PHY1530F - "Graduate fluid mechanics" (FM)

What is it?

FORMALLY (mathematically) $\Rightarrow$ FM is a classical field theory

field theory $\forall \vec{x}, \forall t \longrightarrow$ field(s) $\phi(\vec{x}, t)$

exs: E&M: $\vec{E}(\vec{x},t)$, $\vec{B}(\vec{x},t)$ (vector fields)
(or $\varphi(\vec{x},t)$, $\vec{A}(\vec{x},t)$)

GR (general relativity): $g_{\mu\nu}(\vec{x},t)$ (spacetime metric, tensor field)

FM:
$\rho(\vec{x},t)$ density
$\vec{u}(\vec{x},t)$ velocity
$p(\vec{x},t)$ pressure
$\sigma_{ij}(\vec{x},t)$ stress
- - -

scalar (ρ, p), vector (u_i), tensor (σ_{ij}) fields
$i,j = 1,2,3$ ($= x,y,z$)

"M" is as in mechanics: know $\rho, \vec{u}, p, \ldots$ @ $t=0 \Rightarrow$
$\Rightarrow$ what will happen @ $t>0$?

- need equations of motion (like Newton, Maxwell, Einstein

+ Maxwell - linear (mostly)
+ Einstein - nonlinear } means hard to solve (computers!
+ FM - nonlinear } but still must study

PHYSICALLY : (less formal) :

* a FLUID is a system of many particles and of course a rigid (or not so rigid) body is, too
* to distinguish a FLUID from a body (rigid), define a FLUID by saying that it is a system (conglomerate) of many particles which does not resist a change of its shape (i.e. it doesn't have a definite shape)

** NOT A SHARP FLUID-SOLID DISTINCTION !

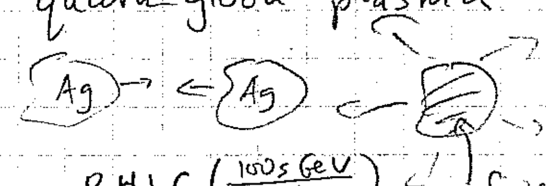
*** the force resisting deformation of a fluid vanishes with the rate of deformation, i.e. a deformation is not prevented

BEST TO LOOK AT EXAMPLES

(≡ WHO NEEDS FM ?)

+ air – atmosphere
+ water – ocean, river, --- , cells ---
+ inner mantle of the Earth – geology, geophysics
+ interstellar medium – astrophysics
+ the Sun, stars
+ plasmas, MHD, $\Longrightarrow$ incl. quark-gluon plasma

say



RHIC $\left(\frac{100\text{s GeV}}{\text{nucleon}}\right)$

relativistic heavy ion collider

electron, proton, neutron, photon plasma when $T_{\text{Universe}} \gtrsim eV$

($\gtrsim 10^5$ K — now 2.7 K)

