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"Center symmetry and anomaly matching"

[aka: new 't Hooft anomaly matching conditions involving 0-form/1-form symmetries & how they constrain strong dynamics]

- relatively new development (2014 – ...)
- still an active research area ... no theorems (examples!)
- will give you a flavor & some technical backgrd in not-too-mathematical terms (if you are so inclined/able, translate ...)

Plan:
1) reminder of 't Hooft anomaly matching (1980's); summary of main points of "new stuff"
2) what is center symmetry? what good for? (lattice & continuum descriptions of "1-form" center)
3) 2dim examples of mixed 0-form/1-form anomalies & matching
4) some overview of 4d results, inflow, etc...

- strongly coupled gauge theories are usually hard to solve

lattice | supersymmetry | experiment (QCD)

- having general constraints on RG flow very useful

Energy

UV → free "quarks" & "gluons" (in asymptotically free QFT)

RG flow

IR → complicated mess
how are global symmetries realized? broken or not?
is there a mass gap?

If we know some quantities that can be computed in the UV free theory (reliable!) & we have arguments that they <u>remain same in IR</u>, then this places strong constraints on any "fantasy" one might have about the IR behavior!

Such things are rare! ('t Hooft 1980)

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= Suppose we have a theory, say AF, (asympt. free) w/ a <u>global symmetry G</u> (G could be a product group); G is an <u>exact symmetry</u>, i.e. it has no gauge anomaly

= imagine now weakly gauging G; in many cases G will have an anomaly (e.g. G^3)

the coeff of the anomaly (a#) in UV determined by the charges & reps of fields under G;

't Hooft argued this # is RG invariant,

therefore the IR dynamics of the theory has to produce same # : "anomaly matching"

this places constraints on possible IR effective

('t Hooft)

(Frishman, Schwimmer, Banks, Yankielowicz)

(Coleman, Grossmann)

1980s

I'll give an example and some handwaving arguments in favor, tailored for the discussion to follow

④

consider massless $SU(n_c)$ w/ n_f flavors of quarks

(QCD w/ $m_q = 0$, $n_f = 2, 3 \ldots$)
$n_c = 3$

	$SU(n_c)$	$SU(n_f)_L$	$SU(n_f)_R$	$U(1)_B$
q_{L_α}	$\square$	$\square$	1	$1/3$
q_{R_α}	$\overline{\square}$	1	$\overline{\square}$	$-1/3$

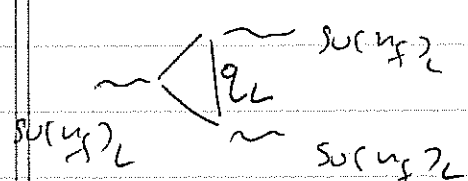
$SL(2, C)$ (gauged, no anomaly $SU(n_c)^3$)

(q_{R_α} are R-handed antiquarks in this notation)

($m\, q_L^\alpha q_{R_\alpha} + h.c.$)

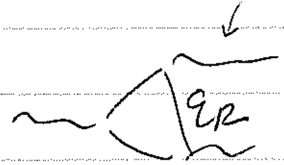
"'t Hooft anomalies" in UV

if gauge $SU(n_f)_L \rightarrow$ have only Weyl $\square$'s, so would be theory anomalous



$\sim n_c$ (there are n_c fundam. of $SU(n_f)_L$)

if gauge $SU(n_f)_R$ (no $\overline{\square}$'s)



$\sim n_c$ (same)

(also ~~...~~ $U(1)_B\, SU(n_f)_L^2$, $U(1)_B\, SU(n_f)_R^2$, don't consider)

5 * gauging $SU(n_f)_{L,R}$ means introduce gauge fields' backgrounds $A^a_{L\mu}$, $A^a_{R\mu}$ for $SU(n_f)_{L,R}$

* quark path-$\int$ measure then is not invariant under $SU(n_f)_{L,R}$ background gauge transforms

(A_L, A_R are not dynamical gauge fields, not Σ over in path $\int$, just backgrounds)

in particular, under $SU(n_f)_L$ (, say) $\Gamma_{eff}(A_L)$

we have $\delta_{\omega_L} \Gamma_{eff}(A_L) \sim tr(\omega_L F_{\mu\nu_L} F_{\alpha\beta_L} \epsilon^{\mu\nu\alpha\beta} + \dots)$

$SU(n_f)_L$ gauge transf. parameter

(the usual expression for $\sim$ triangle with q_L)

(& similar for $SU(n_f)_R$)

this is NOT the variation of a local 4d functional (anomalies $\neq$ variation of local counterterm.

however, it can be written as the variation of a local 5d term, which only depends on $A^a_{\mu_L}$

(there is a whole "science" here that I can't/don't need to/ review → just give a flavor!) ((called "anomaly descent"))

(6) ignoring $\mathcal{O}(A^4)$ terms, we have

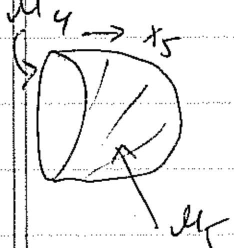
$$S_{5D} \sim \int_{\mathcal{M}_5} d^5x \, tr\left(A^L_\mu \partial_\lambda A^L_\sigma \partial_\alpha A^L_\beta + \ldots\right) \epsilon^{\mu\lambda\sigma\alpha\beta}$$

$$\delta A^L_\mu = \partial_\mu \omega^L + \ldots$$

for us, S_{5D} is formal construction, one really only needs to extend $g = e^{i\omega^L}$ to $\mathcal{M}_5$

$$\delta S_{5D} \sim \int_{\mathcal{M}_5} d^5x \, \partial_\mu \left(tr\, \omega^L \partial_\lambda A^L_\sigma \partial_\alpha A^L_\beta\right) \epsilon^{\mu\lambda\sigma\alpha\beta} + \ldots$$

$$= \int_{\mathcal{M}_4 = \partial \mathcal{M}_5} d^4x \, tr\left(\omega^L \partial_\lambda A^L_\sigma \partial_\alpha A^L_\beta\right) \epsilon^{5\lambda\sigma\alpha\beta} + \ldots$$



$(\mathcal{M}_4 = \partial \mathcal{M}_5)$

't Hooft's argument for RG inv'ce of $SU(n_f)^3$, etc....; slightly re-phrased $\Rightarrow$

$\Rightarrow$ introduce backg'd $A^a_{\mu L}$,

$\Gamma^{4d}_{eff}(A_L)$ is not gauge inv't, to make it so, introduce

$S_{5d}(A_L)$ to cancel variation of 4d Γ^{4d}_{eff} ('t Hooft: add spectators whose anomaly cancels that of 4d; spectator fermions only charged under $SU(n_f)_L$, not QCD), thus the combined theory QCD + spectators now is $SU(n_f)_L$ invariant

aside

FYI:

$$Q_5 = tr\left(AF^2 - \tfrac{1}{2}A^3F + \tfrac{1}{10}A^5\right)$$

$$\delta Q_5 = d Q'_4$$

$$Q'_4 = tr\left(\omega\, d\left(AdA + \tfrac{1}{2}A^3\right)\right)$$

...

further, if $\mathcal{M}_5$ is closed then

$$e^{i\delta S_{5d}} = e^{i 2\pi \mathbb{Z}} = 1$$

has variation only on $\partial \mathcal{M}_5$, whenever it erupts

(($SU(n_f)$ gauge theory in UV consistent ~~consistent~~ at all scales))

is $SU(n_f)$ invariant & will remain so at all scales

($S_{5D}(A_{\mu L})$ doesn't care about what 4d strong coupled gauge theory "does"; S_{5D} represents 'spectators') ; so in IR,

whatever the 4d physics is, its

$\Gamma_{eff}^{4d, IR}(A^a_{\mu L})$ should have the same variation

under $SU(n_f)_L$ as it had in UV, same

variation under $SU(n_f)_L$ means same "anomaly"

in IR, there can be massless fermions (in QCD "baryons") that carry $SU(n_f)_L$ charges, or

there can be scalar goldstones — massless pions, that ~~cancel the~~ ~~an~~ reproduce the anomaly ($SU(n_f)_L$ broken)

can't discuss more, in QCD 2nd option

$SU(n_f)_L \times SU(n_f)_R \to SU(n_f)_V$

anomalies 'matched' by pions, kaons, via "WZ term" in chiral Lagrangian,

(— famous $\pi^0 \to 2\gamma$ decay etc—)

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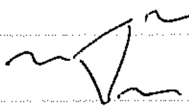
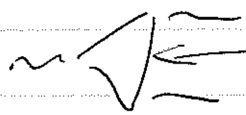
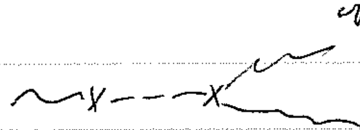
Moral :

- global symmetry, exact (anomaly free)
- if, when gauged, anomalous
  $\equiv$ " 't Hooft anomaly "
- 't Hooft anomaly computed in UV AF theory via  , exact (RG inv.)
- in IR theory must be "matched"

continuous anomaly in 2d, 4d associated w/ IR $\frac{1}{q^2}$ pole ---

by
 $\longleftarrow$ massless (composite) fermions (unbroken)

or

 $\longleftarrow$ massless Goldstone bosons (broken)

= all kinds of examples in "theory space"
= in real world QCD : N_f flavor broken

I talked about $SU(N_f)_L^3$ — there's mixed as well

$U(1)_B\, SU(N_f)_L^2$: $\frac{1}{3} \times N_c$ in UV

(--- in IR: WZ..)

= often 't H. anomalies only check on conjectured IR behavior (Seiberg duality etc..)

