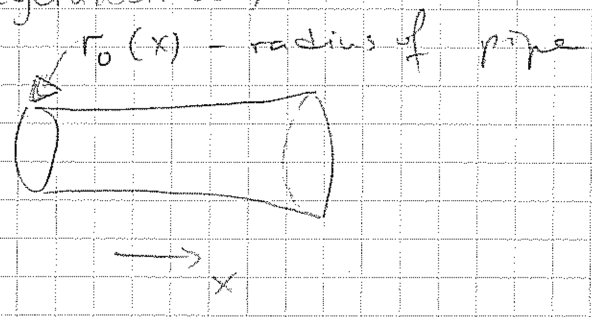
→ a pipe of varying x-section (cylindrical)



let, as before $p = p_0 + p_1 x$, p_1 = constant pressure gradient

if r_0 were constant, we'd have (p. 101)

$$u(r) = \frac{p_1}{4\nu\rho}(r_0^2 - r^2)$$

now we have $r_0 = r_0(x)$, so let's try

$$u(r,x) = \frac{p_1}{4\nu\rho}(r_0^2(x) - r^2)$$

↓ p. 102

$\forall x$, the flux is $Q = \int_0^R 2\pi r\,dr\,\rho\,u(r,x) = \frac{\pi}{8}\frac{p_1}{\nu} r_0^4(x)$

So of course, this would only make sense if flux is same $\forall x$ (since steady-state)

$\Rightarrow$ $p_1(x) = \frac{const}{r_0^4(x)}$ (to have flux x-independent)

So really, looks like soln is $p = p_0 + p_1(x)\,x$

→ need more work — $u(r,x) = \frac{1}{4\nu}\frac{const}{r_0^4(x)}(r_0^2(x) - r^2)$

let total flux be $Q = \frac{\pi}{8}\frac{const}{\nu}$ $\Rightarrow$ $const = \frac{8\nu Q}{\pi}$ →

const — determined by Q, ν, ρ

so $p_1(x) \approx \left(\frac{8\nu Q}{\pi}\right) \frac{1}{r_0^4(x)}$

or, if Δp @ ends is known — can get Q

$$u(r,x) \approx \frac{2Q}{\pi\rho} \frac{r_0^2(x) - r^2}{r_0^4(x)}$$

(e.g. central velocity (at $r=0$) $\sim \frac{1}{r_0^2}$)

not an exact solution ... $\approx$

how $\approx$?

- we used soltn for pipe w/ fixed r_0
- imagined that u is one appropriate for given $r_0(x)$ (at $x \Rightarrow u(r) \to u(r,x)$)
- made this self consistent by imposing that no fluid is lost (constancy of flux thru pipe)

what did we neglect?

nonlinear term $(\bar{u}\nabla)\bar{u} \neq 0$ anymore but $\sim u\frac{\partial u}{\partial x}$

(since $\partial_x u_x \neq 0$)

must be $\ll -\left(\frac{\nabla p}{\rho}\right), \nu\Delta^2\vec{u}$

these are $\approx$ same order (since solution obeys these)

so $\left(u\frac{\partial u}{\partial x}\right)_{\max} \ll \frac{1}{\rho} p_1$ if neglecting it OK

$$\frac{\partial u}{\partial x} = \frac{2Q}{\pi\rho}\frac{\partial}{\partial x}\left(\frac{1}{r_0^2(x)} - \frac{r^2}{r_0^4(x)}\right) = \frac{2Q}{\pi\rho}\left(-\frac{2}{r_0^3} + \frac{4r^2}{r_0^5}\right) r_0'(x)$$

max abs value @ $r=0$ $\to$ $\left|\frac{\partial u}{\partial x}\right| \sim \frac{4Q}{\pi\rho}\frac{r_0'}{r_0^3}$

$$u\left|\frac{\partial u}{\partial x}\right|_{r=0} = \frac{r_0'}{r_0^5}\frac{8Q^2}{\pi^2\rho^2} \ll \frac{1}{\rho}\frac{8\nu Q}{\pi r_0^4} \Rightarrow r_0' \ll \frac{\nu\pi\rho}{Q} r_0$$