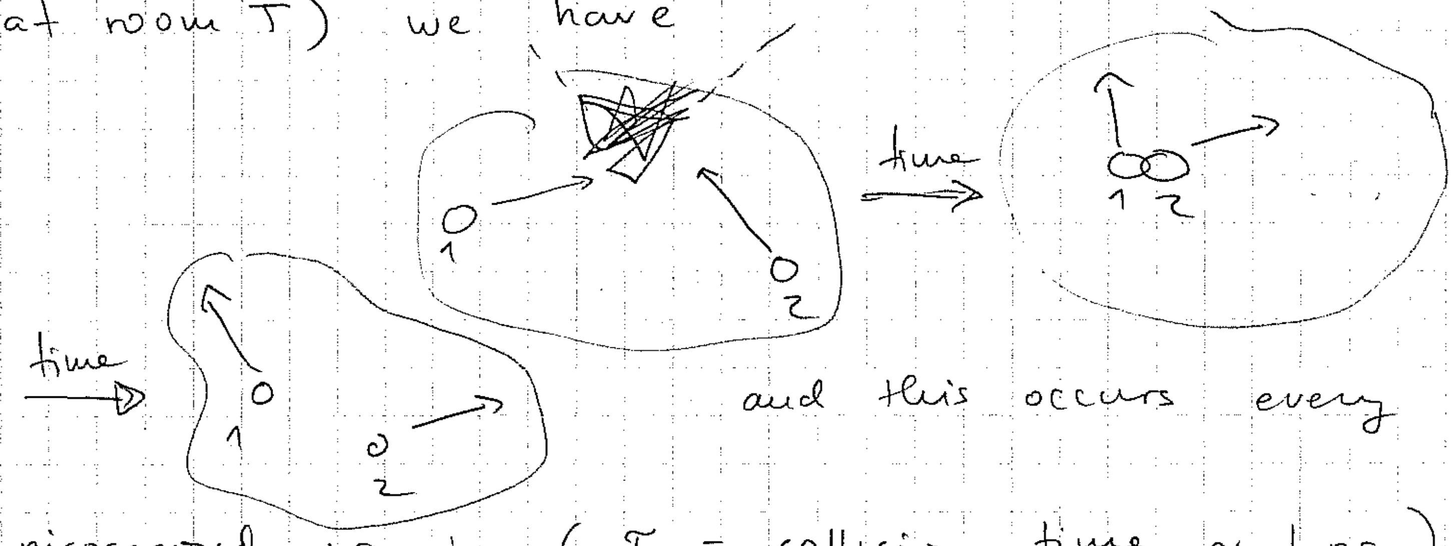
(— cosmic microwave bkgd —)

- liquid crystals
- liquid He, superfluid or not
- Bose-Einstein condensates

<u>Note</u>: #s of particles varies ⇒ from ~ 10^3 (RHIC)
to ~ 10^5 (BEC)
to ~ 10^{23} (most of the rest)

<u>However, in each case we approximate fluid as continuum</u>,

i.e. by considering the density or velocity of the fluid at given $\vec{x}, t$; however, microscopically, things are rather complicated — e.g. in a gas (air at room T) we have



and this occurs every picosecond or so (τ = collision time ~ 1 ps) a typical particle flies a distance $\lambda \sim \bar{v}\tau$

(λ = mean free path, ~ nm in air)

$$\left(\bar{v} \sim \frac{10^{-9}\,m}{10^{-12}\,s} \sim 10^3 \frac{m}{s} \text{ as we know from } \bar{v} \sim \sqrt{\frac{k_B T}{m}} \right)$$

So when we say that the

$$\begin{pmatrix} \text{density} \\ \text{or} \\ \text{velocity} \end{pmatrix} \text{ of the fluid @ } (\vec{x}, t) \equiv \begin{pmatrix} \rho(\vec{x}, t) \\ \vec{u}(\vec{x}, t) \end{pmatrix}$$

we really mean by " $\vec{x}$ " a "physical point"

$\Longrightarrow$ $\vec{x}$ denotes the location of a "physical volume element" containing many particles whose average density is what we call $\rho(\vec{x}, t)$ & average velocity is — " — $\vec{u}(\vec{x}, t)$

thus, the length scales we're allowed to consider are such that $\Delta L \gg \lambda$ and the time scales we can follow the state of the fluid are such that

$$\Delta t \gg \tau$$

(If we have a measuring aparatus able to "see" distances $\lesssim \lambda$, there'll be no meaningful ρ or $\vec{u}$ observed.)

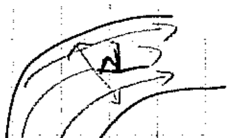
As we'll see (and we already know from other branches of physics), dimensionless #s play great role in FM. So here we go, we just saw our first one:

$$\text{Knudsen \#} : \frac{\lambda}{\Delta L} \ll 1$$

ΔL: typical scale of the fluid motion

$\ll 1$: if FM is to be useful

e.g.



if $\gtrsim 1$ – need to study <u>kinetic theory</u> (dynamics of collisions, how thermodynamic equilibrium is approached ...)

good for gases, hard to impossible for liquids

N.B.: very different time scales are described by TD, KT & FM:

TD $\Leftrightarrow$ $\Delta t = \infty$ (equilibrium, nothing ever changes)

KT $\Leftrightarrow$ $\Delta t \sim \tau$ (rate of approach to equil.)

