What about momentum?

let's go back to our "$f = ma$" equ. of p. 15:

$$\int_V dV \rho(\bar{x},t) \frac{D\vec{u}(\bar{x},t)}{Dt} \equiv \int_V dV \rho(\bar{x},t) \vec{F}(\bar{x},t) + \int_{S=\partial V} d^2S(\vec{n}) \vec{\Sigma}(\vec{n},\vec{x},t)$$

recall what $\frac{D\vec{u}}{Dt}$ was

$$\int_V dV \rho(\bar{x},t)\left(\frac{\partial \vec{u}(\bar{x},t)}{\partial t} + (\vec{u}(\bar{x},t)\cdot\vec{\nabla})\vec{u}(\bar{x},t)\right)$$

now this is <u>NOT</u> $\int_V dV \frac{\partial}{\partial t}(\rho\vec{u})$ which we would naturally call "the change of momentum of fluid inside a fixed V: $\frac{\partial}{\partial t}\underbrace{\int_V dV \underbrace{\rho\vec{u}}_{\text{momentum density}}}_{\text{momentum inside } V}$ =

$$\equiv \text{since } V \text{ is fixed} \equiv \int_V \frac{\partial}{\partial t}(\rho\vec{u})\, dV$$

but it is close. Let's compute the change of momentum:

$$\frac{\partial}{\partial t}(\rho \vec{u}) = \underbrace{\frac{\partial \rho}{\partial t}}\cdot \vec{u} + \rho \frac{\partial \vec{u}}{\partial t} =$$

$$= -\vec{\nabla}\cdot(\rho\vec{u})\ \vec{u} + \rho\frac{\partial \vec{u}}{\partial t}$$

hence: $$\int_V \frac{\partial}{\partial t}(\rho u^i)\, dV = \int_V \rho \frac{\partial u^i}{\partial t} dV - \int_V \underbrace{\nabla^j (\rho u^j)}_{\sum_j \text{ understood}} u^i dV$$

$$= \int_V \rho \frac{\partial u^i}{\partial t} dV - \int_V dV \underbrace{\left(\nabla^j(\rho u^j u^i) - \rho u^j \nabla^j u^i\right)}_{\equiv\ \nabla^j(\rho u^j)\, u^i \text{ (identity!)}}$$

$$= \int_V \rho \underbrace{\left(\frac{\partial u^i}{\partial t} - u^j \nabla^j u^i\right)}_{\equiv\ \frac{Du^i}{Dt}} dV - \underbrace{\int_V dV\, \nabla^j(\rho u^j u^i)}_{\oint_{S=\partial V} d^2\sigma^j \rho u^j u^i}$$

hence:

$$\frac{\partial}{\partial t}\int_V \rho u^i dV = \begin{pmatrix} \text{i-th component} \\ \text{change of} \\ \text{momentum} \\ \text{of fluid} \\ \text{in } V \end{pmatrix}$$

$$= \int_V \rho \frac{Du^i}{Dt} dV - \oint_{S=\partial V} d^2 S^j \rho u^j u^i$$

Proof: $$\int_V dV\, \vec{\nabla}\cdot\underbrace{(\rho\vec{u}\, u^i)}_{\vec{A}_i} = \oint_{\partial V = S} d^2\vec{S}\, \vec{A}_i$$

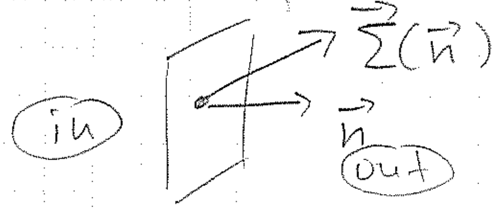
hence: , use "$\vec{f} = m\vec{a}$" equ.

top of p (21) for $\int_V \rho \frac{Du^i}{Dt} dV$:

$$\frac{\partial}{\partial t}\left(\int_V \rho u^i dV\right) = \int_V \rho \frac{Du^i}{Dt} dV - \oint_{S=\partial V} \rho u^j u^i d^2S^j$$

$$= \int_V dV \rho F^i - \oint_S \rho u^j u^i d^2S^j + \underbrace{\oint_S d^2S(\vec{n}) \Sigma^i(\vec{n}, \bar{x}, t)}$$

let's stare at this.
recall what $\Sigma^i(\vec{n}, \bar{x}, t)$
was: the i-th component of
the force on the fluid



