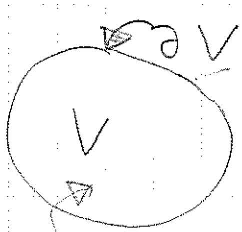
As in classical mechanics, in FM "$\bar{f} = m\bar{a}$" can be cast as a consequence of momentum (non-) conservation

(i.e. $\vec{p} = m\vec{v}$, $\dot{\vec{p}} = m\vec{a} = 0$ - conserved $\vec{p}$ if no reason for it to change, i.e. a force acts)

In FM, there are, in the nonrelativistic limit, several important conservation laws – mass, momentum, energy, angular momentum.

(mass alone is not (necessarily) conserved in relativistic fluids).

<u>mass conservation</u>



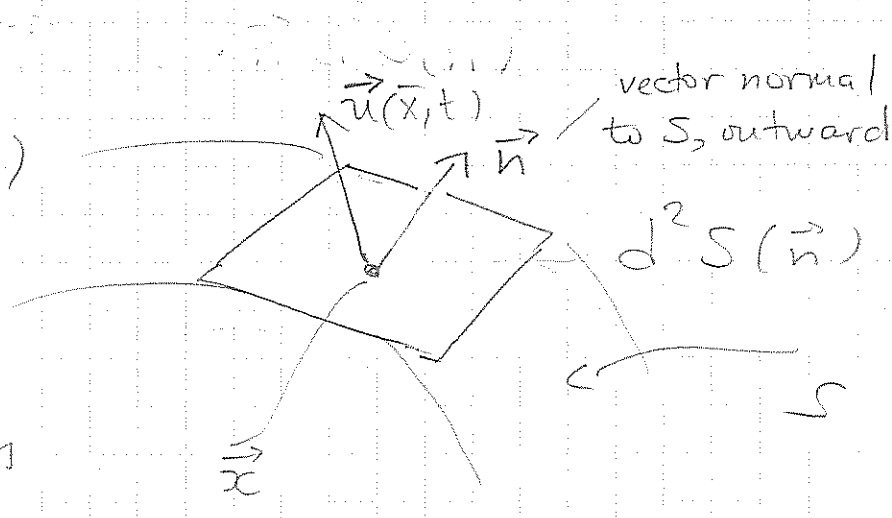
- <u>fixed</u> volume in fluid w/ mass density $\rho(\bar{x}, t)$ & velocity $\vec{v}(\vec{x}, t)$

total mass inside V can change only due to fluid coming into V or going out of V through its boundary ∂V

(replace "mass" by "charge" => get charge conservation in E & M) total mass inside is $\int_V \rho(\bar{x},t)\, d^3x$

now $\rho(\bar{x},t)\, \vec{u}(\bar{x},t)\cdot\vec{n}\, d^2S(\vec{n})$

is the mass which flows through $d^2S(\vec{n})$ in unit time:



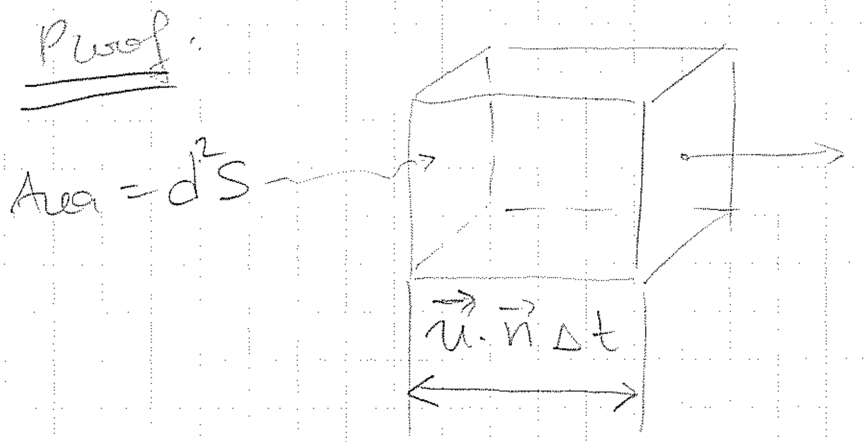
$\vec{u}(\bar{x},t)\cdot\vec{n} =$

= component of velocity $\perp$ to $d^2S(\vec{n})$

$$\rho\,\vec{u}\cdot\vec{n} = (\text{mass going through } d^2S(\vec{n}) \text{ in unit time})$$

= flux of mass:

Proof:



all particles inside will go through $d^2\vec{S}(\vec{n})$ in time Δt

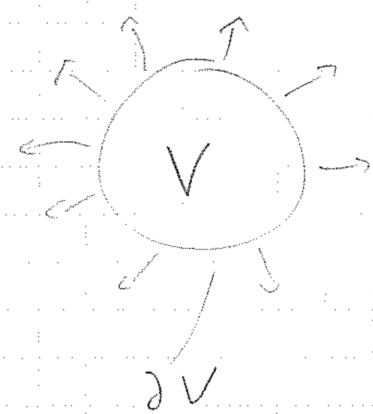
So total mass inside  is $\rho\,\vec{u}\,\vec{n}\cdot\Delta t$

and in Δt it will go through $d^2\vec{S}(\vec{n}) \Rightarrow$

$\Rightarrow$ flux of mass is $\rho\,\vec{u}\cdot\vec{n}$ $\boxed{}$

So now we have learned that, given an arbitrary closed volume V (fixed) of fluid:

$$\frac{d}{dt}\int_V \rho(\bar{x},t)\, d^3x = -\oint_{\partial V = S} \rho(\bar{x},t)\,\vec{u}(\bar{x},t)\cdot\vec{n}\, d^2S(\vec{n})$$



mass in V changes since fluid leaves V (as in pic.) hence "$-$" sign:

$\vec{u}\cdot\vec{n} > 0 \Leftrightarrow$ flux leaves

$\vec{u}\cdot\vec{n} < 0 \Leftrightarrow$ flux comes in

---

now some math: $\vec{n}\, d^2S(\vec{n}) \equiv d\vec{S}$

$$\oint_{\partial V = S} \vec{A}\cdot d\vec{S} = \int_V \vec{\nabla}\cdot\vec{A}\, dV \qquad (dV \equiv d^3x)$$

(Proof by integration by parts: $\text{rhs} = \int_a^b \frac{d}{dx} f(x)\, dx \leftrightarrow \int dV$

$\text{lhs} = f(b) - f(a) \leftrightarrow \oint_{\partial V} dS$

$$\vec{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \equiv \left\{\frac{\partial}{\partial x^i}\ i=1,2,3\right\} = \{\nabla^i\}$$

In the future it will be very useful to write using indices

$$\vec{n}\, d^2 S(\vec{n}) = d\vec{S} \equiv \underbrace{\{ d^2 S^i ,\ i = 1, 2, 3 \}}_{\text{i-th comp surface element}}$$

(-onent of)

$$\underbrace{\oint_{S=\partial V} \vec{A} \cdot d\vec{S} = \oint_{S=\partial V} A^i d^2\sigma^i \equiv \int_V \nabla_i A^i \, dV}_{\Sigma \text{ over } i \text{ understood}}$$

it may happen that A carries another index —

then $\oint_{S=\partial V} A^i_j \, d^2\sigma^i = \int_V \nabla^i A^i_j \, dV$

index j "goes for the ride"

---

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$$\oint_{\partial V = S} \rho \vec{u}\, d^2\vec{S} = \oint_{\partial V = S} \underbrace{\rho u^i}_{\equiv A^i} d^2 S^i = \int_V \nabla^i (\rho u^i)\, dV$$

hence

$$\frac{d}{dt}\int_V \rho(\bar{x},t)\, d^3x = -\oint_{\partial V = S} \rho(\bar{x},t)\, \vec{u}(x,t)\cdot d^2\vec{S}$$

Vis fixed ($\frac{d}{dt}$ goes thru!)

$$\int_V \frac{d}{dt}\big(\rho(\bar{x},t)\big)\underbrace{d^3x}_{\equiv dV} = -\int_V \vec{\nabla}\cdot\big(\rho(\bar{x},t)\,\vec{u}(\bar{x},t)\big)\, dV$$

$$= -\int_V \nabla^i\big(\rho(\bar{x},t)\, u^i(\bar{x},t)\big)\, dV$$

$$\int_V dV\left[\frac{d}{dt}\big(\rho(\bar{x},t)\big) + \nabla^i\big(\rho(\bar{x},t)\, u^i(\bar{x},t)\big)\right] = 0$$

V is arbitrary : $\int_V dV\, f(\vec{x}) = 0 \;,\; \forall V \Rightarrow f \equiv 0$.

hence

$$\boxed{\frac{d}{dt}\rho + \nabla^i(\rho u^i) = 0}$$

• $\rho u^i \leftrightarrow$ flux of fluid

continuity equation $\equiv$ mass conservation

(E& M : $\rho \leftrightarrow$ charge density
$\rho\vec{u} \leftrightarrow$ current density)

All conservation laws have this kind of form
(differential expression)

an <u>incompressible</u> fluid is when

ρ = const (doesn't change w/ $\vec{x}$, t, that is)

continuity equ. sez:

$$\underbrace{\cancel{\frac{d}{dt}\rho}_{0} + \cancel{\nabla^i \rho}_{0}\, u^i}_{\rho\text{-indep of } \bar{x},\, t} + \underbrace{\rho\, \nabla^i u^i}_{\text{since } \rho \neq 0} = 0$$