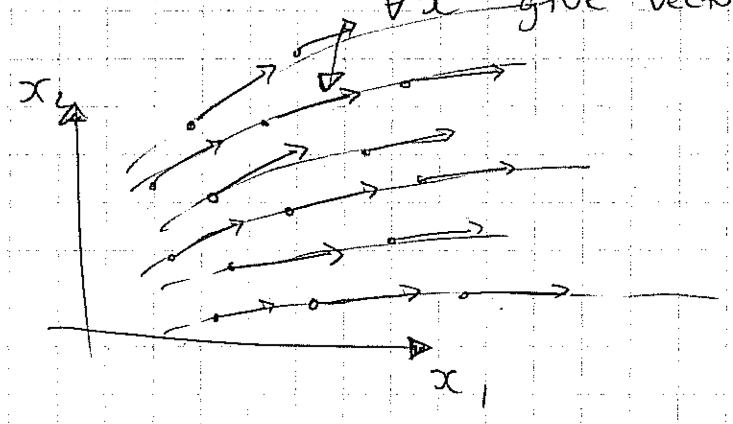
FM $\Leftrightarrow$ $\Delta t \gg \tau$ (macroscopic flows)

The above discussion shows that there's a close relation between thermodynamics, kinetic theory, and fluid mechanics. In fact, input from TD & kinetic theory will be needed ($\rightarrow$ later). For now, let's do some <u>simple dynamics</u>.

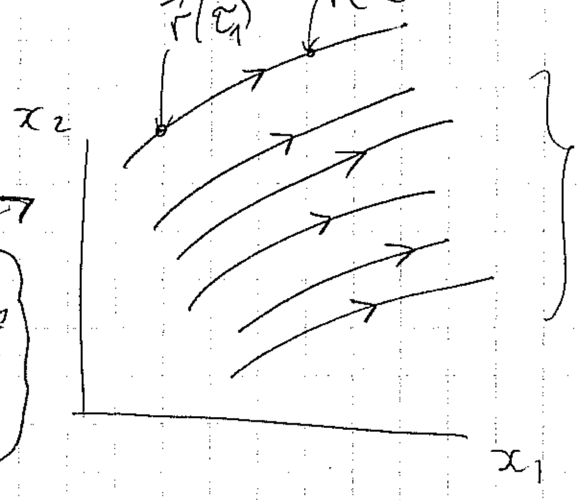
fluid dynamics $\equiv$ " $\vec{f} = m\vec{a}$ " – need a few concepts

velocity field in space (time) $\equiv$ velocity of fluid @ $\vec{x}, t$.

$\forall t$: $\forall \vec{x}$ give vector $\underbrace{\vec{u}(\vec{x}, t)}_{\text{"vector field"}}$



"streamlines" at given t: lines s.t. $\forall \vec{x}$ they are tangent to $\vec{u}(\vec{x}, t)$



immersed died particles will follow streamlines

$\vec{u}(\vec{x}, t)$ are the tangent vectors to these lines (generally $\vec{u}$ changes w/ t, unless steady flow!)

a line in space ($\mathbb{R}^3$) is $= \vec{x}(\tau) =$

$= (x_1(\tau), x_2(\tau), x_3(\tau))$

τ is parameter along the line

any streamline is given by specifying: $\{x_i(\tau), \; i = 1, 2, 3\}$

tangential vector to streamline $\equiv \dfrac{dx_i}{d\tau}$

__math:__ given $\vec{u}(\vec{x},t)$ the streamlines solve the equations; $\forall t$:

$$\frac{dx_i(\tau)}{d\tau} = u_i(\vec{x}(\tau),t), \quad i=1,2,3$$

(or, which is the same, since all $d\tau$ are equal:

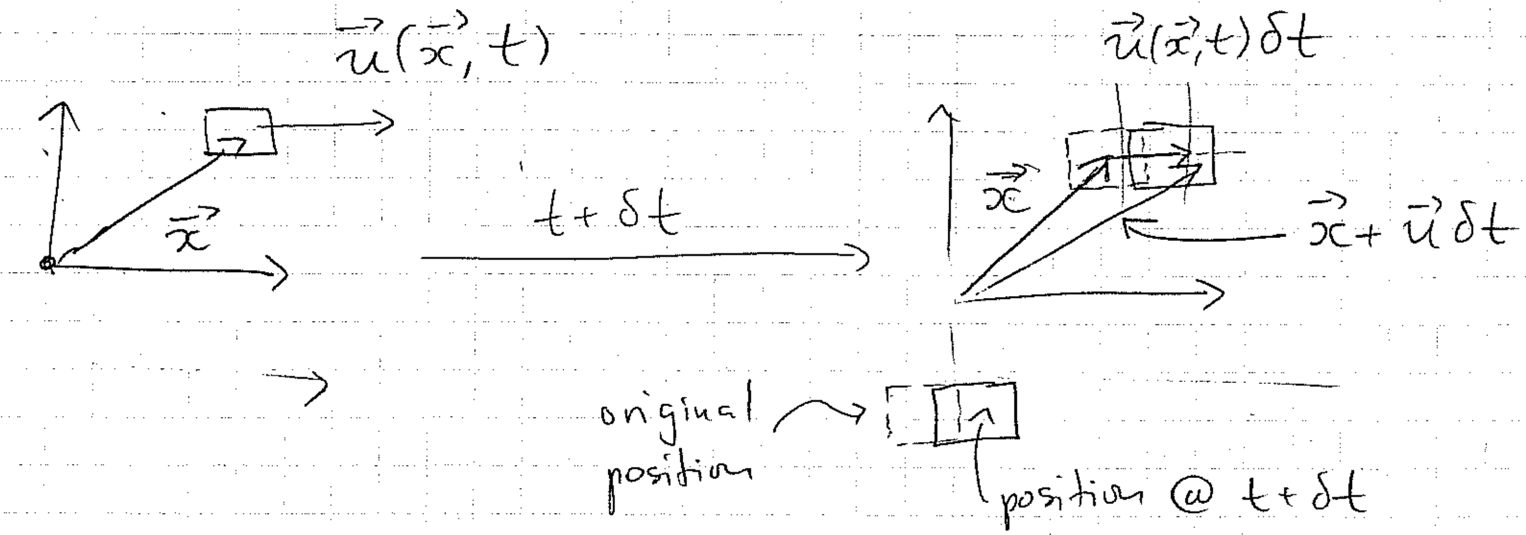
$$\frac{dx_1}{u_1(\vec{x},t)} = \frac{dx_2}{u_2(\vec{x},t)} = \frac{dx_3}{u_3(\vec{x},t)}$$ ((will use LATER)))

__Note:__ $\vec{u}(\vec{x},t)$ is an "Eulerian description"

$\vec{x}$ refers to a fixed point in space

NOT a fixed element of a fluid

at time $t+\delta t$ (SMALL) an element of the fluid which is at $\vec{x}$ at t is now at $\vec{x}+\vec{u}(\vec{x},t)\delta t$



So if at $(\vec{x}, t)$ velocity of a material fluid element is $\vec{u}(\vec{x}, t)$, its velocity @ $t + \delta t$ is (since it's now at $\vec{x} + \vec{u}\,\delta t$)

$$\vec{u}(\vec{x} + \vec{u}\,\delta t,\ t + \delta t)$$

since time is now $t + \delta t$

So, "$\vec{a}$" – the acceleration of this material element is

$$\vec{a} = \left.\frac{(\text{velocity at } t+\delta t) - (\text{velocity at } t)}{\delta t}\right|_{\delta t \to 0}$$

$$= \left.\frac{\vec{u}(\vec{x} + \vec{u}\,\delta t,\ t+\delta t) - \vec{u}(\vec{x}, t)}{\delta t}\right|_{\delta t \to 0}$$

since δt is small, we have

$$\underbrace{\vec{u}(\vec{x} + \vec{u}\,\delta t,\ t+\delta t)}_{} = \vec{u}(\vec{x}, t) + \frac{\partial \vec{u}}{\partial t}(\vec{x}, t)\,\delta t + \sum_{i=1}^{3} \frac{\partial \vec{u}}{\partial x^i}\, u^i\,\delta t$$