New stuff: (Gaiotto, Kapustin, Komargodski, Seiberg, Willett 2014 –)

- mixed 0-form/1-form symmetry anomalies & their matching
- also impose nontrivial IR constraints

Ex: SYM $\mathcal{N}=1$ $SU(N_c)$ w/ $\lambda_\alpha^{a=1\ldots N_c^2-1}$

$\uparrow SL(2,\mathbb{C})$

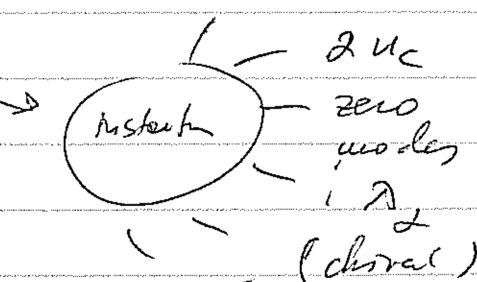
adjoint Weyl massless

classically $\lambda \to e^{i\alpha}\lambda$ $U(1)$ "R" symmetry (chiral)

instanton background

"$e^{-\frac{8\pi^2}{g^2}}\left(\mathrm{tr}\,\lambda^\alpha\lambda_\alpha\right)^{N_c}$"

↑ schematic 't Hooft vertex



under $U(1)$: $\left(\mathrm{tr}\,\lambda\lambda\right)^{N_c} \to e^{i\alpha 2N_c}\left(\mathrm{tr}\,\lambda\lambda\right)^{N_c}$

so only $\alpha = \frac{2\pi}{2N_c}$ $\Rightarrow$ $U(1) \to \mathbb{Z}_{2N_c}$

anomalous $U(1)$ (anomaly free chiral symmetry)

there is a mixed $\mathbb{Z}_{2N_c}^{\text{"0-form"}} \times \left(\mathbb{Z}_{N_c}^{\text{"1-form"}}\right)^2$ 't Hooft anomaly

↑ ordinary symmetry

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$\mathbb{Z}_{N_c}^{1\text{-form}}$ = center symmetry, 1-form: acts on line operators, not local fields

(gauging $\mathbb{Z}_{N_c}^{1\text{-form}}$ center breaks discrete chiral ≡ ≡ 't Hooft anomaly)

- by similar arguments (S_{Γ_d} etc.) must be matched in IR

$\Downarrow$

here "Goldstone" mode

- $\mathbb{Z}_{2N_c}^{0\text{-form}}$ broken

"Goldstones" or Domain Walls match anomaly (IR TQFT!

plays role of chiral lagrangian

- ~~[illegible]~~ 't Hooft anomaly constrains physics on DW's, too ...

- hope to give you flavor
- no theorems!!

(- all still pretty much being worked on!)

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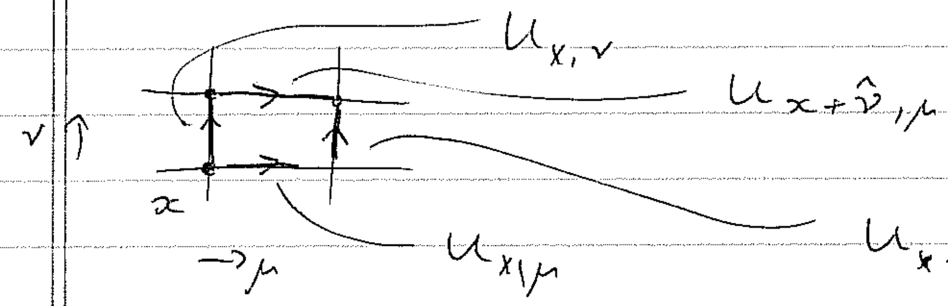
What is center symmetry? What is it good for?

- 1st intro best done on lattice:



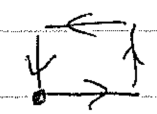
$$U_{x,\mu} \to g_x\, U_{x,\mu}\, g^{+}_{x+\hat\mu}\,, \quad g_x \in G$$

$$\left(\begin{array}{l} U_{x,\mu} \approx e^{iaA_\mu(x)} \approx 1 + i a A_\mu(x) \\ a \sim \text{lattice spacing} \\ A_\mu = \text{Lie algebra element} \end{array} \right)$$

$\underbrace{U_{x\mu} \in G}$

- $SU(N_c)$ $N_c \times N_c$ unitary
- $U(1)$: $e^{i\alpha}$, $\alpha \in (0, 2\pi]$

$$\prod_{\Box_{x(\mu\nu)}} U \equiv \underbrace{U_{x,\mu}\, U_{x+\mu,\nu}\, U^{+}_{x+\nu,\mu}\, U^{+}_{x,\nu}}$$



$$\prod_{\Box_{x(\mu\nu)}} U \to g_x \left(\prod_{\Box_x} U\right) g_x^{+} \quad ; \text{ so } \operatorname{tr} \prod_{\Box_x} U \ \underline{\text{invt}}$$

$$\left(\prod_{\Box_{x(\mu\nu)}} U \approx e^{i a^2 F_{\mu\nu}(x)} \right)$$

(more generally $\operatorname{tr}(\prod U) = \text{invt.}$ closed loop)

$$S_{gauge} = \frac{1}{e^2} \sum_{x} \sum_{\mu > \nu} \left(tr\left(\prod_{\square_{x_{\mu\nu}}} U \right) + h.c. \right)$$

$\uparrow$ all points

$$\sim \frac{1}{e^2} \sum_{x} \sum_{\mu>\nu} \left(tr\, F_{\mu\nu}(x) F^{\mu\nu}(x) \right)$$

+ const.

Consider 2 cases of interest:

1) ✓ $SU(n_c)$ gauge theory, say no matter, or matter in adjoint

2) $U(1)$ gauge theory, $\underbrace{\text{w/ fields}}_{\text{matter}}$ of charge n_c

$$\begin{tikzcd} \phi_x \quad x \arrow[r, "U_{x,\mu}"'] & \phi_{x+\hat{\mu}} \quad x+\hat{\mu} \end{tikzcd}$$

$$\phi_x \to g_x^{n_c} \phi_x \,, \quad g_x \in U(1)$$

$\Uparrow$ charge n_c

kinetic term

$$\phi_x^* \left(U_{x,\mu}\right)^{n_c} \phi_{x+\hat{\mu}} \longrightarrow$$

$$\left(\to \phi_x^* (g_x^*)^{n_c} (g_x)^{n_c} (U_{x,\mu})^{n_c} (g_{x+\hat{\mu}}^\dagger)^{n_c} g_{x+\hat{\mu}}^{n_c} \phi_{x+\hat{\mu}} \right)$$

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for 1) matter in adjoint

$$\phi_x \to g_x \phi_x g_x^+ \,, \quad g_x \in SU(n_c)$$

$$tr\left(\phi_x \, U_{x,\mu} \, \phi^*_{x+\hat{\mu}} \, U^+_{x,\mu}\right) \to$$

$$\Big(\to tr\big(g_x \phi_x g_x^+ \, g_x U_{x,\mu} g^+_{x+\hat{\mu}} \, g_{x+\hat{\mu}} \phi^+_{x+\hat{\mu}} g^+_{x+\hat{\mu}} \cdot$$

$$\cdot \, g_{x+\mu} U^+_{x,\mu} g^+_x\big)\Big) \text{ invariant}$$

__Claim__

both theories 1 & 2 possess a

"one form Z_{n_c} center symmetry"

global

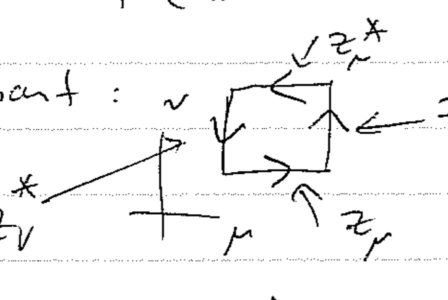
$$U_{x,\mu} \to z_\mu U_{x,\mu} \,, \qquad z_\mu \equiv e^{i\frac{2\pi}{n_c}z} , \quad \underline{\mu = 1..d}$$

$z \in \mathbb{Z} (\mathrm{mod}\ n_c)$

__1st__ · both U & zU are elements of G ($SU(n_c)$ or $U(1)$, respectively)

__2nd__ · path $\int$ measure invariant (Haar measure)

__3rd__ $SU(n_c)$ action invariant:



"1-form symmetry"

(≡ parameter is a "vector")

adjoint fermion/scalar hopping terms clearly invariant, has $\underset{U_\mu}{\overset{U^+_\mu}{\rightleftarrows}}$

4th U(1) action also inv't $\phi^*(U)^{N_c}\phi$

(inv't under $U \to zU$)

So lattice gauge theories of the kind just described have this Z_{N_c} 1-form "center symmetry"

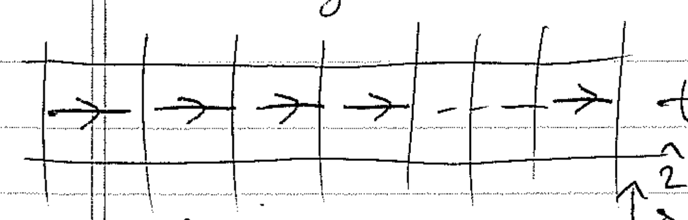
(name: Z_{N_c} is center of $SU(N_c)$)

- it is a global 1-form symmetry $\equiv Z^{(1)}_{N_c}$

- does not act on local (0-form) fields, but only on $U_{x,\mu}$ ≡ a tiny Wilson line

only gauge inv't observables that transform under $Z^{(1)}_{N_c}$ ← indicate one-form

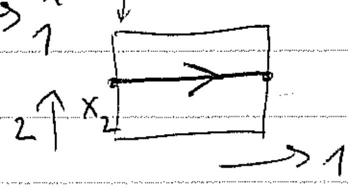
are Wilson lines wrapping around the world (we shall imagine world is a $T^d = (S^1)^d$)



$\mathrm{tr}\left(\prod\left(U_{\text{line},1}\right)\right) \equiv W_1(x_2)$

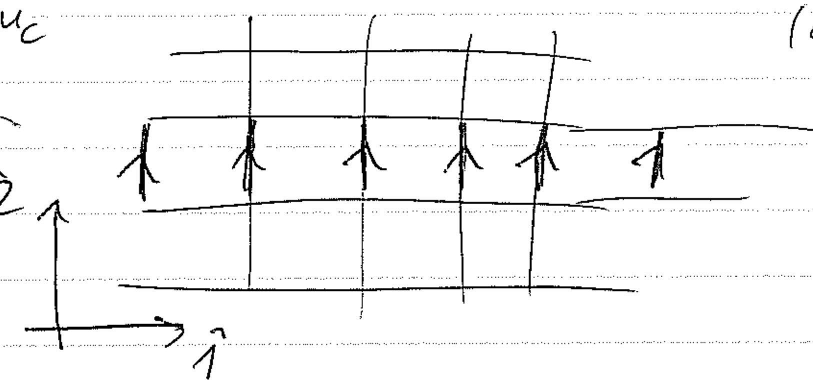
(transforms under $Z^{(1)}_{N_c}$)

2d example →



skip in lectures:

a slight refinement of lattice def of $Z^{(1)}_{N_c}$ (agrees well w/ bundle (continuum description) as a symm. acting on transition functions...)



