Supersymmetry, multi-instantons, and the necessity of Lefshetz thimbles

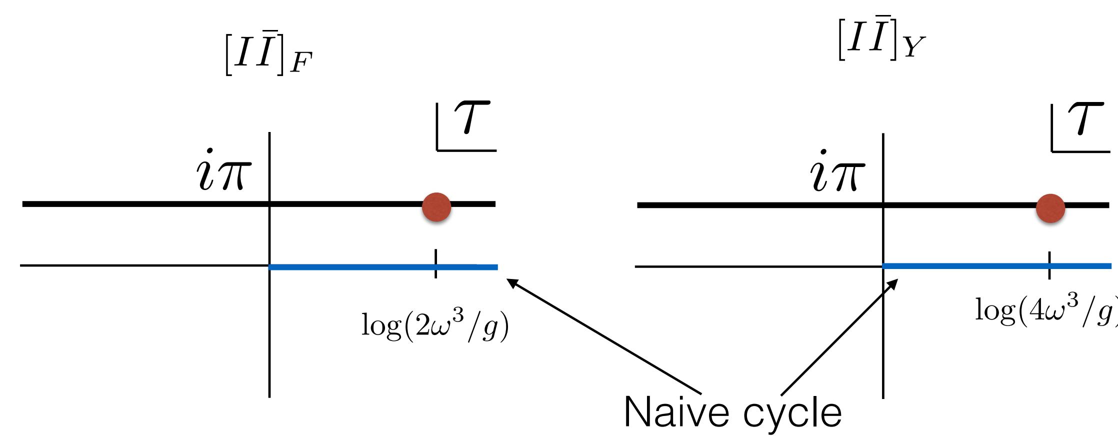
Erich Poppitz (Toronto)

with Alireza Behtash, Tin Sulejmanpasic, Mithat Unsal (NC State)

1507.04063 and earlier related works

“Thimble and buttons”

in “Fashion District,” Toronto



Instanton—anti-instanton thimble

in N=2 SUSY QM

This is about using SUSY as a tool to study QFT

Instantons play a role in many physical problems.

Key to understanding important physics, e.g.:

N=1 SUSY theories: nonperturbative superpotentials.

N=2 SUSY theories: Seiberg-Witten curves.

Phenomenological "instanton liquid" models
of chiral symmetry breaking in QCD. . . .

more recent and closer to my point: **Unsal w/ Shifman,Yaffe, EP, Argyres… 2007+**

mass gap, confinement & center stability in a controlled manner!

QCD(adj)/SYM & deformed Yang-Mills theory on $R^{1,2} x S^1$, small L

despite weak coupling, a major difficulty:
"How to define & calculate instanton—anti-instanton contributions?"

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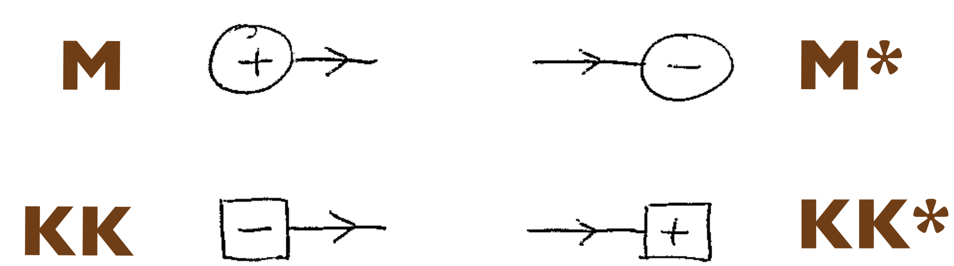
Ex. 1: SYM, mass gap (confinement) and center stability due to such configurations: vacuum is a dilute gas of “magnetic bions” and “neutral bions.” both are different types of I-I* “molecules”



for SU(2)

BPST instanton ‘falls apart’ into constituents with magnetic charge under U(1) part of SU(2)
[string theorists/lattice people…late 1990’s]

(from talk at SUSY2013 on work with Schafer/Unsal)

