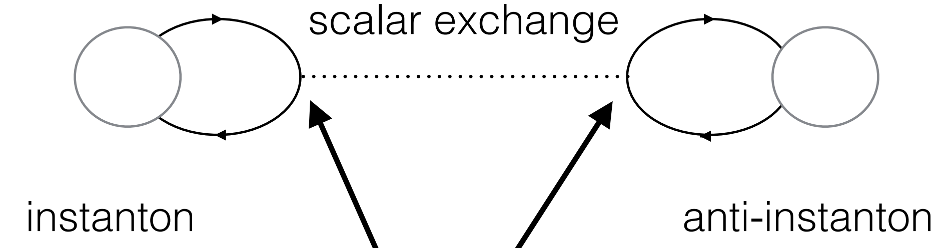

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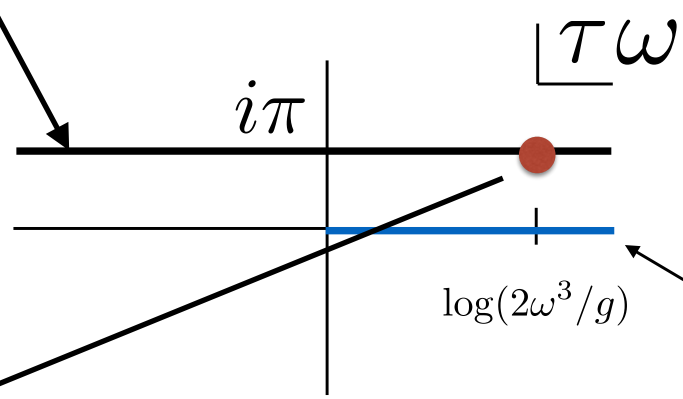

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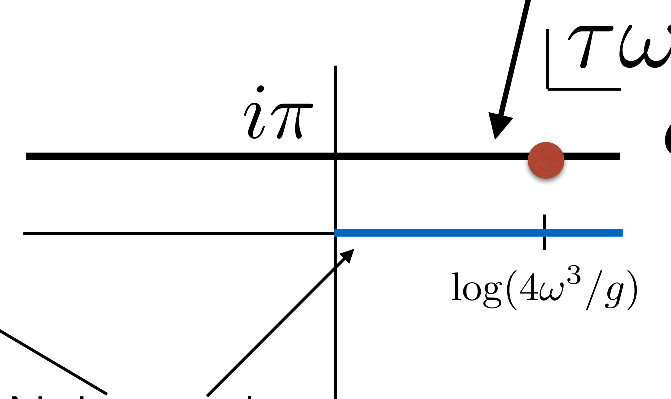

Naive cycle

$$e^{-\omega\tau_{crit.}} = e^{-i\pi + \log\frac{4\omega^3}{g}} = -\frac{4\omega^3}{g}$$

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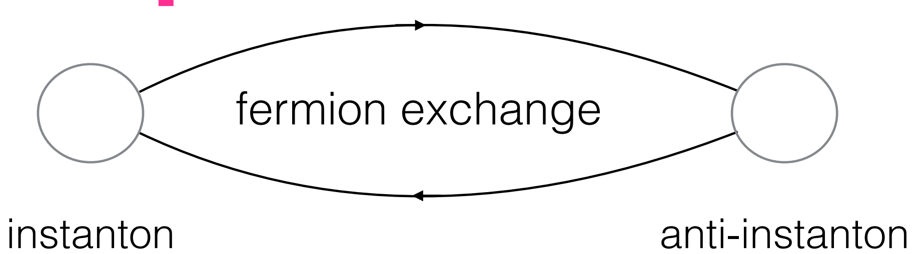

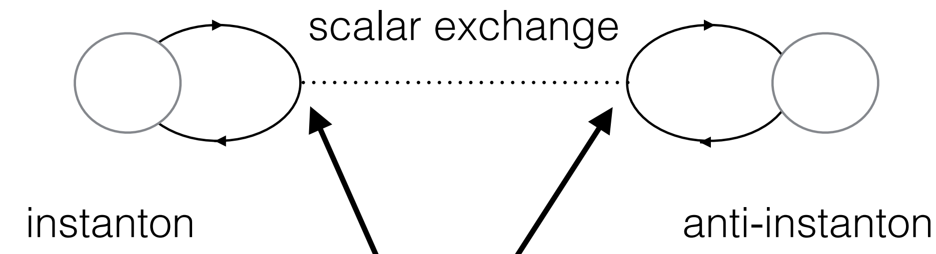

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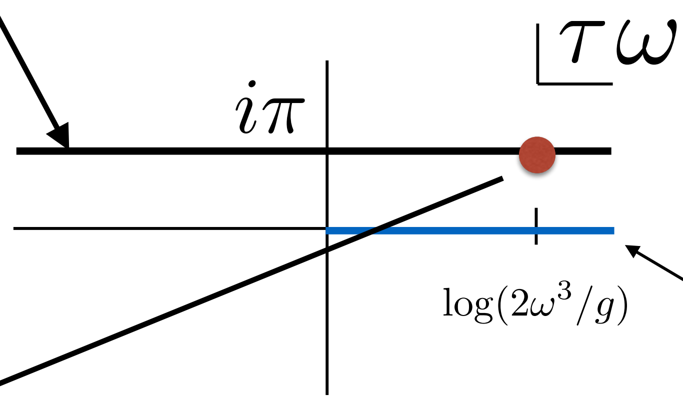

