Since $\rho = \text{const}$, $\vec{u}$ & $\frac{p}{\rho}$ are the natural variables to describe the flow

so then

$$\left| \begin{array}{l} (\vec{u}\vec{\nabla})\vec{u} = \vec{F} - \vec{\nabla}\left(\frac{p}{\rho}\right) + \nu \Delta \vec{u} \\ \vec{\nabla}\vec{u} = 0 \end{array} \right.$$

as already said, 4 equs for 4 variables (given $\vec{F}$)

but also, only depend on one parameter characterizing liquid

$\nu = \frac{\mu}{\rho}$ (kinematic viscosity)

$$\left(\begin{array}{l} \nu_{air} \sim .15 \frac{cm^2}{s} \\ \nu_{water} \sim .001 \frac{cm^2}{s} \\ \text{etc.} \end{array} \right)$$

the other characteristics of the flow are typical $|\vec{u}| \sim U$ & typical distance scale L across which flow changes considerably

$[\nu] = \frac{m^2}{s}$, $[L] = m$, $[U] = \frac{m}{s}$

$$\frac{UL}{\nu} \equiv \text{dimensionless } \# = \underbrace{R = \frac{UL}{\nu}}_{\text{Reynolds } \#}$$

In the absence of body forces, equ. reads
(but allow t-dependence, still incompressible)

$$\rho \frac{\partial \vec{u}}{\partial t} + \rho\, \vec{u} \cdot \frac{\partial}{\partial \vec{r}}\, \vec{u} = - \frac{\partial}{\partial \vec{r}}\, p + \mu \frac{\partial^2}{\partial \vec{r}^2}\, \vec{u}$$

use scales L & U to introduce dim-less variables like this:

$$\left.\begin{aligned} \vec{u} &= U\, \vec{u}' \\ \vec{r} &= L\, \vec{r}' \\ p &= \rho U^2 p' \\ t &= \frac{L}{U}\, t' \end{aligned}\right\}$$

primed variables are dimensionless

plug into equ:

$$\rho \frac{U^2}{L} \frac{\partial \vec{u}'}{\partial t'} + \rho \frac{U^2}{L}\, \vec{u}' \cdot \frac{\partial}{\partial \vec{r}'}\, \vec{u}' =$$

$$= - \frac{\partial}{\partial \vec{r}'}\, p' \, \frac{\rho U^2}{L} + \mu \frac{U}{L^2} \frac{\partial^2 \vec{u}'}{\partial \vec{r}^{2\prime}}$$

divide by $\rho \frac{U^2}{L}$:

$$\frac{\mu}{\rho L U} = \frac{\nu}{L U} = \frac{1}{R}$$

$$\frac{\partial \vec{u}'}{\partial t'} + (\vec{u}' \cdot \vec{\nabla}')\vec{u}' = - \vec{\nabla}' p' + \left(\frac{\mu}{\rho L U}\right) \Delta' \vec{u}'$$

$$\frac{\partial \vec{u}'}{\partial t'} + (\vec{u}' \cdot \vec{\nabla}')\vec{u}' = - \vec{\nabla}' p' + \frac{1}{R} \Delta' \vec{u}'$$

N.S. eqn, then, has only a dependence

on $R = \frac{LU}{\nu}$

So, soltn i.t.o $\vec{u}', \vec{r}', t', p'$ can only depend on R, when time & length are measured in units of L & $\frac{L}{U}$, respectively.

$$\vec{u}' = \vec{u}'(\vec{r}', t', R)$$

$$p' = p'(\vec{r}', t', R)$$

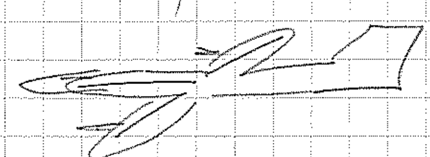
if there are body forces, e.g. $\vec{F} = \vec{g}$ there's a new scale (acceleration) so NEW dimenless # → e.g.

$$\frac{U^2}{gL} = \text{"Froude \#"}$$

((but in many cases g not important))

(e.g) if ν is fixed (air)

R is the same for

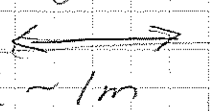


real aircraft

$L \sim 30$ m

& $U \sim 500 \frac{km}{h}$

model in wind tunnel



(! same shape!)