> center symmetry associated w/ dir'n $\hat{2}$: only multiply shaded links by $z_{\hat{2}}$,

this ensures $W_2(x_1) \to z_{\hat{2}}\, W(x_1)$

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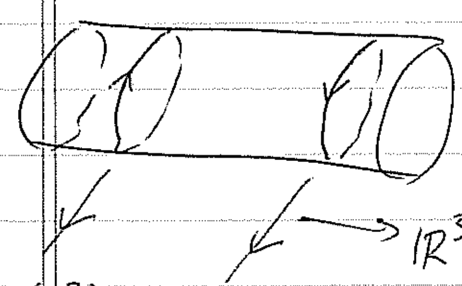
historically $Z^{(1)}_{N_c}$ has been associated w/ confinement - deconfinement transition



$\beta = \frac{1}{T}$, "x^0" = Euclidean, periodic

$\to \mathbb{R}^3$, say

$W_0(\vec{x})$ $\quad W_0^*(\vec{y}) \longrightarrow$ W_0 is tr in Fund. rep (our U's are $N \times N$)

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= static fundamental quark probes in $SU(n_c)$ pure YM (~~or~~ adjoint YM) theory

$$\langle W_0(\vec{x})\, W_0^*(\vec{y})\rangle_\beta \Big|_{|\bar{x}-\bar{y}|\to\infty} \sim \text{heuristically}$$

$$\sim e^{-\beta V(\vec{x}-\vec{y})}$$

(quark-antiquark potential)

low T phase, $T < T_c$ "confining phase"

$$\sim \begin{cases} e^{-\beta|\vec{x}-\vec{y}|\sigma_{string}} \to 0, & |x-y|\to\infty \\ e^{-\beta \frac{e^{-m_D|\bar{x}-\bar{y}|}}{|\bar{x}-\bar{y}|}} \to 1, & |x-y|\to\infty \end{cases}$$

high T, Debye screening, ('deconfined' phase)

$\langle W_0(\bar{x})\, W_0^*(\bar{y})\rangle \to 0$ in confined phase $(\langle W_0\rangle = 0)$ $Z_{n_c}^{(1)}$ unbroken

$\langle W_0(x)\, W_0^*(y)\rangle \to$ const in deconfi. $\to \underbrace{\langle W_0\rangle \neq 0}_{}$!

$Z_{n_c}^{(1)}$ broken phase

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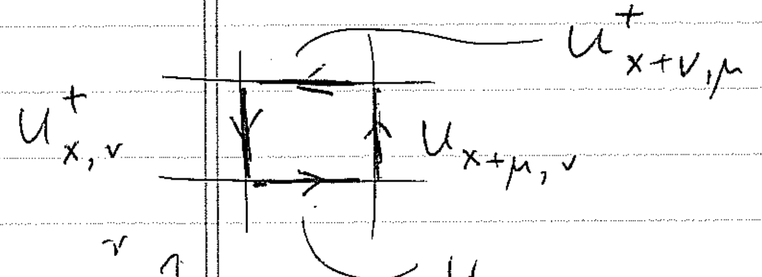
The "broken" or "unbroken" $Z^{(1)}_{N_c}$ refers to the x_0-direction part of the 1-form symmetry

($Z^{(1)}_{N_c}$ "broken" at high T --- — unusual)

But our subject is 't Hooft anomalies; to study, recall discussion of QCD $SU(N_f)^3$ etc — — gauge symmetry!

How to gauge 1-form $Z^{(1)}_{N_c}$?

— very easy! — let $U_{x,\mu} \to z_{x,\mu} U_{x,\mu}$, $\quad z_{x,\mu} = e^{i\frac{2\pi}{N_c} n_{x,\mu}}$



(different integer ∀ link x,μ ||| "local" $Z^{(1)}_{N_c}$)

Then $\square \to e^{i\frac{2\pi}{N_c}(n_{x,\mu} + n_{x+\mu,\nu} - n_{x+\nu,\mu} - n_{x,\nu})} \times \square$

$D_{x,\mu\nu} \leftrightarrow U_{D_{x,\mu\nu}}$ $\qquad$ $D_{x,\mu\nu} = U_{D_{x,\mu\nu}}$

introduce a plaquette $\Box_{x,\mu\nu}$ - based

$Z^{(2)}_{N_c}$ field $b_{x,\mu\nu} \in \mathbb{Z} \pmod{N_c}$

& modify action

$$\sum_x \sum_{\mu>\nu} \mathrm{tr}\left(U_{\Box_{x,\mu\nu}} e^{i \frac{2\pi}{N_c} b_{x,\mu\nu}} + h.c. \right)$$

where under $Z^{(1)}_{N_c}$

$$b_{x,\mu\nu} \to b_{x,\mu\nu} - n_{x,\mu} - n_{x+\mu,\nu} + n_{x+\nu,\mu} + n_{x,\nu}$$

s.t. above is invariant (cancels trf of $U_{\Box_{x,\mu\nu}}$)

<u>Notice</u>: • now one has various options (one could Σ over $b_{x,\mu\nu}$ in path $\int$, modifying the theory, and in the process decide if a "kinetic" term for $b_{x,\mu\nu}$ should be included $\longrightarrow$ not here!)

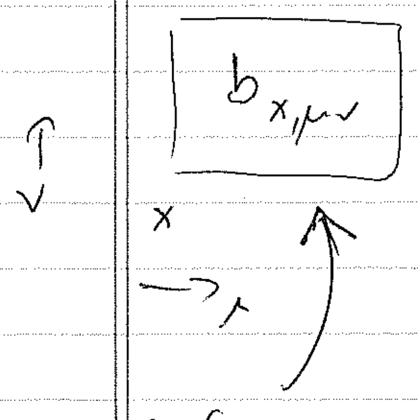
• we're interested in $b_{x,\mu\nu}$ backgrounds as probes of the original G ($SU(N_c)$ or $U(1)$) gauge theory

• we'll consider "<u>topological $b_{x,\mu}$ backgrounds</u>"

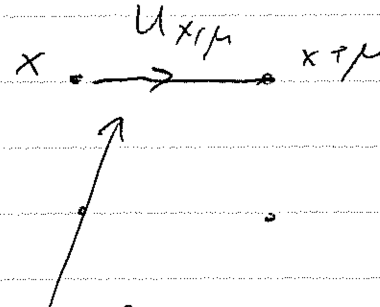
let me explain/summarize:

• $SU(n_c)$ w/ adjoints or $U(1)$ w/ charge n_c have a $\mathbb{Z}^{(1)}_{n_c}$ global symmetry

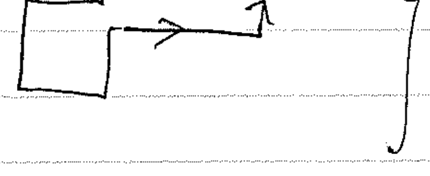
• to gauge 1-form global symmetry, introduce a 2-form gauge field (as shown); as symmetry is $\mathbb{Z}_{n_c}$, so is the gauge field



2-form gauge field gauges 1-form symmetry acting on links (line operators)



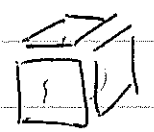
one-form gauge field gauges 0-form symmetry (acting on local fields)



product of U's on closed loops are gauge invariant! (traced if nonabelian) similarly, one can make closed surfaces from plaquettes (like one makes closed loops from links)

similarly, sums of $b_{x,\mu\nu} \equiv b^{(2)}$ over plaquettes forming a closed surface are $\mathbb{Z}^{(1)}_{N_c}$ gauge invariant

e.g.



flux of $b^{(2)}$ thru cube is gauge invariant

a topological $b^{(2)}$ background is one where flux of $b^{(2)}$ thru any contractible 2-surface vanishes

such as a cube

for a toroidal lattice, clearly, there is a topological $\underbrace{\text{invariant}}_{\text{gauge}}$ which is nontrivial → flux of $b^{(2)}$ thru $\mu\nu$ plane

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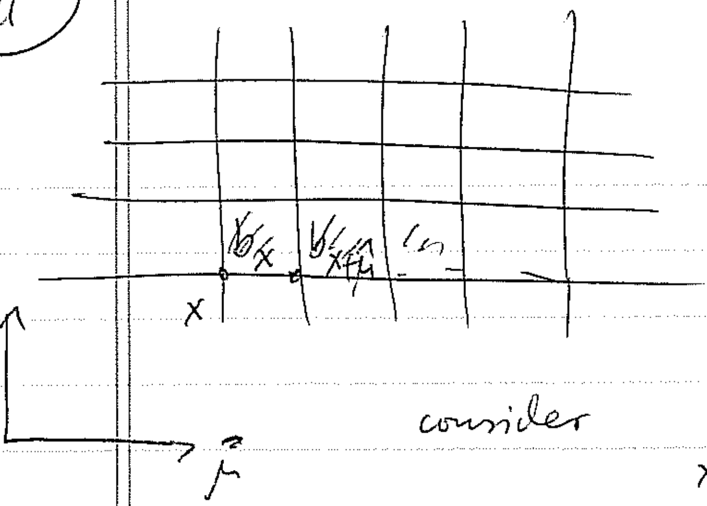


some integer mod N_c

consider $\sum_{x_\mu, x_\nu = (0,0)}^{(L_\mu, L_\nu)} b_{x\,\mu\nu} \equiv n_{\mu\nu}$