RHS: a Taylor expansion of fn of (3+1) variables

$$F(y^A + \delta y^A) = F(y^A) + \sum_B \frac{\partial F(y^A)}{\partial y^B}\,\delta y^B$$

here: $A = (x^1, x^2, x^3, t)$

$\delta y^A = (u^1 \delta t, u^2 \delta t, u^3 \delta t, \delta t)$

so we have

$$\vec{a} = \frac{\left(\sum_{i=1}^{3} \frac{\partial \vec{u}(\vec{x},t)}{\partial x^i} u^i \delta t + \frac{\partial \vec{u}(\vec{x},t)}{\partial t} \delta t \right)}{\delta t}$$

$$\vec{a} = \frac{\partial \vec{u}(\vec{x},t)}{\partial t} + \sum_i \frac{\partial \vec{u}(\vec{x},t)}{\partial x^i} u^i(\vec{x},t)$$

acceleration of a material element of fluid $\forall \vec{x}$ $\forall t$

in index notation:

$$a_i = \frac{\partial u_i}{\partial t} + \underbrace{\sum_j \frac{\partial u_i}{\partial x_j} u_j}$$

In this course will NOT matter wheter u_i or u^i (same)

Einstein notation: omit $\sum_j$ but always imply whenever repeated indices!

$$a_i(\vec{x},t) = \frac{\partial u_i(\bar{x},t)}{\partial t} + \frac{\partial u_i(\bar{x},t)}{\partial x_j} u_j(\bar{x},t) \equiv \frac{D u_i(\bar{x},t)}{Dt}$$

$\frac{D}{Dt} \leftrightarrow$ " material derivative"

PLEASE "INTERNALIZE"!

YOU WON'T UNDERSTAND ANYTHING ABOUT FLUID MECHANICS W/OUT FIRST DEEPLY APPRECIATING $\frac{D}{Dt}$!

So we have:

$$\vec{a}(\bar{x},t) \equiv \frac{D\vec{u}(\bar{x},t)}{Dt} \equiv \frac{\partial \vec{u}(\bar{x},t)}{\partial t} + \left(\vec{u}(\bar{x},t)\cdot\vec{\nabla}\right)\vec{u}(\bar{x},t)$$

in normal vector notation

or using indices $\left\{\begin{array}{l}\text{imagine you have to code equations}\\ \text{- must use indices} = 1,2,3!\end{array}\right.$

$$a_i(\bar{x},t) = \frac{\partial u_i(\bar{x},t)}{\partial t} + u_j(\bar{x},t)\frac{\partial u_i(\bar{x},t)}{\partial x_j}$$

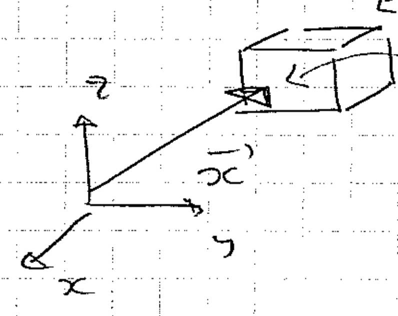
Why so important? → well — $m\vec{a} = \vec{f}$

and we've got the $\vec{a}$!!