So when we say "radius is slowly varying"

we mean $\dfrac{r_0'}{r_0} \sim \dfrac{1}{\begin{pmatrix}\text{distance on} \\ \text{which } r_0 \\ \text{changes} \\ \text{appreciably}\end{pmatrix}} \ll \dfrac{\pi \nu \rho}{Q}$

Q = mass going thru tube in unit time $\left(\dfrac{kg}{s}\right)$

$\nu\rho = \mu$ - viscosity $\left(\dfrac{kg}{ms}\right)$

In other words, if flux is too high may not be so good

$$\underbrace{\begin{pmatrix}\text{distance where } r_0 \\ \text{changes appreciably}\end{pmatrix} \gg \frac{Q}{\mu}}$$

criterion of validity of our approximation

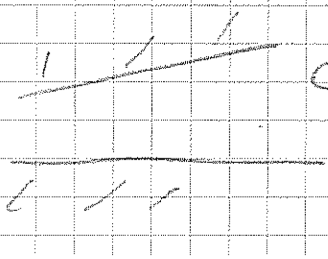
(⟶ "Bohr-Oppenheimer" approx in QM --
-- something changes slowly -- always must know how slowly, though)

> i.e. dim-less # ~ $\lambda \sim \dfrac{r_0'}{r}$

$$\left(\frac{Q}{\lambda \mu} \ll 1\right)$$

Good for other cases, too!

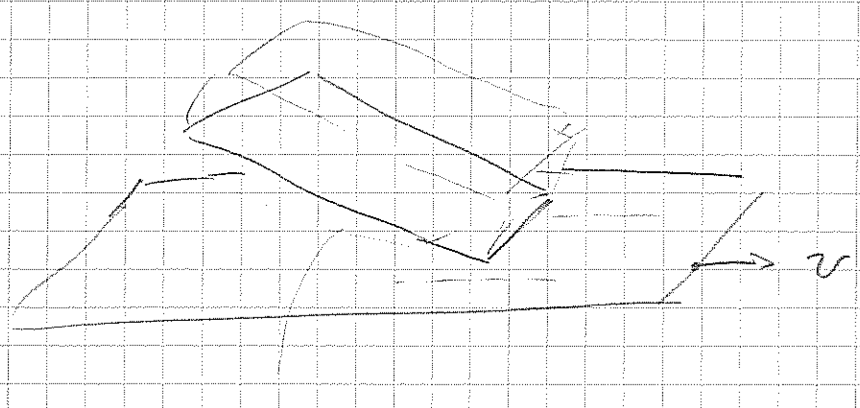
(any channel that's ≈ varying)



} places not exactly parallel

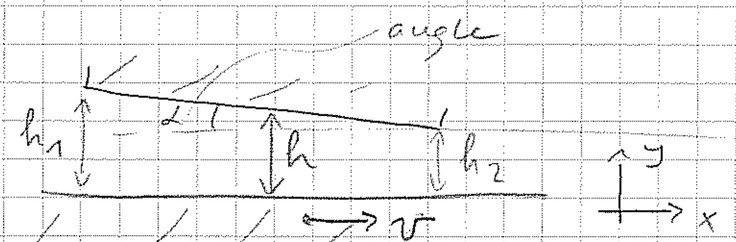
Another important ex of similar flavor:

-- "lubrication" --



thin layer -- so strain between two planes large
(fluid dragged to left by relative motion)
- large stress, too, due to viscosity

fluid resists being squeezed out --
(carry high loads...)



Imagine using our soltn for parallel plates $\underline{h}$ apart w/ pressure gradient $\underline{p_1}$ & $\underline{v}$ - velocity of lower plate:

$$u(y, x) \simeq \frac{p_1}{2\mu} y(h-y) + v\frac{h-y}{h}$$

see p. 98 except now lower plate's moving

but NOW $h = h(x)$

-- & of course p_1 will be fn of x, too

So then flux of mass (per unit $\hat{z}$-width) ← which we imagine "very wide" (infinite, etc)

is $Q = \int_0^h \rho\, u(y,x)\, dy =$

$$= \frac{\rho p_1}{2\mu} \int_0^h y(h-y) + \frac{\rho v}{h} \int_0^h (h-y)\, dy =$$