$L \sim 1$m

$U \sim 15{,}000 \frac{km}{h}$

same $U \cdot L$

if need to reduce U to keep R same ↑ L

⇒ this scaling of solutions obtained using different U, L, ν but same R

⇒ "dynamic similarity"

Generally, from any given solution of NS (for incompressible flow) ⇒ get a triply infinite family of solutions

(same R, change ν, L)

→ useful in practice – more convenient to study model than real plane ...

R measures relative importance of inertia vs viscous forces acting on fluid

$R \ll 1$ inertia ≪ viscosity

$R \gg 1$ inertia ≫ viscosity

Our flows so far do not do justice to importance of R: e.g. flow thru pipe of radius r_0 (there we called it R, but don't want to confuse w/ Reynolds #) $u(r) = \frac{p_1 r_0^2}{4\nu}\left(1-\left(\frac{r}{r_0}\right)^2\right)$

take $U = \frac{p_1 r_0^2}{4\nu}$, $L = r_0$

so $u(r) = U(1-r'^2)$, $u' = 1-r'^2$

i.e. because our eqn linearized so nicely, its solution in dim-less coordinates is actually R#-independent $u' = 1 - r'^2$
(however, STABILITY ANALYSIS DEPENDS ON R !!)

Another comment on R# importance is that we assumed $t = \frac{L}{U} t'$ — so no other parameter w/ dimension of time would enter the solution (e.g. some externally determined frequency that drives flow). If there's such an external characteristic time τ, we can form another dimless parameter

$$S = \frac{U\tau}{L} \quad \text{"Strouhal \#"}$$

together w/

$$F = \frac{U^2}{Lg} \quad \text{"Froude \#"}$$

&

$$R = \frac{UL}{\nu} \quad \text{"Reynolds \#"}$$

→ can have many dimless parameter

a solution can depend on. then, dynamical similarity between two flows of different b.c. requires that all three parameters of the

flows are equal between the two flows. --
(less useful then. --)

But -- for stationary flows w/out $\vec{g}$
R# is enough!

Now it turns out that the character of less simple flows changes w/ R. The 1st one we'll consider is the flow around a body at small R#.

At small R#, N.-S. equ simplifies a bit -- in dimless variables we have for stationary flow

$$(\vec{u}' \cdot \vec{\nabla}')\, \vec{u}' = -\vec{\nabla}' p' + \frac{1}{R} \Delta' \vec{u}'$$

so drop this one

this one more important as $R \to 0$

same argument in "normal" variables

$$(\vec{u}\,\vec{\nabla})\, \vec{u} = -\vec{\nabla}\left(\frac{p}{\rho}\right) + \nu \Delta \vec{u}$$

$$\sim \frac{U^2}{L} \xleftarrow{\text{ratio}} \longrightarrow \sim \frac{UL}{\nu}$$

$\left(\frac{U^2}{L} / \frac{\nu U}{L^2}\right) \sim R$

$\to$ so drop inertial $(\vec{u}\,\vec{\nabla})\,\vec{u}$ term.

110

- It may be tempting to use same argument to also rid the $\vec{\nabla} p$ term in the equation — but this would be empty of content (as then flow'd be $\nu \Delta \bar{u} = 0$).

- So, we'll drop $\frac{D\bar{u}}{Dt}$ from N.-S. eqn. at small R (we'll have to come back & reconsider consistency after we find solution!)

If we do so, we find

$$\begin{cases} \vec{\nabla} p = \mu \, \Delta \vec{u} \\ \vec{\nabla} \cdot \vec{u} = 0 \end{cases}$$

- If boundary conditions depend on $\vec{u}$ only (e.g. velocity field vanishing @ body), these eqns. are equivalent to:

$$\vec{\nabla} \cdot \vec{u} = 0$$

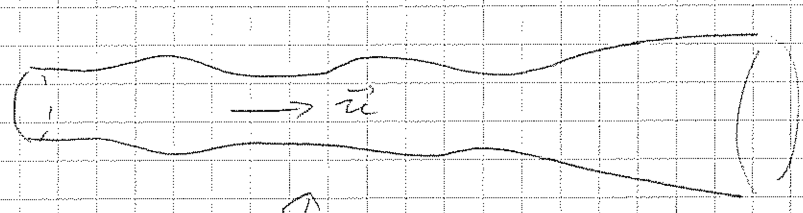
$$\underbrace{\vec{\nabla} \times (\vec{\nabla} p)}_{=0 \text{ (curl of gradient } \equiv 0)} = \mu \, \Delta (\vec{\nabla} \times \vec{u})$$