(at given $x^\alpha x^\beta$

$\alpha\beta \neq \mu\nu$

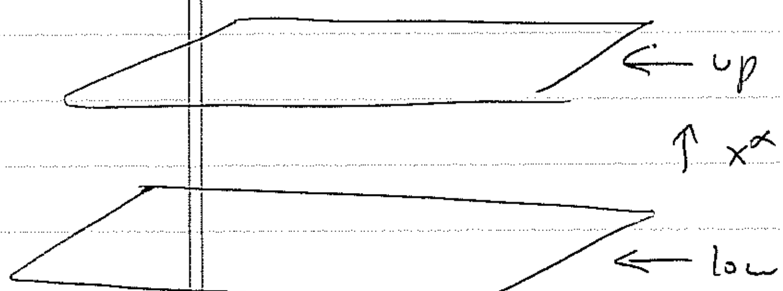
e.g. $\mu\nu = 1,2$

$\alpha\beta = 3,4$)

this is gauge invariant, as gauge trfs of neighboring plaquettes cancel out

if the flux of $b^{(2)}$ thru any cube $= 0$, then

$n_{\mu\nu}$ does not depend on x^α, x^β $(\alpha, \beta \neq \mu\nu)$

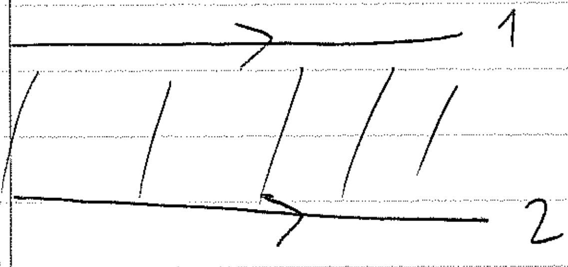


$b^{(2)}$ flux (up) $-$ $b^{(2)}$ flux (low) $=$

$=$ volume $\int$ of $d b^{(2)}$

$= 0$ (by topological nature)

(similar statement for Wilson loops, abelian



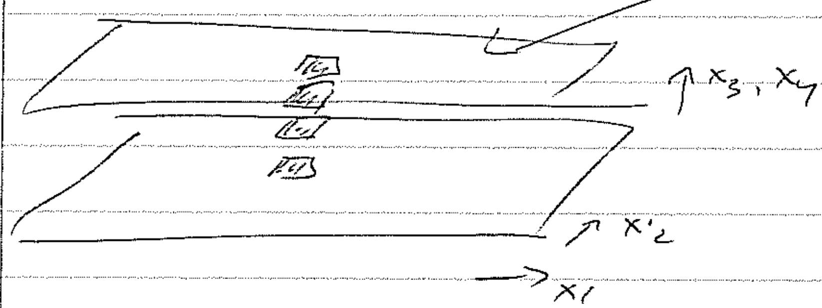
$\int_1 - \int_2 A = \int dA = 0$)

These mod N_c integers $n_{\mu\nu}$ are topological invariants characterizing $b^{(2)}$ backgrounds

in 2d: n_{12}, in 4d $n_{\mu\nu}, \mu<\nu$
(6 2-planes)

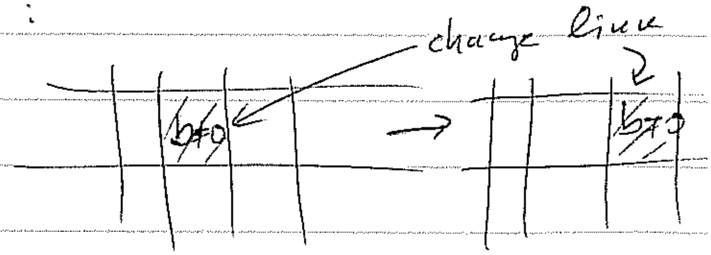
- they are a.k.a. "'t Hooft fluxes" and the above was their lattice implementation

- simplest ex. of topological bckd: modify a single plaquette in the x^1-x^2 plane, for all x^3 & x^4



(aka thin "center vortex")

- position in $x_1 x_2$ plane can be moved around by redefining link fields:
- only $\oint b^{(2)}$ matters!



What this has to do w/ anomalies?

-- start in 2d.

need a little reminder of topological charge

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- in a 2d compact U(1) gauge theory

(such as our lattice theory,)

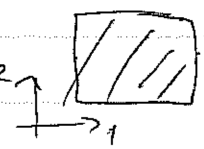
$$\int \frac{dx^1 dx^2}{2\pi} F_{12} \in \mathbb{Z}$$

- closed 2 manifold (S^2, T^2)

(or closed by boundary conditions: $\mathbb{R}^2$ w/ $F_{12}|_\infty \to \phi$)

- one proof (take $\mathbb{R}^2$, w/ $F_{12}|_\infty = 0$)

$$\int dx^1 dx^2 F_{12} = \int dx^1 dx^2 (\partial_1 A_2 - \partial_2 A_1) =$$



$$= \oint d\vec{x} \cdot \vec{A} = \oint d\vec{x} \cdot \vec{\nabla}\omega = \omega(|\vec{x}| \to \infty, \varphi = 2\pi) - \omega(|\vec{x}| \to \infty, \varphi = 0)$$

at ∞

$\vec{A} = \vec{\nabla}\omega$ @ ∞ ($F_{12} = 0$)

$= $ winding # of $\omega = 2\pi\mathbb{Z}$

$e^{i\omega}$ is in $U(1)$, $e^{i\omega(2\pi)} = e^{i\omega(0)}$

so $\omega(2\pi) - \omega(0) = 2\pi\mathbb{Z}$

($e^{i\omega}$ is single valued)

- other proofs exist, perhaps in "tutorial"

? on the lattice ?

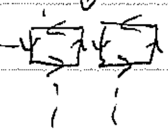
continuum limit

we said $U_{\square_x} \approx e^{i a^2 F_{12}(x)}$

so $\prod_{\text{all } x} U_{\square_x} \simeq e^{i a^2 \sum_x F_{12}(x)} = e^{i \oint_{T^2} F_{12} dx^1 dx^2}$

↑ all space, i.e. T^2 in 1-2

on the other hand, $\prod_{\text{all}_x} U_{\square_x} = 1$ since



all links cancel out betw neighbor plaquett

hence $1 = e^{i \oint_{T^2} F_{12} dx^1 dx^2} \Rightarrow \oint_{T^2} F_{12} dx^1 dx^2 = 2\pi \mathbb{Z}$

Note: can one ever get nonzero integer on r.h.s ?

define $Q_{top} = \frac{1}{2\pi i} \sum_x \arg\log U_{\square_x}$; $-\pi < \arg\log \leq \pi$,

branch cut of log makes it possible --

e.g. take $U_{\square_x} = e^{i \frac{2\pi}{L_1 L_2} n}$, clearly $\prod_x U_{\square_x} = 1 = e^{i 2\pi n}$

but $Q_{top} = n$ ....

— that this is sensible, see Lüscher '80's

now, if we gauge center: $U_{\square_x} \to U_{\square_x} e^{i \frac{2\pi}{n_c} b^{(2)}(x)}$

in order that $\mathbb{Z}^{(1)}_{n_c}$ gauge invariance holds, so then

we replace

$$e^{i a^2 \sum_x F_{12}(x) + i \sum_x \frac{2\pi}{n_c} b^{(2)}(x)} = \prod_x U_{\square_x} e^{i \frac{2\pi}{n_c} b^{(2)}(x)}$$

as before we say $Q_{top} = \dfrac{a^2 \sum_x F_{12}(x) + \frac{2\pi}{n_c} \sum_x b^2(x)}{2\pi} = \dfrac{\oint F_{12} dx^1 dx^2}{2\pi} + \dfrac{\sum_x b^{(2)}(x)}{n_c}$.

numerator is our topological invariant n_{12}

so we have

fractional $\Longrightarrow$ $Q_{top}(\mathbb{Z}^{(1)}_{n_c} \text{ gauge inv}) = \mathbb{Z} + \frac{n_{12}}{n_c}$

notice that gauging $\mathbb{Z}_{n_c}^{(1)}$ center effectively says $U_{link} \approx U_{link}\, e^{i\frac{2\pi}{n_c}}$

$$\downarrow \qquad\qquad \downarrow$$

$$e^{i\alpha_{link}} \approx e^{i\left(\alpha_{link} + \frac{2\pi}{n_c}\right)}$$

(so range of α_{link} can be restricted to $\frac{2\pi}{n_c}$)

this means that group elements ($U(1)$) are identified up to multiplication by $\sqrt[n_c]{1}$ ($U(1)/\mathbb{Z}_{n_c}$)

(in $\mathbb{Z}_{n_c}^{(1)}$ gauged case)

But recall discussion of Q_{top}, we said

$$Q_{top} = \frac{1}{2\pi} \oint_{C_\infty} \partial_\mu \omega \, dx^\mu$$

$$= \frac{i}{2\pi} \oint_{C_\infty} e^{i\omega} \partial_\mu e^{-i\omega} \, dx^\mu$$

$$= \frac{1}{2\pi} \left(\omega(2\pi) - \omega(0) \right)$$

if group is $U(1)$, $e^{i\omega(2\pi)} = e^{i\omega(0)}$, so $\omega(2\pi) - \omega(0) = 2\pi\mathbb{Z}$ & $Q_{top} \in \mathbb{Z}$, but

if $e^{i\omega(2\pi)} = e^{i\frac{2\pi}{n_c}\mathbb{Z}} e^{i\omega(0)}$, then

$$\omega(2\pi) - \omega(0) = \frac{2\pi}{n_c}\mathbb{Z} \quad \& \quad Q_{top} \in \frac{\mathbb{Z}}{n_c}$$

this squares well with the usual continuum description of center symmetry as acting on "transition functions" of the gauge bundle ...