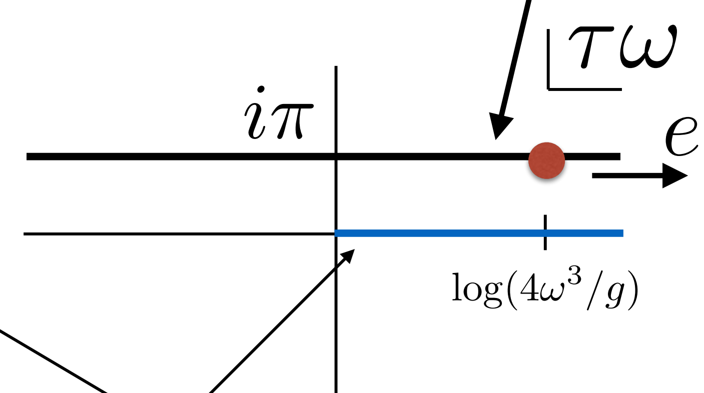

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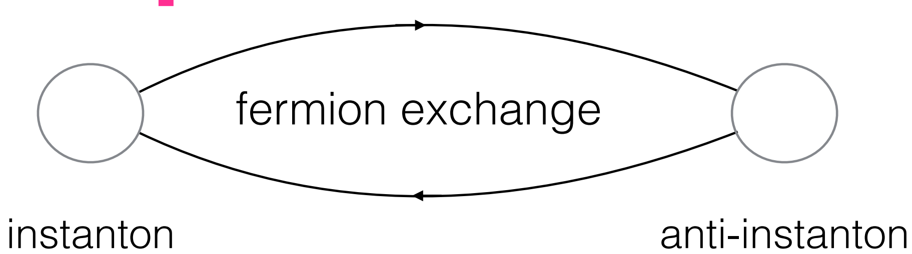
$$I_2 = -\frac{g}{4\omega^4} = \frac{4\omega^3}{g}(e^{i\pi} I_1)$$

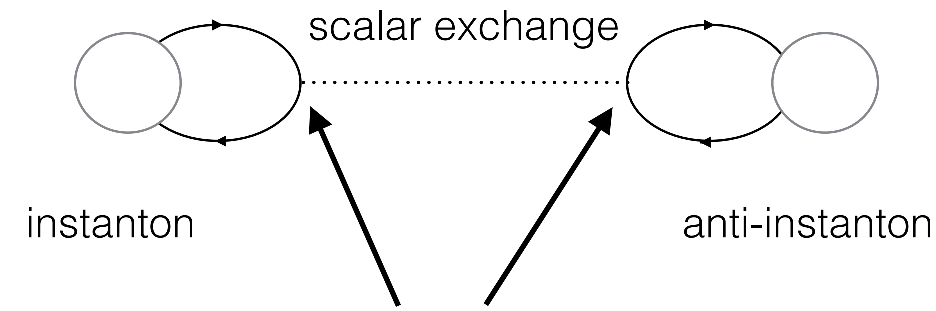
Imaginary part, *Im* $\omega\tau_{1,2} = i\pi$ of critical separation responsible for change of relative sign (one vs. two "massive propagators")

+ two vs. one "massive propagators" at saddle compensates for relative g

(nontrivial interplay of complexification and perturbation theory)

Main part of talk:





Yukawa squared = g

$$E_0 \propto -e^{-2S_0}(4\omega^3 I_1 + g I_2)$$

$$I_1 = \int_{\mathcal{J}_1} d\tau\, e^{\frac{4\omega^3}{g} e^{-\omega\tau} - 2\omega\tau} \qquad I_2 = \int_{\mathcal{J}_2} d\tau\, e^{\frac{4\omega^3}{g} e^{-\omega\tau} - \omega\tau}$$

THE TWO "LEFSHETZ THIMBLES"

$$I_1 = \frac{g^2}{16\omega^7} \qquad I_2 = -\frac{g}{4\omega^4} = \frac{4\omega^3}{g}(e^{i\pi} I_1)$$

$$4\omega^3 I_1 + g I_2 = 0$$

Comments/Results:

1 **Imaginary part, *Im* $\omega\tau_{1,2} = i\pi$ of critical separation responsible for change of relative sign - one vs. two "massive propagators"; g-order!**

2 **Absolute value of separation is large at small g - self consistent!**
I and I* are never on top of each other: complex separation

3 Integrating over the *entire* (…?) thimble gives E_vac = 0!

Final slides: **N=2 SUSY QM = 4d WZ model reduced to 2d, Witten index≠0**

Goal: **Understand E_vac = 0 from plain next-order semiclassics**
… no localization, no deformation invariance…

Upshot: **It’s not completely trivial.** **{Relation to motivation: complexification!}**

Found that complexifying the quasi-zeromode crucial. I and I* “live” a complex & large separation apart; consistent next-to-leading order semiclassics.

Final slides: **N=2 SUSY QM = 4d WZ model reduced to 2d, Witten index≠0**

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Comments:

All was done to I-I* order… not immediately clear how to proceed to higher orders.

Showed that "quasi-zeromode" complexification crucial; notice that this is just one direction in field space (the most relevant for this case!).

Suggests that complexification of path integral important.

Magnetic and neutral bions in SYM can be seen to emerge in a similar way, at (generally) complex separations. (Recall SYM is only SUSY w/out scalars … YM)

Solving analogous puzzles in SW theory harder… but worthwhile, beyond QM?

status: "theoretical experiment" in search of a theory… **finite dimensional thimbles (lattice)? mathematics?**