"inside" surface due to the
fluid outside

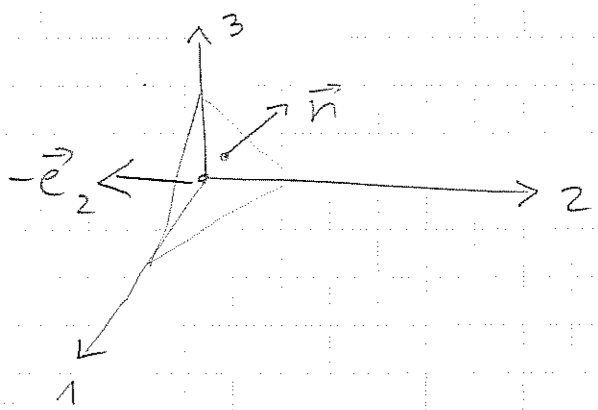
let us instead of $\vec{\Sigma}(\vec{n})$ use
$\{\Sigma^i(\vec{n}), i = 1,2,3\}$ $\rightarrow$ but then $\vec{n} = \{n^i, i=1,2,3\}$

let $\Sigma^i(\vec{n}) \equiv \sigma^{ij} n^j$, (since $\vec{n}$ can be arbitrarily oriented)
since $\vec{\Sigma}$ has 3 components & so does $\vec{n}$ $\Longrightarrow$
$\Rightarrow$ really 9 quantities.

let's dwell on this a bit ---
(the σ^{ij}'s are very important quantities, so worth the attention)

we said force on $d^2S(\vec{n})$ is $d^2S(\vec{n})\,\vec{\Sigma}(\vec{n})$.

Consider a small fluid element $d^2S(\vec{n})$, choose arbitrary oriented coordinates 1,2,3 w/ unit vectors ($\vec{e}_i$, $i=1,2,3$).



Together w/ $d^2S(\vec{n})$ form a tetrahedron w/ four faces. Outward normals are $-\vec{e}_1$, $-\vec{e}_2$, $-\vec{e}_3$
(only $-\vec{e}_2$ shown).

Force on $d^2S(-\vec{e}_2)$ is $\underbrace{d^2S(-\vec{e}_2)}\,\vec{\Sigma}(-\vec{e}_2)$
(etc. for $\vec{e}_1, \vec{e}_3$).

this is the area of the triangle in 1-3 plane $\equiv$ area of $d^2S(\vec{n})$ $\times$
$\times$ cos (angle between plane $\perp \vec{n}$ & plane $\perp \vec{e}_2$) =

$$\Rightarrow d^2S(-\vec{e}_2) = d^2S(\vec{n})\,\vec{n}\cdot\vec{e}_2$$

So, total of surface forces is

$$d^2S(\vec{n})\,\vec{\Sigma}(\vec{n}) + \sum_{i=1,2,3} \underbrace{d^2S(-\vec{e}_i)\,\vec{\Sigma}(-\vec{e}_i)} = d^2S(\vec{n})\,\vec{\Sigma}(\vec{n}) + $$
$$+\ \vec{n}\cdot\vec{e}_i\, d^2S(\vec{n})\,\underbrace{\vec{\Sigma}(-\vec{e}_i)}_{=-\vec{\Sigma}(\vec{e}_i)}$$

(since $\vec{\Sigma}(-\vec{a}) = -\vec{\Sigma}(\vec{a})$)

and then total surface force on tetrahedron becomes

$$d^2S(\vec{n})\left[\vec{\Sigma}(\vec{n}) - \underbrace{\vec{\Sigma}(\vec{e}_i)(\vec{e}_i\cdot\vec{n})}_{\text{sum over } i=1,2,3}\right]$$

now we have that

$$\left(\begin{matrix}\text{mass} \times \text{acceler.}\\ \text{in } V\end{matrix}\right) = \left(\begin{matrix}\text{sum of body}\\ \text{forces in } V\end{matrix}\right) + \left(\begin{matrix}\text{sum of}\\ \text{surface}\\ \text{forces}\end{matrix}\right)$$