--- maybe in "tutorial"? ---

essentially above "ω" can be thought of as a transition fn; ← gauge transform @ ∞, in our derivation

(but it can be taken to be one around one of the cycles of a torus)

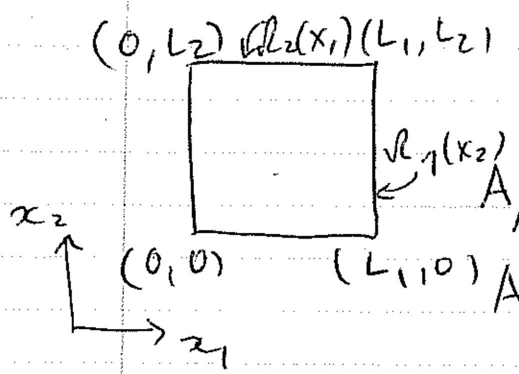
("tutorial"?)

skip in lecture

"bundle on $\mathbb{T}^2$ (light)" 't Hooft, van Baal } '80s → generalizes to $\mathbb{T}^4$ & $SU(N_c)$!

only one coordinate patch (not many) but nontrivial b.c.

$0 \le x_1 \le L_1$
$0 \le x_2 \le L_2$

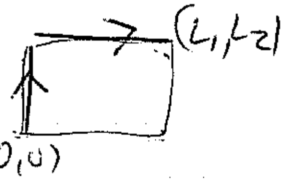


$$A_\mu(L_1, x_2) = \Omega_1(x_2)\left(A_\mu(0, x_2) + i\partial_\mu\right)\Omega_1^+(x_2)$$

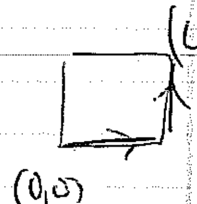
$$A_\mu(x_1, L_2) = \Omega_2(x_1)\left(A_\mu(x_1, 0) + i\partial_\mu\right)\Omega_2^+(x_1)$$

$\Omega_{1,2}$: transition fns ($\in G$)

"cocycle conditions"



$$A_\mu(L_1,L_2) = \Omega_1(L_2)\left(A_\mu(0,L_2)+i\partial_\mu\right)\Omega_1^+(L_2)$$



$$= \Omega_1(L_2)\Big(\Omega_2(0)\left(A_\mu(0,0)+i\partial_\mu\right)\Omega_2^+(0) + i\partial_\mu\Big)\Omega_1^+(L_2)$$

$$= \Omega_1(L_2)\,\Omega_2(0)\left[A_\mu(0,0)+i\partial_\mu\right]\Omega_2^+(0)\,\Omega_1^+(L_2)$$

$$A_\mu(L_1,L_1) = \Omega_2(L_1)\left(A_\mu(L_1,0)+i\partial_\mu\right)\Omega_2^+(L_1)$$

$$= \Omega_2(L_1)\Big(\Omega_1(0)\left(A_\mu(0,0)+i\partial_\mu\right)\Omega_1(0)^+ + i\partial_\mu\Big)\Omega_2^+(L_1)$$

$$= \Omega_2(L_1)\,\Omega_1(0)\left(A_\mu(0,0)+i\partial_\mu\right)\Omega_1^+(0)\,\Omega_2^+(L_1)$$

so we have to have "cocycle" consistency condition

$$\boxed{\Omega_1(L_2)\,\Omega_2(0) = \Omega_2(L_1)\,\Omega_1(0)}$$

(where Ω's are $\in G$) on the transition functions

for $U(1)$, $\Omega_i(x) = e^{i\omega_i(x)}$

(ω_1 dep. on x_2, ω_2 dep. on x_1)

$$e^{i\omega_1(L_2)}\,e^{i\omega_2(0)} = e^{i\omega_2(L_1)}\,e^{i\omega_1(0)}$$

or $\quad \omega_1(L_2) + \omega_2(0) - \omega_2(L_1) - \omega_1(0) = 2\pi\mathbb{Z}$

26.2 26.2

- Show that for A_μ's obeying conditions of bottom p. 26,

$$\oint \frac{F_{12}\, dx^1 dx^2}{2\pi} = \frac{1}{2\pi}\left(\omega_1(L_2) - \omega_1(0) - \omega_2(L_1) + \omega_2(0)\right)$$

$$= \mathbb{Z} \text{ for } U(1)$$

Where's order here?

matter fields also obey consistency condition (like A_μ, they are related by $\Omega_{1,2}$ at the opposite sides of $\square$)

- Show for a charge n_c field ϕ:

$$\phi(L_1, L_2) = \left(\Omega_1(L_2)\, \Omega_2(0)\right)^{n_c} \phi(0,0)$$

$$= \left(\Omega_2(L_1)\, \Omega_1(0)\right)^{n_c} \phi(0,0)$$

Show ⇓ same consistency condition, but $(--)^{n_c}$.

- Gauge field b.c. & matter field b.c. are invariant under

$$\begin{cases} \Omega_1 \to e^{i\frac{2\pi}{n_c} l_1}\, \Omega_1 \\ \Omega_2 \to e^{i\frac{2\pi}{n_c} l_2}\, \Omega_2 \end{cases}$$

(26.3)

(26.3)

- Show ↴

if $A_\mu \to g(A_\mu + i\partial_\mu)g^+$ — a gauge trf. $g \in G$ defined on □

$\Omega_1(x_2) \to g(L_1, x_2)\, \Omega_1(x_2)\, g^+(0, x_2)$

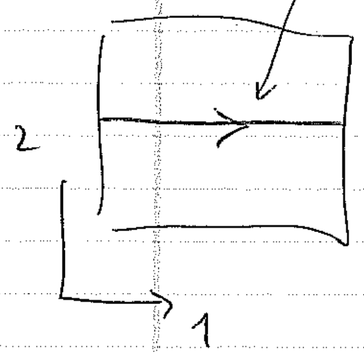
$\Omega_2(x_1) \to g(x_1, L_2)\, \Omega_2(x_1)\, g^+(x_1, 0)$

(& matter b.c. invariant, too)

(transition fns gauge transf. rule.)

Consider now a Wilson loop:

$$W_1(x_2) = e^{i\int_0^{L_1} A_1(x, x_2)\, dx^1}$$



Under gauge transf.

$$W_1(x_2) \to g(0, x_2)\, W_1(x_2)\, g^+(L_1, x_2)$$

these are not same, def'd on □

to make gauge inv.: (show ⇓)

$W_1(x_2)\,\Omega_1(x_2)$; but $\Omega_1(x_2) \to e^{i\frac{2\pi}{N_c}\ell_1}\,\Omega_1(x_2)$ under center.

So $W_1(x_1) \to e^{i\frac{2\pi}{n_c}\ell_1} W_1(x_2)$

If we "gauge center" $\to$ we have a more general consistency "cocycle" condition

$$\Omega_1(L_2)\,\Omega_2(0) = e^{i\frac{2\pi}{n_c}n_{12}}\,\Omega_2(L_1)\,\Omega_1(0)$$

then $\omega_1(L_2) + \omega_2(0) - \omega_2(L_1) - \omega_1(0) = \frac{2\pi}{n_c}\,n_{12}$

& then $Q_{top} = \mathbb{Z} + \frac{n_{12}}{n_c}$.

- this product is unity up to a center element
- matter consistency condition does not care $(\ )^{n_c}$ ✓
- gauge field as well, since (see p. 27)

$$A_\mu(L_1 L_2) = \Omega_2(L_1)\,\Omega_1(0)\,(A + id)\,\Omega_1^+(0)\,\Omega_2^+(L_1)$$
$$= \Omega_1(L_2)\,\Omega_2(0)\,(A + id)\,\Omega_2^+(0)\,\Omega_1^+(L_2)$$

center element cancels out.

---

see van Baal, CMP ~1982 for $SU(n_c)$ & $SU(n_c)/\mathbb{Z}_{n_c}$ on T^4 — end skip

(27)

2 dim example of anomaly & its IR matching

Ex: 2d massless Schwinger model w/ charge "n_c" fermion

(self-servingly, one I worked on w/ Anber, 2018)

matter lagrangian: $i\psi_+^\dagger(\partial_+ + iA_+ n_c)\psi_+ + i\psi_-^\dagger(\partial_- + iA_- n_c)\psi_-$

$A_\pm$: lightcone components, U(1) (Schwinger model)

$\psi_\pm$: Weyl components of Dirac

$U(1)_V$: $\psi_\pm \to e^{i\alpha}\psi_\pm$, this symmetry is gauged
(Dirac mass term would be
$\sim \bar{\psi}_+\psi_-$, $\bar{\psi}_-\psi_+$)

$U(1)_A$: $\psi_\pm \to e^{i(\pm)\chi}\psi_\pm$, acts oppositely on L & R
moving spinors

this is anomalous ; in 2d Jacobian $\sim e^{\pm i Q_{top}\chi q}$

+ for ψ_+
− for ψ_-

parameter of $U(1)_A$

charge of ψ under $U(1)_{gauged}$

so we have
Jacobian under $U(1)_A$ = $e^{2i\chi n_c Q_{top}}$.