Now → $\vec{a}(\vec{x},t)$ is the acceleration of a fluid element @ $\bar{x},t$

let $\rho(\vec{x},t)$ be the mass density of the fluid element @ $\bar{x},t$,

ie. let $\rho(\vec{x},t)\,d^3x$ be the mass inside

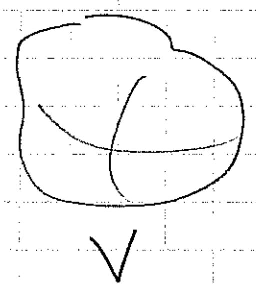


$d^3x \equiv dx_1\,dx_2\,dx_3$
$(\equiv dx\,dy\,dz)$

So, "$m\vec{a}$" of fluid inside $d^3\vec{x}$ is, then

$$\vec{a}(\vec{x},t)\,\rho(\vec{x},t)\,d^3x$$

For any volume V



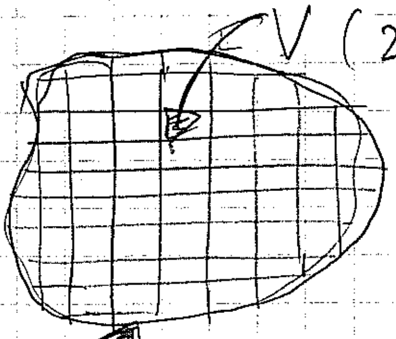
we have that "$m\vec{a}$" is the Σ "$m\vec{a}$" over its small parts:

$$\int_V d^3x\;\vec{a}(\vec{x},t)\,\rho(\vec{x},t) = \int_V d^3x\,\frac{D\vec{u}(\bar{x},t)}{Dt}\,\rho(\bar{x},t)$$

By Newton's "$\bar{f} = m\bar{a}$" this should equal the total force acting on the fluid in V.

<u>What is it?</u>

V (2d so I can draw)



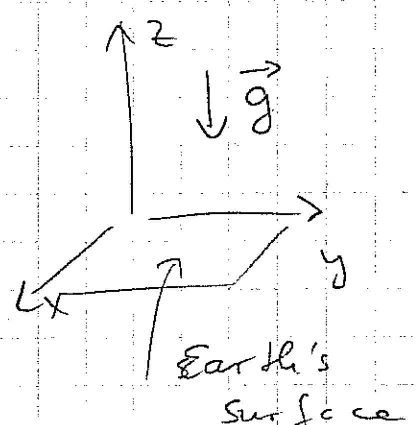
$S = \partial V$

S - surface = boundary of V $(\equiv \partial V)$

what's the total force?

1. suppose there's a gravitational potential — then every element of the fluid will experience a force $\vec{g}\,\rho(\vec{x},t)\,d^3x$

assumed a constant $\vec{g}$ above Earth, i.e.

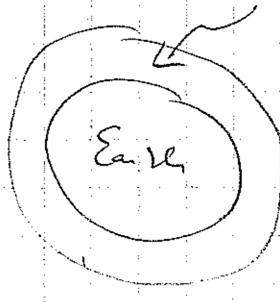


then total force is

$$\int_V \vec{g}\,\rho(\vec{x},t)\,d^3x$$

Now more generally, there can be a force that depends on $\vec{x}$ (and even t):

$\vec{g}$ not a constant, really



This is an example of <u>long-range</u> forces (gravity; E&M - if NOT screened) that act on every element of the fluid → the force on a unit mass is $\vec{F}(\vec{x},t)$, most generally. (13)

Force on mass in d^3x element is

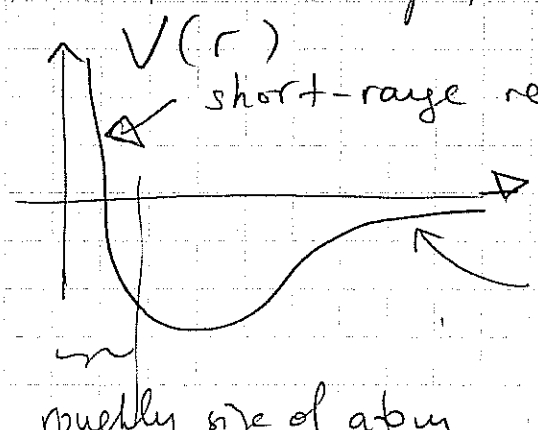
$$d^3x \; \rho(\bar{x},t)\, \bar{F}(\bar{x},t)$$

Force on fluid in V is

$$\int_V d^3x \; \rho(\vec{x},t)\, \vec{F}(\vec{x},t)$$

these (long-range forces) are called "volume forces" (1)