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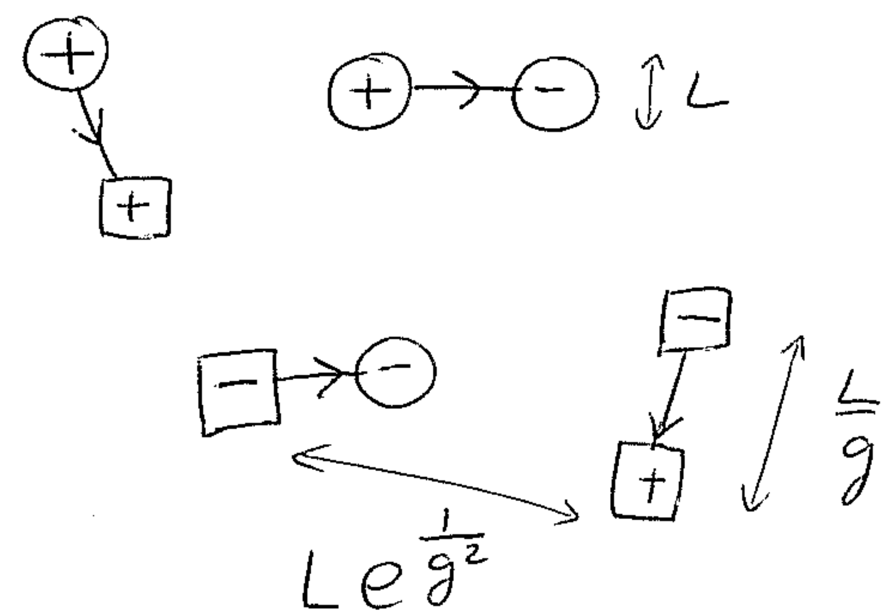

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separation of scales at small g(L)

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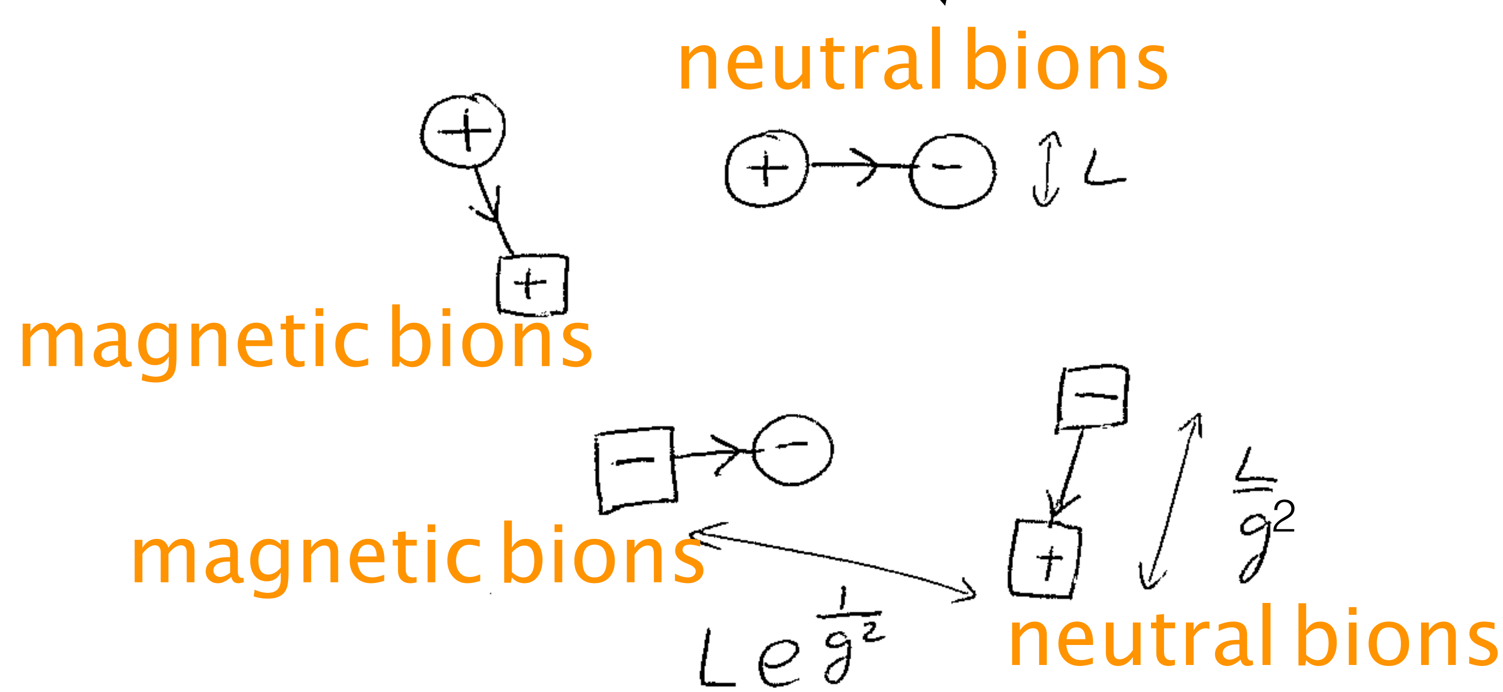
center stability
(confinement/deconfinement transition)



