So let's now study the Stokes problem
— a body moving w/ $\vec{U}$ = const
thru a viscous incompressible fluid
— take a sphere now (explicit sol'tn exists)
— low R ≡ "Stokes' equations" – neglect inertia & stationary

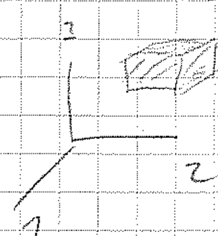
Stokes' eqns
$$\begin{cases} \mu^2 \Delta \vec{u} = \vec{\nabla} p \\ \bar{\nabla} \cdot \bar{u} = 0 \end{cases}$$

assume no body force for now

N.B.: Stokes eqtn's are linear
- any boundary motion or body force affects sol'tns linearly
- take a solid body moving w/ $\vec{U} = (U_1, U_2, U_3)$
then force on it is $\vec{F} = \bar{F}_1 + \bar{F}_2 + \bar{F}_3$

force due to motion w/ $\vec{U} = (U_1, 0, 0)$
force due to $\vec{U} = (0, U_2, 0)$
force due to $\vec{U} = (0, 0, U_3)$

- take it a cube oriented along



by symmetry same α

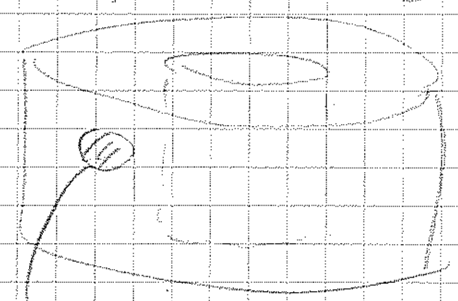
then $\vec{F_1} = (\alpha U_1, 0, 0)$, $\vec{F_2} = (0, \alpha U_2, 0)$,
$\vec{F_3} = (0, 0, \alpha U_3)$

$$\Rightarrow \quad \vec{F} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 = \alpha \, \vec{U}$$

<u>Moral:</u> at low R, the drag force on a cube is independent on its orientation.

(at higher R, not true anymore)

- also because of linearity, if sign of velocity @ boundaries is reversed — the sign of solution is reversed
  - "reversibility" of very viscous (low - R) flows
  - great <u>videos</u> (see links to be provided



dried blob $\xrightarrow{\quad}$ blob stretched long

blob goes back to original shape $\xrightarrow{\quad}$ dried blob

- Finally, as opposed to N-S. in general, soltns of Stokes' eqns: (a) exist (b) are unique when velocity field is specified on ∂V of some volume V, if $R < \pi\sqrt{3}$ (!)

There are various ways to write Stokes' equ:

$$\text{take } \vec{\nabla}\cdot(\ldots) \Rightarrow \Delta p = 0$$

$$\text{take } \vec{\nabla}\times(\ldots) \Rightarrow \nabla^2 \vec{\omega} = 0$$

$$(\bar{\omega} = \bar{\nabla}\times\vec{v})$$

$\Rightarrow$ pressure field & vorticity field are both harmonic (their Laplacian vanishes)

INTERLUDE, NEEDED:

Especially simple are "2d" incompressible flows!

Ex. 1: a "genuine" 2d flow $\vec{u} = (u_x(x,y,t), u_y(x,y,t), 0)$

$$\bar{\nabla}\cdot\bar{u} = 0 \Rightarrow \partial_x u_x + \partial_y u_y = 0$$

* Clearly, if $u_x(xyt) = \partial_y \psi(x,y,t)$

& $u_y(xyt) = -\partial_x \psi(x,y,t)$

continuity equ is satisfied, as

$$\partial_x u_x + \partial_y u_y = \partial_x\partial_y\psi - \partial_y\partial_x\psi = 0$$

* Conversely, every solution of $\partial_x u_x + \partial_y u_y = 0$ can be locally written as

$$u_x = \partial_y \psi$$

$$u_y = -\partial_x \psi$$

(Poincaré' lemma

(there can be global obstructions)

Given u_x, u_y, ψ can be found as follows $\longrightarrow$

$$\frac{\partial}{\partial y}\psi(x,y) = u_x(x,y)$$ (a set of 2 linear diff. equs)

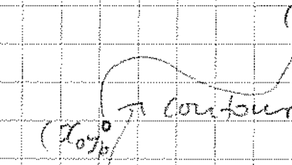
$$\frac{\partial}{\partial x}\psi(x,y) = -u_y(x,y)$$

From 1st $\psi(x,y) = \int_{y_0}^{y} u_x(x,y')\,dy' + (y\text{-indep.})$

2nd $\psi(x,y) = -\int_{x_0}^{x} u_y(x',y)\,dx' + (x\text{-indep.})$

& these can be combined into

$$\psi(x,y) - \psi(x_0,y_0) = \int_{(x_0,y_0)}^{(x,y)} (u_x\,dy - u_y\,dx)$$ contour