$$\sim V \sim L^3 \qquad \sim V \sim L^3 \qquad \sim S \sim L^2$$

take $V \to 0$: l.h.s $\sim L^3$

rhs $\sim L^3 + \underbrace{L^2}_{\gg L^3 \text{ when } L \to 0}$

only way this can work when $L \to 0$ is if

Σ of surface forces on tetrahedron $\to 0$ $\to$

hence $\vec{\Sigma}(\vec{n}) = \vec{\Sigma}(\vec{e}_j)\, \vec{e}_j \cdot \vec{n}$

or $\Sigma^i(\vec{n}) = \underbrace{\Sigma^i(e_j)}\cdot n_j$

$\Sigma^i(\vec{n}) \equiv \underbrace{\sigma^{ij}}\, n_j$

- true for any $\vec{n}$
- true for any choice of axes $e_1\, e_2\, e_3$!

- independent on $\vec{n}$
- depends on $\bar{x}, t$, of course.

Now back to top of 23

$$\frac{\partial}{\partial t}\int_V \rho u^i dV = \int_V dV \rho F^i - \oint_S \rho u^i u^j d^2S^j - \oint_S \underbrace{d^2S(\vec{n})\, n^j}_{d^2S^j} \sigma^{ij}$$

as we just showed

$$= \int dV \rho F^i - \oint_S d^2S^j \underbrace{(\rho u^i u^j - \sigma^{ij})}$$

$$\oint d^2S_j A^j{}_i = \int dV\, \partial_j A^j{}_i$$

$$\int dV \frac{\partial}{\partial t}(\rho u^i) = \int dV\left(\rho F^i - \frac{\partial}{\partial x^j}(\rho u^i u^j - \sigma^{ij})\right)$$

sorry, deserves more space! E.P.

drop $dV \Rightarrow$ $$\boxed{\frac{\partial}{\partial t}(\rho u^i) = \rho F^i - \frac{\partial}{\partial x^j}(\rho u^i u^j - \sigma^{ij})}$$

= momentum conservation =

$$\frac{\partial}{\partial t}\begin{pmatrix}\text{momentum}\\ \text{in}\\ V\end{pmatrix} = \begin{pmatrix}\text{force on } V\\ (\text{volume (or body)}\\ \text{force})\end{pmatrix} + \begin{pmatrix}\text{flow of}\\ \text{momentum}\\ \text{through } \partial V\\ (\text{surface force})\end{pmatrix}$$

because momentum is a vector quantity, $\sim \rho u^i$, its flow is a tensor $\sim \underbrace{\rho u^i u^j - \sigma^{ij}}$

$= \Pi^{ij}$ = flow of i-th component of $\vec{p}$ through a surface $\perp$ j

So what do we have for now:

$$\begin{cases} \dfrac{\partial}{\partial t}\rho = -\dfrac{\partial}{\partial x^j}(\rho u^j) \\[2ex] \dfrac{\partial}{\partial t}(\rho u^i) = \rho F^i - \dfrac{\partial}{\partial x^j}(\rho u^i u^j - \sigma^{ij}) \end{cases}$$

- we have 4 equations (differential ones, so given initial conditions can solve for future)
- fluid has $\rho(\bar{x},t)$, $u^i(\bar{x},t)$ – 4 properties we're interested in → so looks promising!
- BUT: will work if F^i is given and σ^{ij} is known and ((expressed thru known quantities!))
  --> what are they?

in h.w. you'll show that $\sigma^{ij} = \sigma^{ji}$, i.e.

σ^{ij} is symmetric $\longleftrightarrow$ not 9 but 6 quantities

$$\begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} \end{pmatrix}$$

(determined by upper ones)

$\underbrace{\text{torques } \vec{r} \times \vec{f}}$

idea: will consider the moments of forces acting on the fluid in V, in particular the moment of the surface forces; total moment $\sim L \times L^3$

surface forces' moment $\sim L \times L^3 + L^3$

condition for this to vanish $\Longleftrightarrow \sigma^{ij} = \sigma^{ji}$

Now recall again def. of σ^{ij}:

$$\underbrace{\Sigma^i(\vec{n}) = \sigma^{ij} \, n^j}$$

- $\vec{n}$ is a vector, fixed in space arbitrarily

- this is an expression in a given coordinate system $\{\vec{e}_i, i = 1,2,3\}$
- how would this look in another one?