confinement

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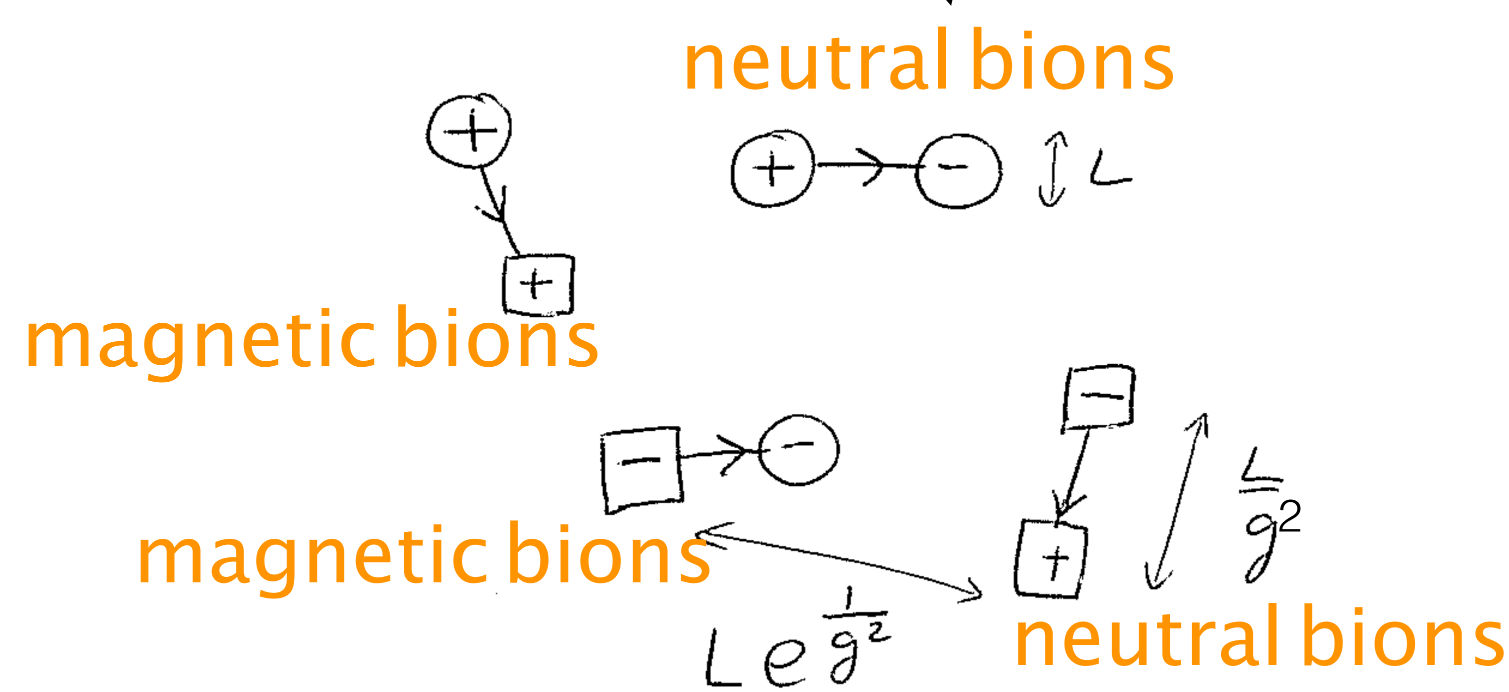
center stability
-confinement/deconfinement transition
-cancel E_vac, gluon condensate!

confinement



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so, neutral bions seem important… but hard to understand!

I-I* ‘bound states’

all interactions attractive!

unlike positronium: no time

“instant”-o=localized in time!