($Q_{top} = \oint F_{12}/2\pi$) which is unity for $\chi = \dfrac{2\pi}{2n_c}$, since $Q_{top} \in \mathbb{Z}$

$\to$ $\mathbb{Z}_{2n_c}^{(0)}$ discrete 0-form chiral anomaly free

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[this is like in SYM in 4d, where there was a $\mathbb{Z}^{(0)}_{2N_c}$ R-symmetry]

we have
$$\psi_\pm \to e^{\pm i \frac{\pi}{N_c}} \psi_\pm \, , \quad \text{exact chiral symmetry } \mathbb{Z}_{2N_c}$$

If we gauge the $\mathbb{Z}^{(1)}_{N_c}$ center (fermions charge N_c, so line we discussed, 1-form global $\mathbb{Z}^{(1)}_{N_c}$)

$$Q_{top} = \frac{\mathbb{Z}}{N_c}$$

but this means that under a $\mathbb{Z}^{(0)}_{2N_c}$ we get a phase $e^{i\frac{2\pi}{N_c}}$.

$$e^{i\Gamma_{eff}}\Big|_{N_{12}\neq 0} \xrightarrow{\mathbb{Z}^{(0)}_{2N_c}} e^{i\frac{2\pi}{N_c}} e^{i\Gamma_{eff}}\Big|_{N_{12}\neq 0} \quad (N_{12}=1 \text{ too})$$

in the background of $N_{12} = 1$ 2-form $b^{(2)}$ flux (gauging $\mathbb{Z}^{(1)}_{N_c}$)

this $e^{i\frac{2\pi}{N_c}}$ phase is the mixed $\mathbb{Z}^{(0)}_{2N_c} - \mathbb{Z}^{(1)}_{N_c}$ anomaly;

How is it matched in the IR theory??

A bunch of claims: (1) the $e^{i\frac{2\pi}{N_c}}$ phase is due to the variation of a 3d CS term (like the 5d $AdAdA$ we talked about), whatever the IR physics of charge-N_c Schwinger model, must match

[technology later]

(2) For the Schwinger model, what happens is that $\mathbb{Z}^{(0)}_{2N_c} \xrightarrow{\text{dynamically}} \mathbb{Z}^{(0)}_{2}$

the anomalous $e^{i\frac{2\pi}{n_c}}$ phase in the IR is from an "IR TQFT" describing the IR physics of the model

(3) exploiting the exact solvability of model, we can actually derive the IR TQFT — but now we'll just postulate it, show that anomaly is matched in IR, — we'll just declare the explicit relation between TQFT fields & UV gauge theory objects

Claim (2) says: $\psi_+^\dagger \psi_-$ - gauge invariant [please take this as given!]

$$\xrightarrow{\text{under } Z_{2n_c}^{(0)}} e^{-i\frac{2\pi}{n_c}} \psi_+^\dagger \psi_-$$

so $\psi_+^\dagger \psi_-$ is an order parameter for $Z_{2n_c}^{(0)}$ breaking

- there are n_c vacua $\langle p | \psi_+^\dagger \psi_- | p \rangle = \text{const}\, e^{i\frac{2\pi p}{n_c}}$ associated w/ the breaking of $Z_{2n_c}^{(0)}$;
- the IR theory is "empty" — no massless states, mass gap $\sim e^2$ (gauge coupling)
- to see the matching of the anomaly, construct a "chiral Lagrangian" -- but since no d.o.f. massless, it simply describes the n_c vacua of the theory →

→ let's see how →

Claim IR on charge n_c Schwinger model is described by the following 2d free TQFT

$$\int \mathcal{D}\varphi\, \mathcal{D}a\, e^{S} \;; \quad S = i\frac{n_c}{2\pi}\int_{M_2} \varphi^{(0)}\, d a^{(1)}$$

- M_2 is, say T^2, all Euclidean for now, or $\mathbb{R}^2$ $\left(\begin{array}{l} d a^{(1)} = f_{12}\, dx^1 dx^2 \\ f_{12} = \partial_1 a_2^{(1)} - \partial_2 a_1^{(1)} \end{array} \right)$
- $\varphi^{(0)}$ is a compact scalar, i.e. $e^{i\varphi^{(0)}}$ is single valued ($\varphi^{(0)} \equiv \varphi^{(0)} + 2\pi\mathbb{Z}$, one can think of these constant $2\pi\mathbb{Z}$ shifts of $\varphi^{(0)}$ as a gauge symm.)
- $a^{(1)}$ is a compact $U(1)$ gauge field, $\oint f_{12}\, dx^1 dx^2 = 2\pi\mathbb{Z}$

gauge invariant operators

$$e^{i\varphi^{(0)}(x)} \;, \quad e^{i\oint_C a^{(1)}}$$

local, line, it is trivial to

show that

$$\left\langle e^{i\varphi^{(0)}(x)}\, e^{i\oint_C a^{(1)}} \right\rangle_{\mathbb{R}^2} = e^{i\frac{2\pi}{n_c}\, \ell_{x,C}}$$

linking number of x w/ C

(0 if x out C
1 if x in C)

(notice $e^{i n_c \varphi^{(0)}}$ & $e^{i n_c \oint_C a^{(1)}}$ have trivial correlation functions)

(31)

+ let's canonically quantize, take space $= S^1$, time $= \mathbb{R}_t$

+ pick $a_0^{(1)} = 0$ gauge, Gauss' law: $\varphi^{(0)} = \cancel{x}$ indedendent

$$\left(a_0^{(1)} \text{ E.O.M} \Rightarrow \partial_1 \varphi^{(0)} = 0 \right)$$

+ then $S = \frac{n_c}{2\pi} \int dt\, \varphi(t) \frac{\partial}{\partial t} \underbrace{\oint_{S^1} dx^1 a_1(x,t)}_{a(t) \,=\, \text{Wilson line on } S^1 \text{ of } a^{(1)}}$

Minkowski

$$= \frac{n_c}{2\pi} \int dt\, \varphi \dot{a}$$

+ canonical commut. relation

$$\left[\frac{n_c}{2\pi} \hat{\varphi}, \hat{a} \right] = -i$$

$$[\hat{\varphi}, \hat{a}] = -i \frac{2\pi}{n_c}$$

<u>algebra</u> $\Rightarrow$ $e^{i\hat{\varphi}} e^{i\hat{a}} = e^{i\frac{2\pi}{n_c}} e^{i\hat{a}} e^{i\hat{\varphi}}$

simplest for $n_c = 2$
$BC = -CB$
two pauli matrices
2d rep's irreducible
$e^{i2\varphi}$ ...
pick ?

[<u>Note</u>: 't Hooft $BC = e^{i\frac{2\pi}{n_c}} CB$ algebra
Wilson-loop/'t Hooft loop in $SU(n_c)$]

no 1-dim representation $\rightarrow$ dim n_c !

n_c dim Hilbert space

$$\begin{cases} e^{i\hat{\varphi}} |p\rangle = e^{i\frac{2\pi}{n_c} p} |p\rangle \\ e^{i\hat{a}} |p\rangle = |p + 1 (\text{mod } n_c)\rangle \end{cases}$$

"$e^{i\hat{\varphi}}$" eigenvalue : phase of $\bar{\Psi}_+ \Psi_-$ condensate, $e^{i\hat{\varphi}}$ } generator of $\mathbb{Z}_{2n_c}^{(0)}$ discrete chiral

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"$e^{i\hat{a}}$" = generator of $\mathbb{Z}_{n_c}^{(1)}$ center symmetry transform (along S^1-spatial) in charge-n_c massless Schwinger model

(this correspondence is explicitly seen in Hamiltonian solution of model)

Moral 1: according to solution of Model, IR physics (n_c ground states, finite dim Hilbert space) governed by a TQFT ("chiral Lagr.")

" T = empty IR "

Moral 2: this matches $\mathbb{Z}_{2n_c}^{(0)}$ $\mathbb{Z}_{n_c}^{(1)}$ anomaly.

to see go back to 2d Euclidean partition f-n

$$\int \mathcal{D}\varphi^{(0)} \mathcal{D}a^{(1)} e^{i \frac{n_c}{2\pi} \int_{M_2} \varphi^{(0)} da^{(1)}}$$

global symmetries? $\varphi^{(0)} \to \varphi^{(0)} + \frac{2\pi}{n_c}$: $\mathbb{Z}_{n_c}^{(0)}$ (broken part of $\in \mathbb{Z}_{2n_c}^{(0)}$)

global chiral$^{(0)}$ → (a) (clearly, since $\oint da^{(1)} = 2\pi\mathbb{Z}$)

center$^{(1)}$ → (b) $a^{(1)} \to a^{(1)} + \frac{1}{n_c}\epsilon^{(1)}$: $\mathbb{Z}_{n_c}^{(1)}$

(clearly symmetry since $d\epsilon^{(1)} = 0$)

a closed one form, w/ $\oint_{\text{closed cycles}} \epsilon^{(1)} = 2\pi\mathbb{Z}$

$$e^{i\oint_C a^{(1)}} \to e^{i\frac{1}{n_c}\oint \epsilon^{(1)}} \times e^{i\oint_C a^{(1)}} = e^{i\frac{2\pi\mathbb{Z}}{n_c}} e^{i\oint_C a^{(1)}}$$

another comment on the interpretation of $e^{i\oint_C a^{(1)}}$

- in canonical quantization $e^{ia} = e^{i\oint_{S_1} a^{(1)}}$, a $Z_{n_c}^{(1)}$-charged Wilson loop around S^1, acting on

$$|p\rangle \xrightarrow{e^{ia}} |p+1 (\mathrm{mod}\, n_c)\rangle$$

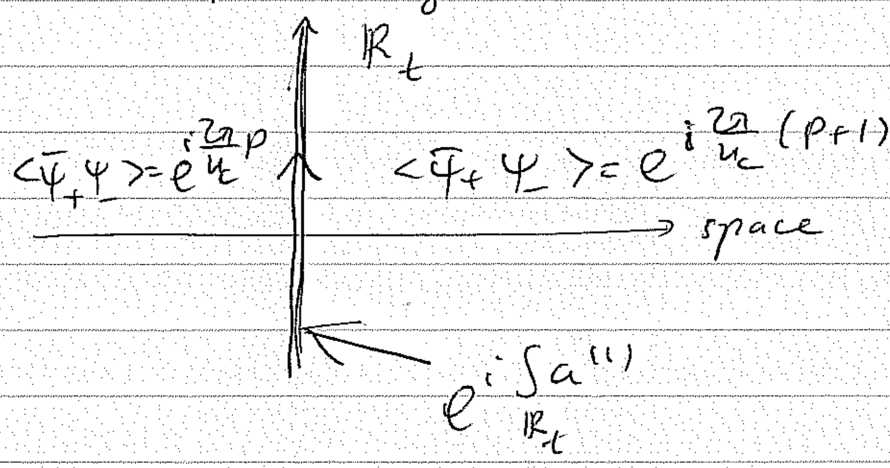
$|p\rangle$: vacuum w/ $\langle \bar\psi_+ \psi_- \rangle = e^{i\frac{2\pi p}{n_c}}$