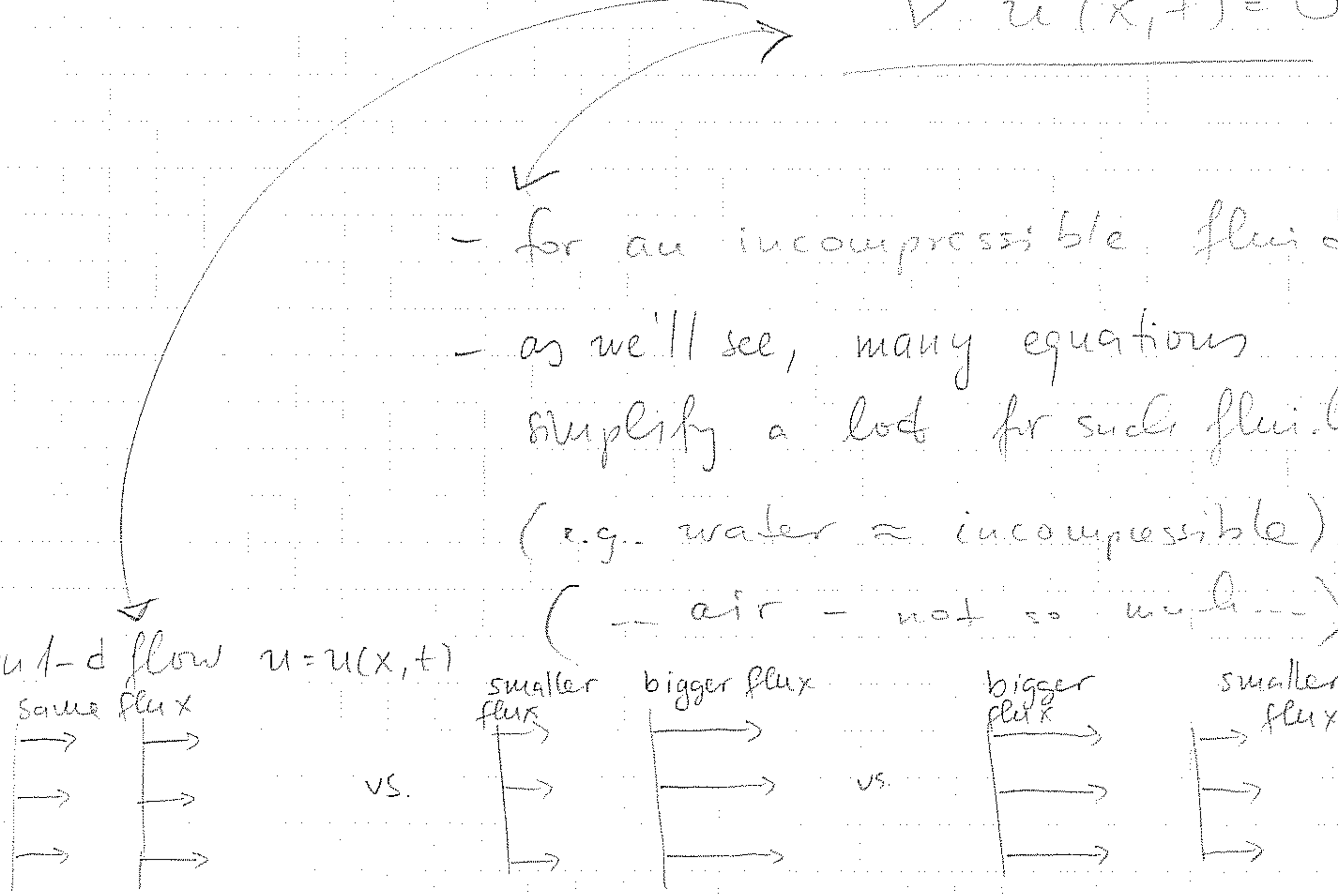
$$\underline{\vec{\nabla}\, \vec{u}(\bar{x}, t) = 0}$$

- for an incompressible fluid
- as we'll see, many equations simplify a lot for such fluids

(e.g. water ≈ incompressible)

(... air - not so much ...)

Ex: in 1-d flow $u = u(x,t)$



$\partial_x u_x = 0$ — <u>no mass pile-up</u> <u>density = const</u>

$\partial_x u_x > 0$ — <u>need source of</u> <u>mass</u>

$\partial_x u_x < 0$ — <u>need sink of</u> <u>mass</u>