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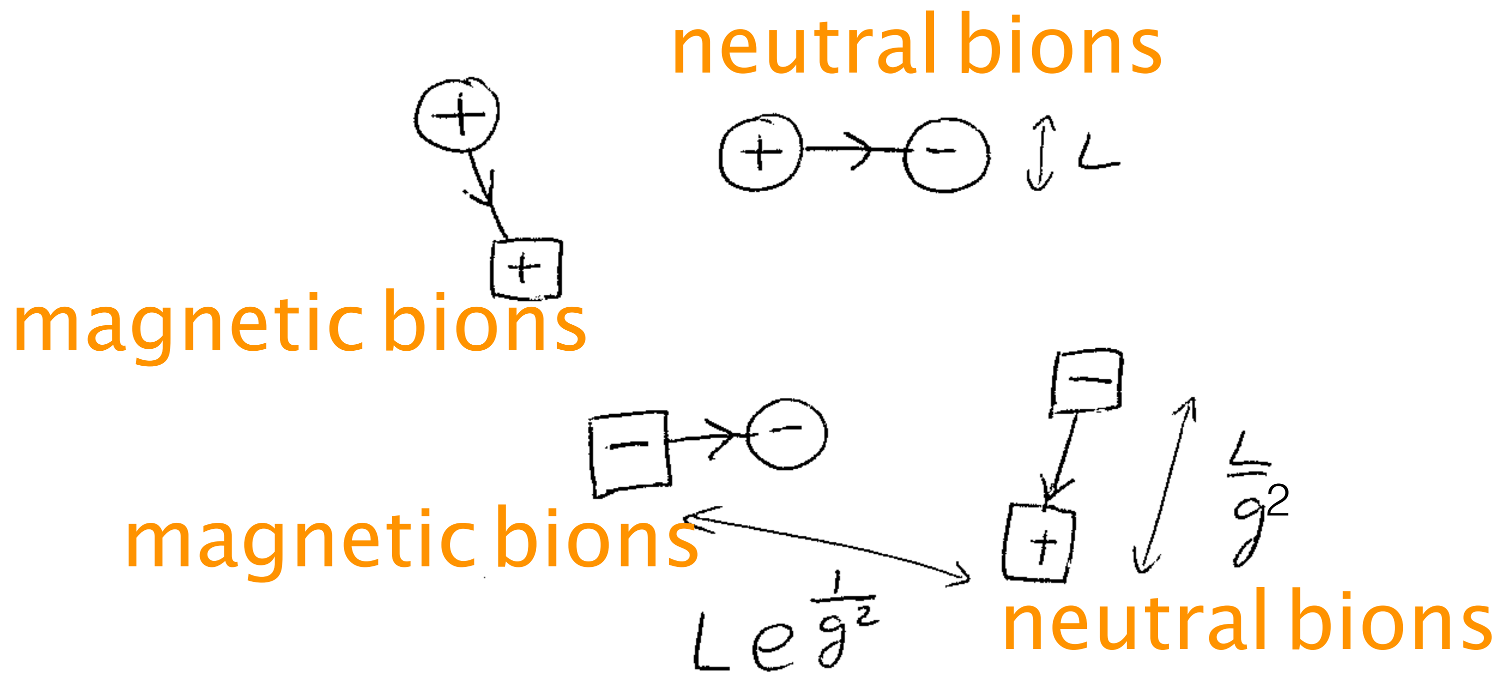

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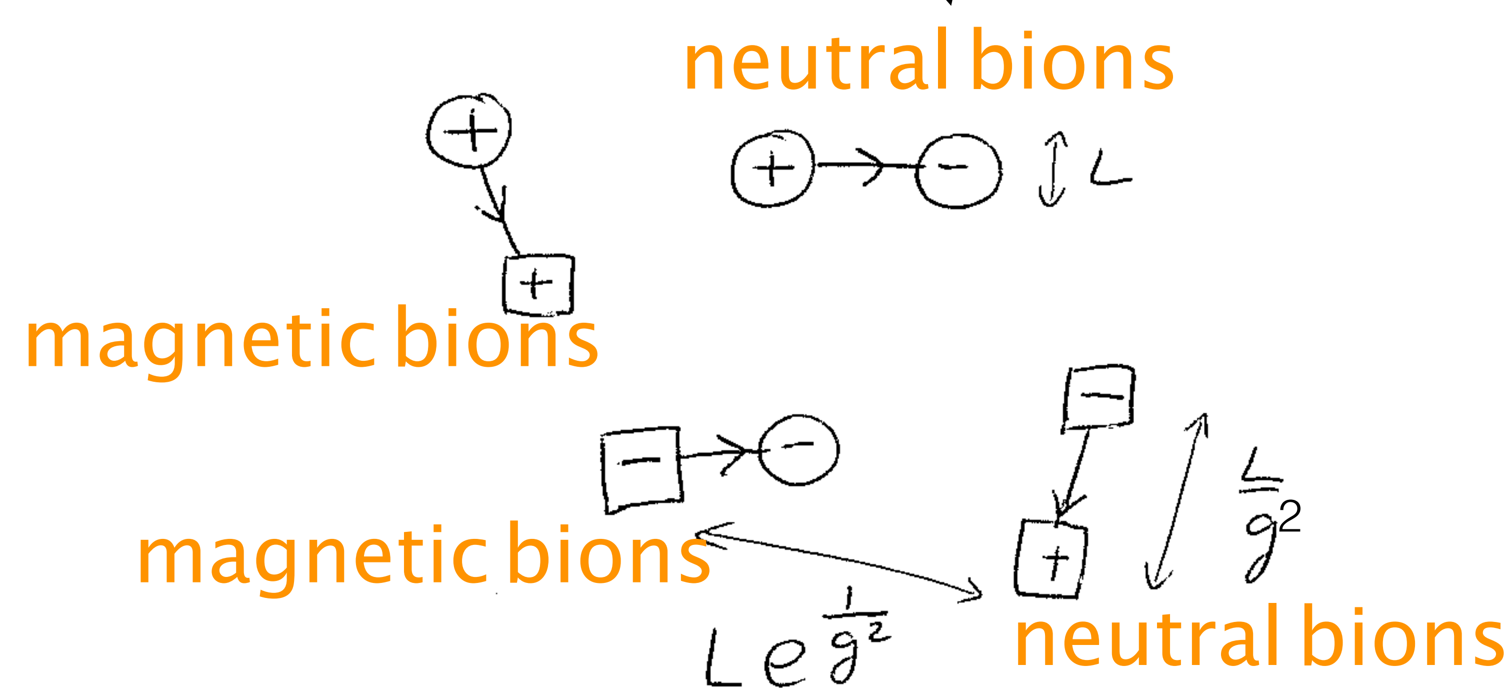
2. analytic continuation:
MM* “live” at complex separation?!