$|p+1\rangle$: vacuum w/ $\langle \bar\psi_+ \psi_- \rangle = e^{i\frac{2\pi (p+1)}{n_c}}$

- if C is taken along time, use Euclidean picture of correlator

$$\langle e^{i\varphi(x)}\, e^{i\oint_C a^{(1)}} \rangle = e^{i\frac{2\pi}{n_c} \ell_{x,C}}$$

to argue that insertion of static $Z_{n_c}^{(1)}$ charged Wilson loop "changes vacuum"



so it acts as a domain wall (DW) separating vacua, DW has unit $\mathbb{Z}_{n_c}^{(1)}$ charge →
→ it's an external fundamental quark
(remember dynamical "quarks" have $\mathbb{Z}_{n_c}^{(1)}$ charge $\emptyset$!)

(33)

last relation identifies correct trf of $e^{i\oint_{S_1} a^{(1)}}$ under $\mathbb{Z}^{(1)}_{N_c}$ (by a $\mathbb{Z}_{N_c}$ phase)

→ (32.1) then ↓

How's anomaly seen? → need to gauge center --

-- like in our previous exercise (bottom of p. 25)

$$a^{(1)} \to a^{(1)} + \lambda^{(1)}, \quad \oint d\lambda^{(1)} = 2\pi\mathbb{Z}$$

$$B^{(2)} \to B^{(2)} + d\lambda^{(1)}, \quad \oint B^{(2)} = \frac{2\pi}{N_c}\mathbb{Z}$$

we have

$$\int \mathcal{D}\varphi^{(0)} \mathcal{D}a^{(1)}\, e^{\overbrace{i\frac{N_c}{2\pi}\int_{M_2} \varphi^{(0)}(da^{(1)} - B^{(2)})}^{S_{TQFT}}}$$

↑ partition fn w/ gauged $\mathbb{Z}^{(1)}_{N_c}$

under chiral $\mathbb{Z}^{(0)}_{N_c}$

$$\delta S_{TQFT} = i\frac{N_c}{2\pi}\frac{2\pi}{N_c}\int_{M_2}(da^{(1)} - B^{(2)}) = -i\oint B^{(2)}$$

$$= -i\frac{2\pi}{N_c}\mathbb{Z} \quad \text{in } \mathbb{Z} \text{ units of "center flux"}$$

--- match anomaly of UV, as claimed.

Cond-mat folks say "no trivial gapped vacuum" possible - - due to 't Hooft anomaly saturation requirement
(trivial ≡ gapped, unbroken symmetry)

→ unless discrete chiral broken explicitly, of course ←

"Bulk" CS term & anomaly matching

As we saw, in the $\mathbb{Z}^{(0)}_{2n_c} \to \mathbb{Z}^{(0)}_2$ broken phase, the $\mathbb{Z}^{(0)}_{2n_c}\, \mathbb{Z}^{(1)}_{n_c}$ mixed 't Hooft anomaly is captured by the IR TQFT (think of solution of model as evidence for anomaly matching!).

As for continuous 't Hooft anomalies, the general argument for anomaly matching involves adding a "bulk" CS coupling, depending only on backgd fields gauging the global symmetries. This makes (M_2 + bulk) partition function $\mathbb{Z}^{(0)}_{2n_c}$ & $\mathbb{Z}^{(1)}_{n_c}$ gauge invariant; as bulk is same at all scales (CS is topological), variation of M_2 partition f-n must be same (@ all scales) - i.e. 't Hooft anomaly should be matched.

What's this "bulk CS"?
What ways are there for the anomaly to match?

on p. 25, motivated by lattice, we already introduced a 2-form continuum gauge field for $\mathbb{Z}^{(1)}_{n_c}$, $B^{(2)}$ - 2 form

s.t.
$$\underset{\substack{\text{closed}\\ \text{2 cycle}}}{\oint} B^{(2)}_{12}\, dx^1 dx^2 \equiv \oint B^{(2)} = \frac{2\pi}{n_c}\mathbb{Z}$$

$$\left(B^{(2)} = \tfrac{1}{2} B_{\mu\nu}\, dx^\mu \wedge dx^\nu \right)$$

Gauging $\mathbb{Z}^{(0)}_{2n_c}$ proceeds along similar lines. A $\mathbb{Z}^{(0)}_{2n_c}$ symmetry requires a 1-form $\mathbb{Z}_{2n_c}$ gauge field. Call it $A^{(1)}$. What characterizes a $\mathbb{Z}^{(1)}_{2n_c}$-gauge field is that its $\oint$ over closed 1-cycles is $\dfrac{2\pi\,\mathbb{Z}}{2n_c}$; so we have

$$\underset{\text{1-cycle}}{\oint} A^{(1)} = \oint A_\mu\, dx^\mu = \frac{2\pi\,\mathbb{Z}}{2n_c}$$

(so that $e^{i\oint A^{(1)}} = e^{i\frac{2\pi}{2n_c}\mathbb{Z}}$, just like

$$e^{i\oint B^{(2)}} = e^{i\frac{2\pi}{n_c}\mathbb{Z}}\Big)$$

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under $\mathbb{Z}^{(0)}_{2n_c}$, we have $\delta A^{(1)} = d\omega^{(0)}$

w/ $\oint d\omega^{(0)} = 2\pi\mathbb{Z}$

compact scalar
(w/ quantized monodromies)

---

to motivate, (1) $e^{i\oint A^{(1)}}$ must be invariant,

& it is, when $\oint d\omega^{(0)} = 2\pi\mathbb{Z}$

(2) in continuum, one can think of a $\mathbb{Z}^{(0)}_{2n_c}$ topological theory arising as follows:
consider a $U(1)$ gauge theory w/ Higgs w/ charge $2n_c$
the Higgs vev breaks $U(1) \to \mathbb{Z}^{(0)}_{2n_c} \subset U(1)$

$$\phi \xrightarrow[U(1)]{} e^{i\alpha 2n_c} \phi$$

charge n_c

a nonzero vev is still invariant under above, w/ $\alpha = \frac{2\pi\mathbb{Z}}{2n_c}$, precisely $\mathbb{Z}^{(0)}_{2n_c}$!

side note

((Everything is gapped, of course, but the IR theory is a $\mathbb{Z}_{2n_c}$ topological gauge theory; it describes the linking of ANO vortices & charged particles (of charge $1 \dots 2n_c - 1$, or nontrivial $\mathbb{Z}^{(0)}_{2n_c}$ charge)

that theory can be constructed from the $U(1)$ gauge field & the phase of ϕ, $\phi = v e^{i\varphi}$; they couple as $(\partial_\mu \varphi - n_c A_\mu)$; invariant under $\varphi \to \varphi + n_c \omega$

$$A_\mu \to A_\mu + \partial_\mu \omega$$, where ω is compact, as in (1) above.)

see Kapustin, Seiberg 2014

further, as already mentioned (p. 25!)

$$\mathbb{Z}^{(1)}_{n_c} : \quad \delta B^{(2)} = d\lambda^{(1)}, \text{ w/ } \oint d\lambda^{(1)} = 2\pi \mathbb{Z},$$

(compact $U(1)$)

&

$$\mathbb{Z}^{(0)}_{2n_c} : \quad \delta A^{(1)} = d\omega^{(0)}$$;

both $A^{(1)}$ & $B^{(2)}$ are topological: $dA^{(1)} = 0$, $dB^{(2)} = 0$

(enforced via Lagrange not shown)

Consider now (e^S in path $\int$)

$$S_{3D} = i \frac{2\pi}{n_c} \int_{M_3} \frac{2n_c A^{(1)}}{2\pi} \wedge \frac{n_c B^{(2)}}{2\pi}$$

if M_3 has ~~compact,~~ no boundary,

S_{3D} is invariant under both, just like S_{TD} of p. 6

e.g. $Z^{(0)}_{2n_c}$: $\delta S_{3D} = i \frac{2\pi}{n_c} \frac{2n_c}{2\pi} \int_{M_3} d\omega^{(0)} \wedge \frac{n_c B^{(2)}}{2\pi} =$

$$= i\, 2 \int_{M_3} d\left(\omega^{(0)} \frac{n_c B^{(2)}}{2\pi}\right) = 0$$

but if $\partial M_3 = M_2 \neq \emptyset$, we have

$$Z^{(0)}_{2n_c} : \delta S_{3d} = i\, 2\, \omega^{(0)} \underbrace{\int_{M_2} \frac{n_c B^{(2)}}{2\pi}}_{} = i\, 2 \omega^{(0)}\Big|_{M_2} \underbrace{\mathbb{Z}}_{}$$

as noted already $\omega^{(0)}\Big|_{M_2} = \frac{2\pi}{2n_c}$

so $\delta S_{3D} = i \frac{2\pi}{n_c} \mathbb{Z}$

\# of 't Hooft fluxes

this phase of the 3D bulk partition f-n IS the 't Hooft anomaly.