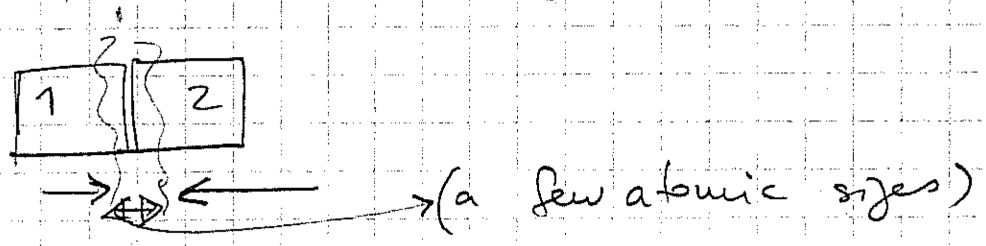
(2) Now — the fluid is made of particles.
In a gas (or fluid) any two particles interact via potentials of the form



long range attraction (weak!)
(van der Waals force ~ $1/r^{6,7}$ —)
"fluctuating dipole", QM ——

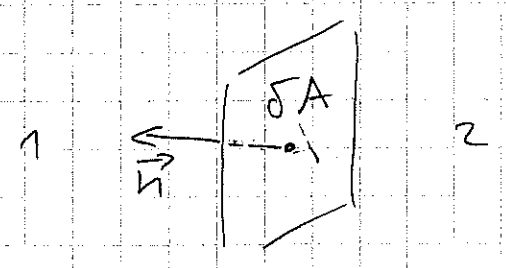
these are very short range and decrease very rapidly w/ distance.

So if we have two elements of fluid



these forces act only on a small layer at the boundary between the fluid elements.

In general, the force of this type from 1 acting on 2 will depend on the orientation of the boundary between 1 & 2 and will be proportional to its area δA:



$\vec{n}$ normal from 2 to 1 ($\perp \delta A$)

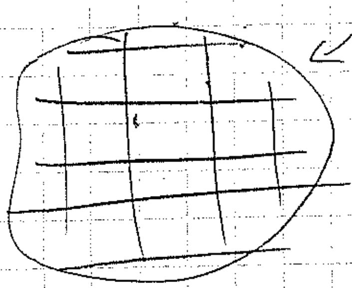
let $\vec{\Sigma}(\vec{n}, \vec{x}, t)\, \delta A$

be the force of particles of element 1 acting on element 2:

These short-ranged forces will be called "surface" forces.

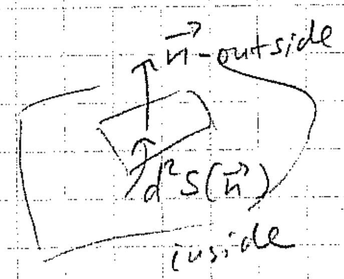
$\vec{\Sigma}(\vec{n}, \vec{x}, t)$ = (force per unit area $\perp \vec{n}$) = local "stress"

Now, for our V



it is clear that these forces between all elements INSIDE V will cancel pairwise (per Newton's 3rd law), but the ones acting on the elements close to $\partial V = S$ from particles outside of V will not cancel. So our equation "$m\bar{a} = \bar{f}$" will be, then

$$\left\{ \int_V d^3x\, \rho(\bar{x},t)\, \frac{D\vec{u}(\bar{x},t)}{Dt} = \int_V d^3x\, \rho(\bar{x},t)\, \vec{F}(\bar{x},t) + \int_{S=\partial V} d^2S(\vec{n})\, \vec{\Sigma}(\vec{n},\vec{x},t) \right\}$$



outward normal to δA

$$\text{last term} \equiv \sum_{\delta A \in S} \delta A\, \vec{\Sigma}(\vec{n},\vec{x},t)$$

"in a sense" we're done ---

$\boxed{\text{Eq. of motion of fluid}}$

but there's much more — need to make sense of it -- not a closed equation at all -- !!

what's $\vec{\Sigma}$ & how to deal with it?

next:
+ conservation laws
+ stress tensor