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Ex. 2: “Resurgent” cancellations: imaginary parts due to Borel resummation of perturbation theory vs imaginary parts of I-I*

high orders of perturbation theory double-well QM, non Borel-summable:

$$E^0_{pert} = -\frac{3}{\pi}\sum_k (3g)^k\, k!$$

ambiguity of Borel sum of pert. series:

$$\Delta E^0[\text{Borel sum}] = -\frac{3}{\pi}\left(\pm\frac{i\pi}{3g}\, e^{-\frac{1}{3g}}\right)$$

II* contribution:
requires analytic continuation
Bogomolnyi, Zinn-Justin

$$E^0_{I\bar{I}} = \left(\pm i\pi + \log\frac{g}{2}\right)\frac{e^{-\frac{1}{3g}}}{g\pi}$$

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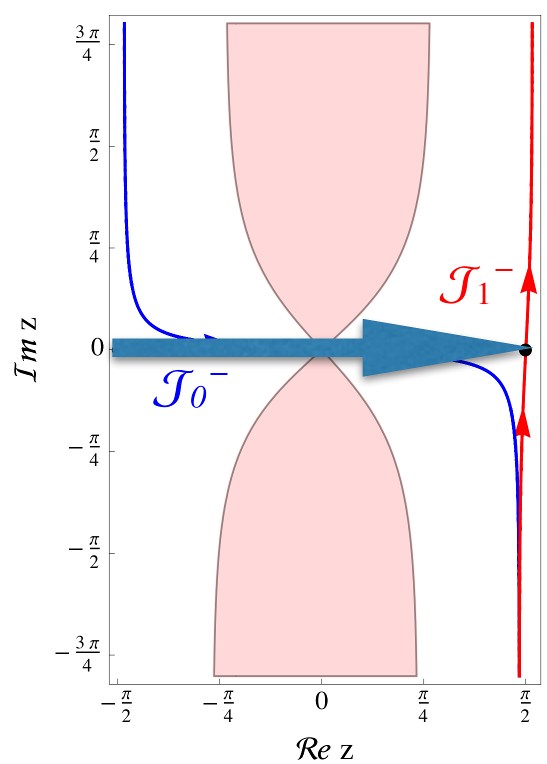
Not merely a question of calculating exponentially suppressed effects.

Instanton—anti-instanton (I-I*) contributions have been found to give the leading effect in many cases: Ex. 1; Ex. 2 above

Complexification seems crucial. **Hypothesis/dream/ is that MM* lie on a different “Lefshetz thimble” from the perturbative vacuum and are distinguished from it by a phase associated with the thimble.**

…**“like” in 1dim integrals, e.g.:**

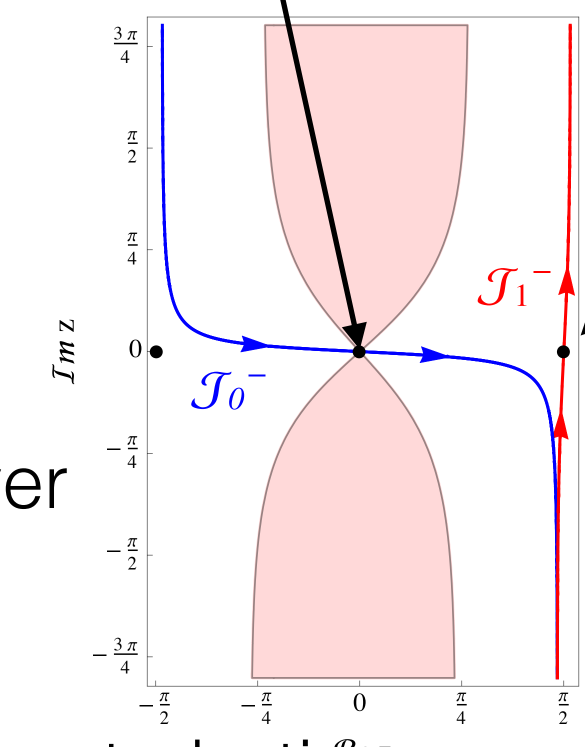
$$Z(\lambda) = \frac{1}{\sqrt{\lambda}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dx\, e^{-\frac{1}{2\lambda}\sin^2(x)}$$