$$\begin{Vmatrix} \vec{\nabla} \cdot \vec{u} = 0 \\ \Delta \vec{\omega} = 0 \end{Vmatrix}$$ + b.c. on $\vec{u}$, then get p by solving $\vec{\nabla} p = \mu \, \Delta \vec{u}$ from solution for $\vec{u}$

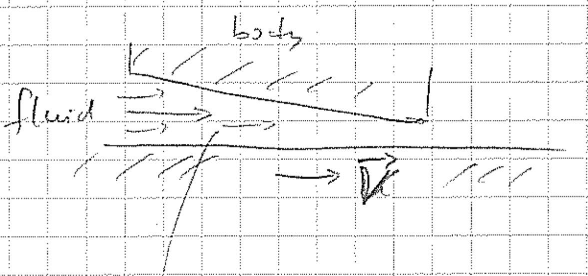
- If b.c. depend on p (e.g. pressure given @ boundaries), then, since $\vec{\nabla} \cdot \vec{\nabla} p = \mu \, \Delta \underbrace{\vec{\nabla} \cdot \vec{u}}_{\text{incompress}} = 0$

then solve $\nabla^2 p = 0 \Rightarrow$ then find $\vec{u}$ from $\vec{\nabla} \cdot \vec{u} = 0$, $\nabla^2 \vec{u} = \frac{1}{\mu} \vec{\nabla} p$.

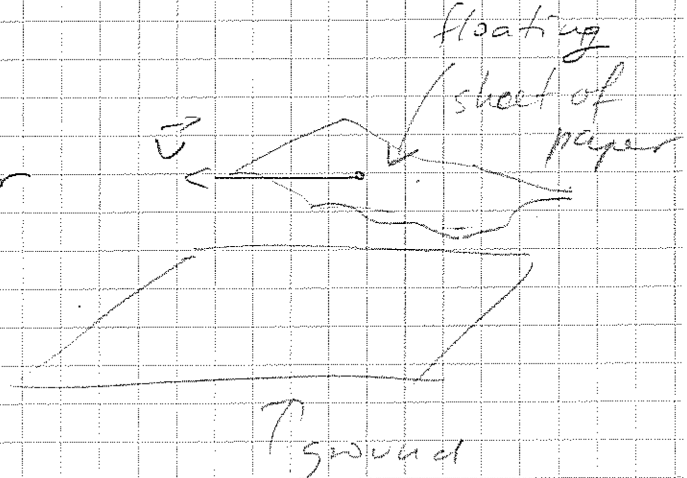
Exs of flows where inertia effects are small ---



a pipe w/ slowly varying x-section



or



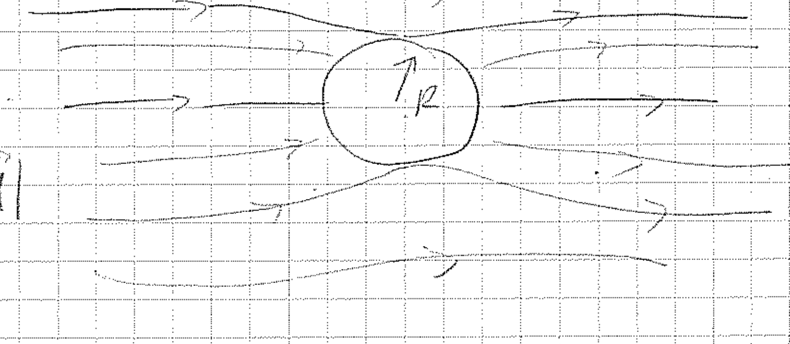
can have huge $\vec{\nabla} p$

if layer thin - support heavy weight

"lubrication theory"

& finally Stoke's formula

$\vec{U}$



drag $= 6\pi R\mu|\vec{U}|$

- will consider these in turn, along w/ discussing when they hold (neglecting inertia, that is) & how to improve (w/out too much detail --)
- then: flows @ large R — instabilities, turbulence —