$$= \frac{\rho p_1}{2\mu} \left(\frac{h^2}{2} h - \frac{1}{3} h^3 \right) + \frac{\rho v}{h} \left(h^2 - \frac{1}{2} h^2 \right)$$

$$= \frac{\rho p_1}{2\mu} \frac{h^3(x)}{6} + v \rho \frac{1}{2} h(x)$$

Again this fixes p_1 as a f-n of x by demanding that Q be x-indep.:

$$\frac{\rho p_1}{2\mu} \frac{h^3(x)}{6} = Q - \frac{\rho h(x)}{2}$$

$$p_1 = \frac{dp}{dx} = \frac{12\mu}{\rho} \left(\frac{Q}{h^3(x)} - \frac{\rho v}{2h^2(x)} \right)$$

mass per unit time unit z-length

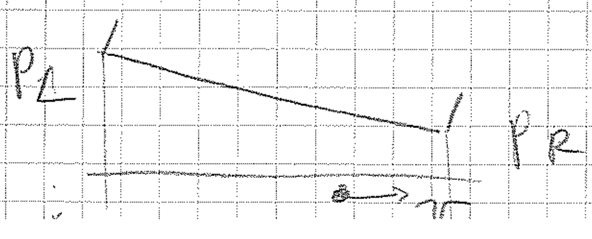
$$[Q] = \frac{kg}{s}\frac{1}{m}$$

$$\longrightarrow p_1(x) = 6\mu \left(\frac{2Q}{\rho\, h^3(x)} - \frac{v}{h^2(x)} \right)$$

$$[\rho] = \frac{kg}{m^3}$$

this gradient of pressure

Now if we know the pressure @ two ends



we can integrate

validity for slow variation not only

$$\frac{d}{d'} \sim \lambda \gg \left(\frac{U d^2}{\nu}\right)$$

$$p_R - p_L = \int_0^L dx\, p_1(x) = \frac{12\mu Q}{\rho}\int_0^L dx \frac{1}{h^3(x)} - 6\mu v \int_0^L dx \frac{1}{h^2(x)}$$

let $h(x) = h_1 - \alpha x$ as shown in picture.

then, if we also assume that object is immersed in fluid & $p_R \approx p_L = p_0$

we can determine

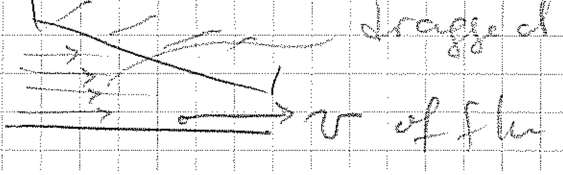
$$p(x) - p_0 = \frac{6\mu}{\alpha}\left(v\left(\frac{1}{h(x)} - \frac{1}{h_1}\right) - \frac{Q}{\rho}\left(\frac{1}{h^2(x)} - \frac{1}{h_1^2}\right)\right)$$

& demanding $p(L) = p_0$ fixes $\frac{Q}{\rho} = v\frac{h_1 h_2}{h_1 + h_2}$

plugging back into $p(x) - p_0$:

$$p(x) - p_0 = \frac{6\mu v}{\alpha}\,\frac{(h_1 - h(x))(h(x) - h_2)}{h^2(x)(h_1 + h_2)}$$

- α is small # (small inclination - large $p(x)$)
- $h(x)$ is a linear f-n in x; note $p_1(x) > p_0$ if $h_2 < h(x) < h_1$: pressure's larges some place in the middle



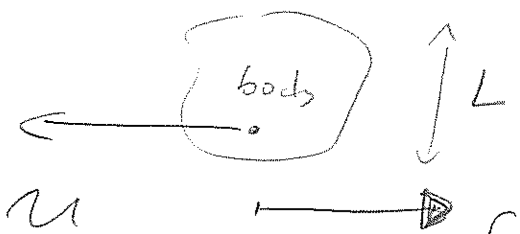
Δp can be large & lead to big normal force!

So, all there were $\approx$,

Further, they're stable @ small $R\,(\ll 1)$.