original contour = sum of integrals over thimbles thru two two saddles: perturbative & nonperturbative

trivial saddle (perturbative)

nonperturbative saddle



(I think) we are far from understanding of what "DEFINING THE PATH INTEGRAL ON LEFSHETZ THIMBLES" **means.**

All I will do is to show you a simple, yet not completely trivial, example supporting the need of complexification.

N=2 SUSY QM = 4d WZ model reduced to 2d

$$g\mathcal{L}_E = |\dot{z}(t)|^2 + |W'(z)|^2 + \begin{pmatrix} \bar{\chi}_1 & \chi_2 \end{pmatrix} \left(-\partial_t + \begin{pmatrix} 0 & \overline{W''(z)} \\ W''(z) & 0 \end{pmatrix} \right) \begin{pmatrix} \chi_1 \\ \bar{\chi}_2 \end{pmatrix}$$

$$W(z) = \prod_{i=1}^{k+1} (z - z_i) \qquad\qquad |I_W| = k$$

Witten index=number of critical points of W(z)

E_vac=0, as opposed to N=1 SUSY QM: well known.

Goal: **Understand E_vac = 0 from next-order semiclassics.**

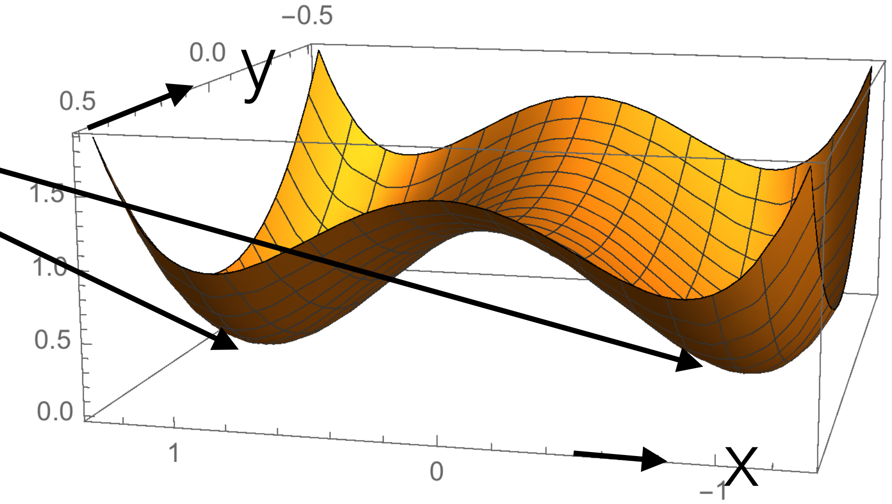
Upshot: **It's not completely trivial. {Relation to motivation: complexification!}**

Repeat again: I want to understand E_vac = 0 'simply', without deformation invariance and localization (i.e. traditional Witten index technology!).

$$W(z) = \frac{1}{3}z^3 - za^2$$

take “a” real (plot for a=1)

potential w/ two minima



BPS (anti)instantons

$$\dot{z} = \pm \overline{W'}$$

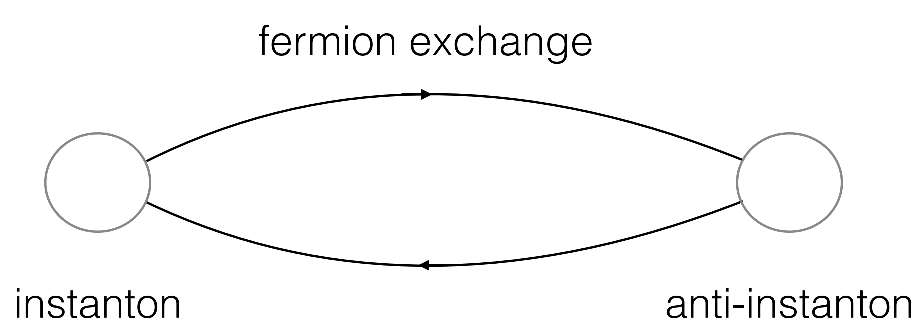
$I \rightleftarrows$

$I^* \leftrightarrows$

I,I*: tunnelling between minima; two fermion zero modes each
(with opposite “chirality” from 4d p.o.v.)