So how can anomaly be matched in the IR ?
(of the M_2 theory)

- we already saw: by symmetry breaking

(I) • if IR theory has no mass gap, the anomaly is matched by a IR TQFT "chiral lagrangian" describing the symmetry broken vacua & the DWs between them

many examples like this exist!

already mentioned 4d SYM, $\mathbb{Z}^{(0)}_{2n_c}$ R-symmetry, $\mathbb{Z}^{(1)}_{n_c}$ -center

anomaly is $\mathbb{Z}^{(0)}_{2n_c} \left(\mathbb{Z}^{(1)}_{n_c}\right)^2$

matched by R-symmetry breaking

requires 't Hooft fluxes at least in $x^1 x^2$ & $x^3 x^4$ planes

(II) • IR theory can be conformal -- e.g. if Banks-Zaks $\Rightarrow$

$\Rightarrow SU(n_c)$ w/, say, 5 massless adjoints

$\Rightarrow$ essentially original fields persist in IR

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14 • anomaly can be matched by
an IR TQFT not associated with
symmetry breaking --- examples in literature

-- Cordova Dumitrescu '18
Bi, Senthil '18

--- list is likely exhaustive, especially given
generality of 3d option above,
but I am not aware of "theorems"

at level of Frishman et al
or Coleman-Grossman

for continuous 't Hooft ---

Final words (for now):

⇒ I gave an example of mixed 0-form / 1-form
anomaly in a theory w/ massless
fermions, using "leftovers" from anomalous
continuous symmetries → but interestingly,
this kind of anomalies occur in purely
bosonic theories, but w/ topological "Θ"
terms

⇒ they may have consequences for theories w/out
1-form center; 1-form may "emerge" after
gauging some 0-form symmetries
(eg QCD, gauge $SU(N_f)_V$ --- Tanizaki '18)

⇒ inflow on DW

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Overview of 4d results, "anomaly inflow" & DW matching

- theories w/out fermions; ex. in 4d

-- YM, pure, at $\theta = \pi$

$$e^{i\theta \int \frac{F\tilde{F}}{32\pi^2}} = e^{i\theta Q_{top}}$$

for $Q_{top} \in \mathbb{Z}$ & $\theta = \pi$

CP: $e^{i\pi Q_{top}} \longrightarrow e^{-i\pi Q_{top}} = e^{i\pi Q_{top}}$

$$\left(\begin{array}{c} \text{for } Q_{top} = n \\ e^{i\pi n} = e^{-i\pi n} \end{array} \right)$$

but as one gauges $\mathbb{Z}_{n_c}^{(1)}$ center symmetry

$$\underbrace{Q_{top} = \frac{\mathbb{Z}}{n_c} \quad \& \quad e^{i\pi\frac{n}{n_c}} \neq e^{-i\pi\frac{n}{n_c}}}_{\text{a } \mathbb{Z}_2^{CP} (\mathbb{Z}_{n_c}^{(1)})^2 \text{ - anomaly.}}$$

in pure YM: CP is broken

(various subtleties involving even-vs odd n_c
see Gaiotto, Kapustin, Komargodski, Seiberg 1703.00501
G, Kanazawa, S. 1708.06806
Tanizaki 1708.01962 ...)

- putting the theory on circles does not alter these anomalies (they thrive on finite volume)

notice however, for $\mathbb{Z}_{N_c}^{(1)}$ always need $\underbrace{\text{2-cycles}}_{\text{nontrivial}}$ to have fractional $Q_{top.}$ (on S^4 — no! or $S^3 \times S^1$ — no!)

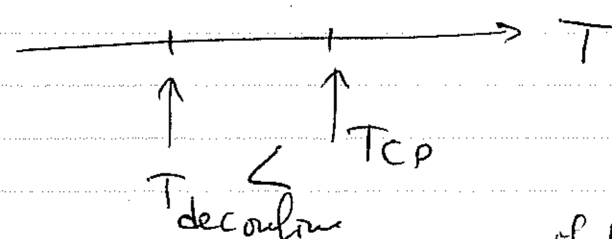
one can use to constrain phase structure as a function of T...

-- in above bosonic YM $\theta = \pi$ example:

$$\begin{cases} \cancel{CP} - \text{low } T - \mathbb{Z}_{N_c}^{(1)} \text{ restored} \\ \cancel{\mathbb{Z}_{N_c}^{(1)}} - \text{high } T - CP \text{ restored } (\theta \text{ irrelevant } T \gg \Lambda) \end{cases}$$

one or two transitions?

-- Gaiotto et al



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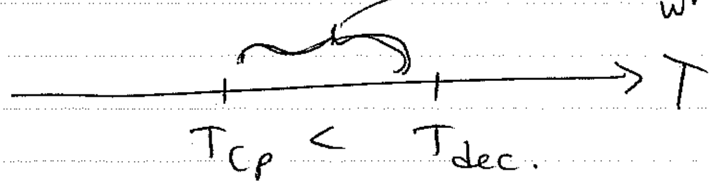


if no massless d.o.f. who will match anomalies

both CP & center unbroken

--- similar for SYM/QCD (adj) Shimizu Yonekura 1706.06104

Unsal, Komargodski, Sulejmanpasic 1706.05731

• "anomaly inflow"... rich DW physics in $\mathbb{Z}^{(0)}_{\#}$ broken phases

low-T $\mathbb{Z}_{\text{chiral-discrete}}$ broken phase

(SYM, QCD(adj), YM $\theta=\pi$)

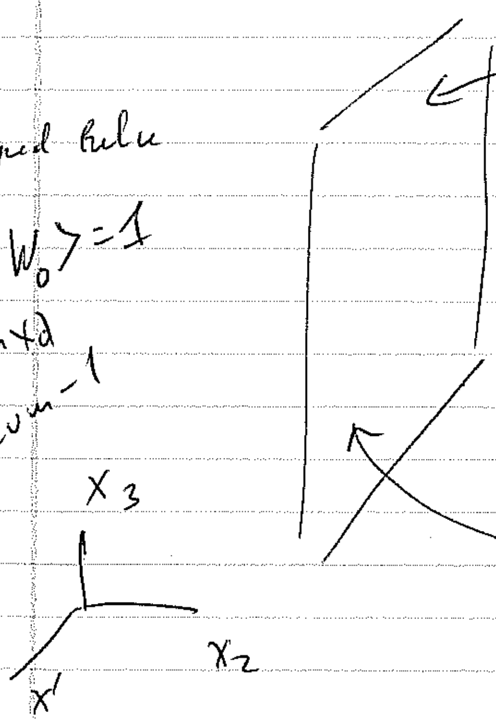
high-T $\mathbb{Z}_{\text{center}}$ - broken along x^4

(SYM, QCD(adj) YM $\theta=\pi$)

very explicit ex: SU(2) SYM, $T \gg T_c$ deconfined phase

Anber, EP 1807.00093

DW (high T center vortex)



gapped bulk

$\langle W_0 \rangle = 1$

$\mathbb{Z}^{(1)}_{2c}$ (in x^0) broken vacuum 1

$\langle W_0 \rangle = -1$

$\mathbb{Z}_2^{(1)}$ broken (in x^0) vacuum 2

$SU(2) \xrightarrow{T} U(1)$ on wall

gapped bulk

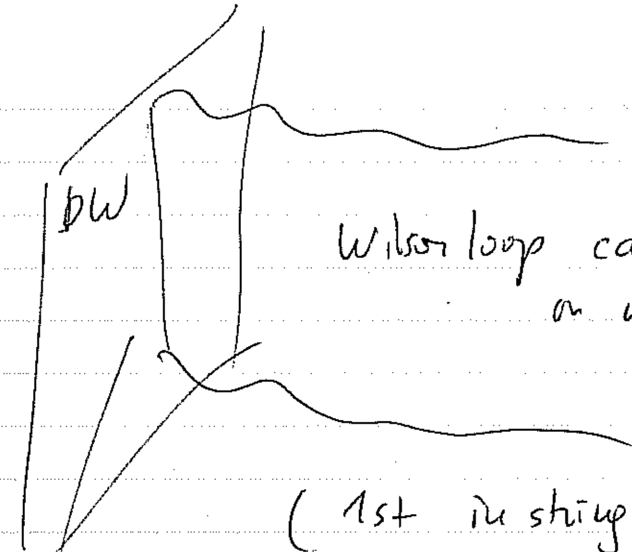
on DW: U(1) Schwinger model w/ charge -2 fermion (adjoint 0 modes on wall!)

$\mathbb{R}^3$, thermal direction x^0 small, not shown

• $\mathbb{Z}_2^{\text{chiral}}$ broken on wall (high T)

$\mathbb{Z}_2^{\text{center in } 1,2,3}$ on DW ⟶ • quarks deconfined on wall

•



Wilson loop can end on wall

(1st in string theory, holography Aharony, Witten '99 $\mathcal{N}=4$)

$\begin{cases} \text{D1 (DW)} \\ \text{F1 (Wilson loop)} \end{cases}$

here QFT -- screening in Schwinger model.

DW in DW

end of W-loop.

that we discussed.

(this nontrivial DW worldvolume required by anomaly inflow from bulk ---)

see papers Unsal, Komargodski, Sulejmanpasic 1706, --- Anber, EP to appear

• very similar to low T DW associated w/ χ SB

similarity due to inflow

• also DW (2d) similar to bulk low T high T (4d)

---.

- new phases ??

Arber EP 1805.12290

Cordova Dumitrescu 1806.10894

Bi Senthil 1808.07495

eg. $SU(n_c)$ w/ $\underbrace{n_f}$ adjoints

- $n_f = 2$ especially interesting
- some unclear lattice results
- various anomalies matched by

massless "baryon" $SU(2)_{flavor}$ doublet

+

TQFT ← somewhat fascinating story; as theory can be 'twisted' & put on non-spin manifolds, new discrete anomalies (changed coeffts ...).

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