Next, we'll consider "Stokes' problem"

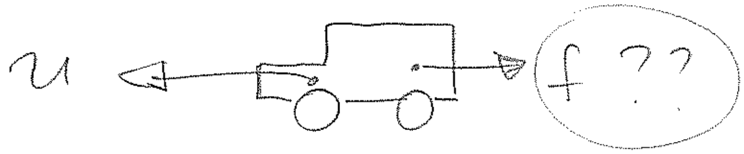
Generally:



incompressible fluid w/ ρ @ rest @ $\pm\infty$.

$f_{\text{resistance}} = ?$

Clearly practically relevant, e.g.



(or swimsuits ...)

Dim analysis: $f \sim \rho u^2 L^2 g(R)$

some function

Reynolds #

$[L^2]$ = area

$[\rho u^2]$ = pressure

pressure × area = force.

Qualitative expectations

$g(R) = ?$

$$f = \rho U^2 L^2 g(R)$$

- small R (large μ @ fixed U, L, ρ)
  - shear should dominate in this case
  - so

$$f \sim \mu \cdot (\nabla U)\big|_{\text{at body surface}} \cdot \underbrace{L^2}_{\text{surface area}} \sim \mu \frac{U}{L} \cdot L^2 \sim \mu U L$$

i.e. $f \sim \mu U L = \rho U^2 L^2 g(R)$

small-R drag linear in velocity

$\underline{f = 6\pi \mu U L}$

$\underline{\text{for sphere}}$

$\rightarrow \quad g(R) \sim \frac{1}{R}$ at small R

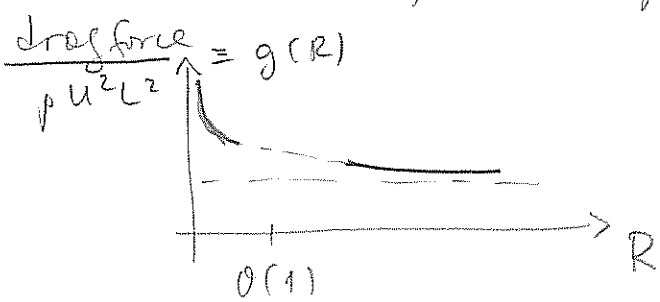
since

$$\rho U^2 L^2 \frac{1}{R} = \frac{\rho U^2 L^2}{UL} \nu \sim \mu U L$$

- large R (small μ @ fixed U, L, ρ)
  - ∇p should dominate, neglect $\frac{1}{R}$ term $\quad (\vec{u}\cdot\nabla)\vec{u} = -\nabla p \frac{1}{\rho}$

so $\nabla p \sim \frac{U^2}{L}\rho$, $p \sim U^2 \rho$

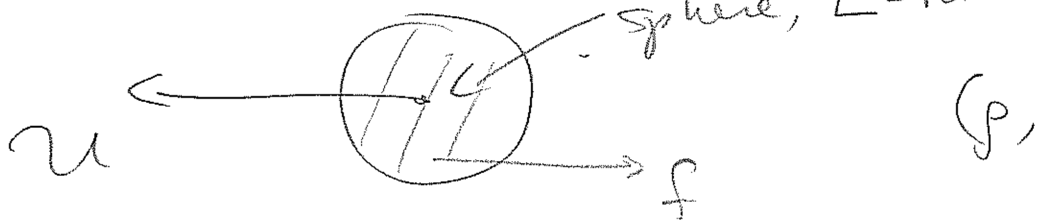
so $f \sim p L^2 \sim \rho U^2 L^2$ so $g(R) \rightarrow$ const at large R



large-R drag $\sim U^2$

$f \sim \rho U^2 L^2$, ~ ideal liquid

We'll start w/ low-R



(ρ, μ)

$\boxed{\text{Stokes' 1st problem } f = ?}$

as $R \to 0$

$$f = 6\pi\mu L U \left(1 + O(R) + \cdots\right)$$

$$R = \frac{LU}{\nu} \to 0.$$

has been called "the hydrogen atom" of fluid mechanics

- $\exists$ small dim-less # R, expand in R
- atom: $\alpha = \dfrac{e^2}{4\pi\hbar c} = \dfrac{1}{137}$ fine structure