To rephrase question:

after all, the far away I* will lift the zero modes of I (and v.v.), e.g.:



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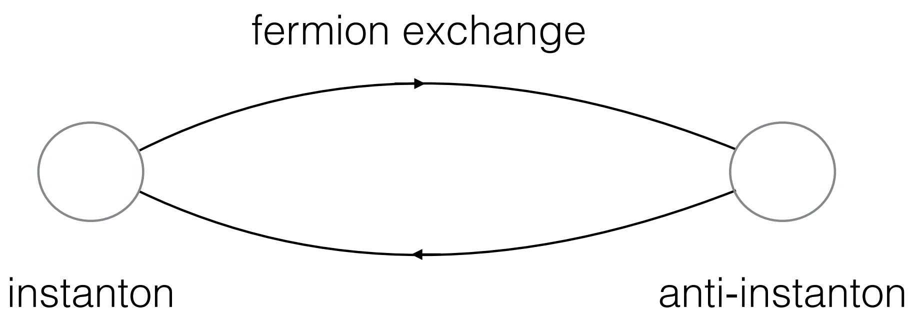
answer: “quasi-zero mode thimble”



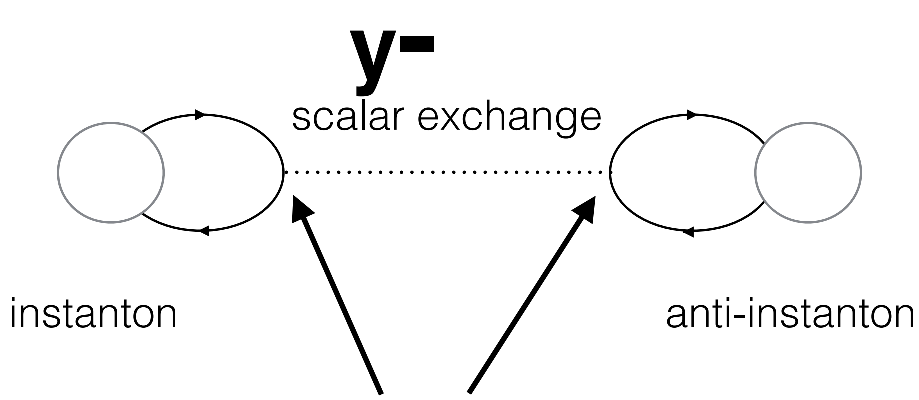
$$I_1 = \int_{\mathcal{J}_1} d\tau \; e^{\frac{4\omega^3}{g} e^{-\omega\tau} - 2\omega\tau}$$

entire story rests on relative factor - somewhat hard calculation

$$E_0 \propto -e^{-2S_0} \left(4\omega^3 I_1 + g I_2 \right)$$

crucial points: **-two contributions, different orders in g!**

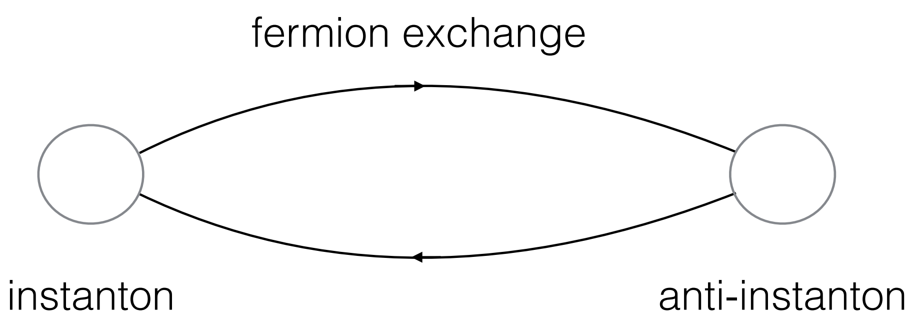
-both come with same (wrong!) sign: how to cancel?

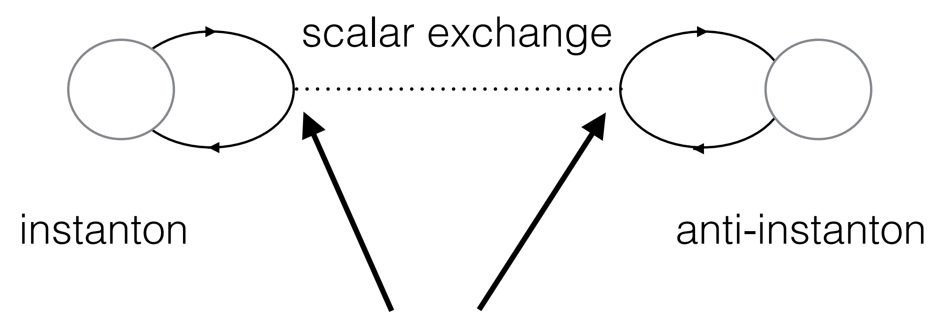


Yukawa squared = g

$$I_2 = \int_{\mathcal{J}_2} d\tau \; e^{\frac{4\omega^3}{g} e^{-\omega\tau} - \omega\tau}$$

so, why does the I-I* contribution to E_vac vanish?
answer: “quasi-zero mode thimble”





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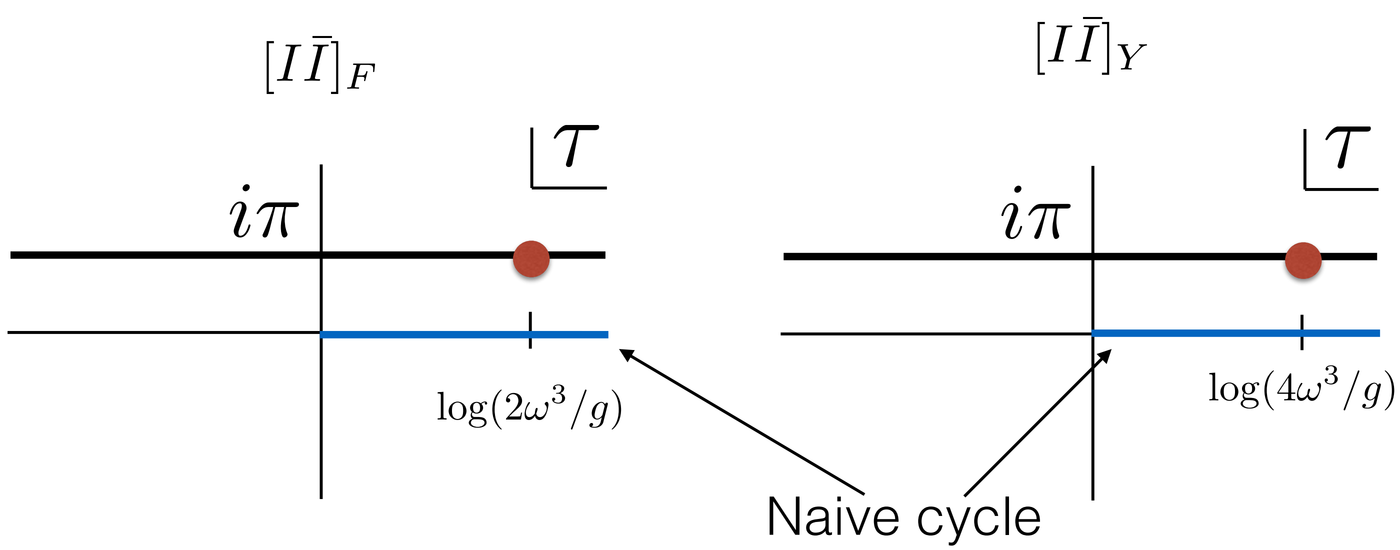

-two contributions, different orders in g
- both come with same (wrong!) sign:
how to cancel?

“quasi-zero mode thimble”
integration gives E_vac = 0

1 **Imaginary part,**
change of relative sign - one vs. two “massive propagators”; g-order!

2 **Absolute value of separation is large at small g - self consistent!**
I and I* are never on top of each other: complex separation

3 Integrating over the thimbles gives E_vac = 0!

Goal: **Understand E_vac = 0 from plain next-order semiclassics**
… no localization, no deformation invariance…

Upshot: **It's not completely trivial.** **{Relation to motivation: complexification!}**

Found that complexifying the quasi-zeromode crucial. I and I* "live" a complex & large separation apart; consistent next-to-leading order semiclassics.

Comments/future:

"Quasi-zeromode" is just one direction in field space (the most relevant for this case!). **Suggests that complexification of path integral important.**

Magnetic and neutral bions in SYM can be seen to emerge in a similar way, at (generally) complex separations. (Recall SYM is only SUSY w/out scalars … YM)

Solving analogous puzzles in SW theory harder… **but worthwhile, beyond QM?**

status: "theoretical experiment" in search of a theory… → **finite dimensional thimbles (lattice)?** **mathematics?**

(subjects of research in various communities)