Proof:

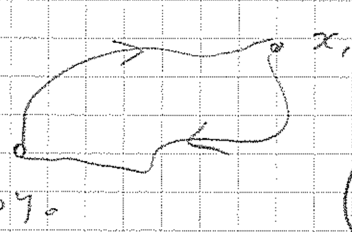
Clearly taking $\frac{\partial \psi}{\partial x} = \frac{1}{\Delta x}\left(\int_{(x_0,y_0)}^{(x+\Delta x,y)} - \int_{(x_0,y_0)}^{(x,y)} (u_x dy - u_y dx)\right)\Big|_{\Delta x \to 0}$

$$= \frac{1}{\Delta x}\int_{(x,y)}^{(x+\Delta x,y)} (-u_y\,dx) = -\frac{\Delta x\, u_y}{\Delta x} = -u_y$$

since y does not change

(same goes for $\frac{\partial \psi}{\partial y}$)

$\psi(x,y)$ is independent of the contour taken $\Longleftrightarrow$



$\Longleftrightarrow$ integral over closed contour $= 0$.

(equivalent to $\nabla\cdot\bar{u} = 0$ inside)

(so there must be only fluid inside)

the reason this introduction of ψ is convenient is that: $*$ it solves 2d $\vec{\nabla}\cdot\vec{u}=0$

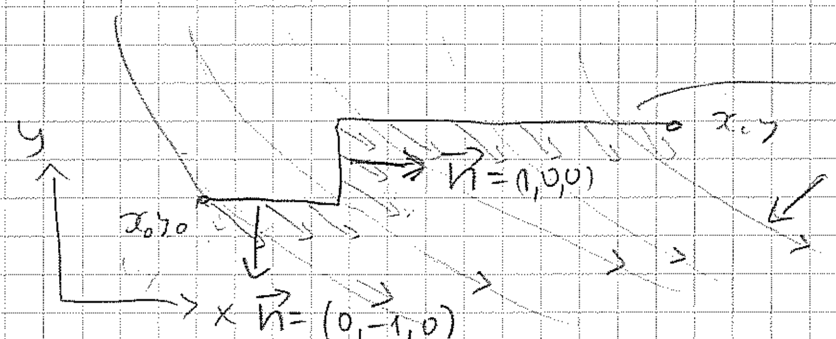
$*$ we get further equ. depending on one scalar ψ, not a vector

---

look @ ψ's def. once more to discern meaning:

$$\psi(x,y) - \psi(x_0,y_0) = \int_{(x_0,y_0)}^{(x,y)} \left(u_x\, dy + (u_y)(-1)\, dx \right)$$

take this contour



an ex. of a velocity field ($\vec{u}$ tangent to lines @ every point)

normal vector to "surface" => above $\int = \underbrace{\vec{u}\cdot\vec{n}\, dl}_{\text{exactly the flux}}$

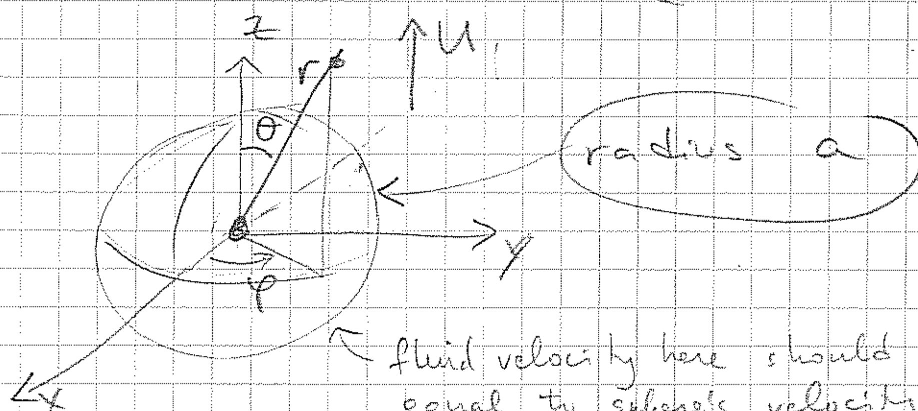
so $\psi(x,y) - \psi(x_0,y_0) \equiv$

$\equiv$ (flux of volume of incompressible fluid thru $\forall$ line joining x,y w/ $x_0 y_0$)

$\psi(x,y) \equiv$ "stream fn"

$\psi(x,y)$ greatly used in 2d
ideal incompressible flows — —

? about our Stokes problem ?



* fix coordinate system w/ spherical particle
* so condition on velocity of fluid @ surface is that it equals U

at instantaneous position !

* we'll take $\vec{U} \parallel \hat{z}$

What to expect?

- linearity says $\vec{F} = \text{const}\ \vec{U}$
- drag $= 0$ if $\mu = 0$
- so $\vec{F} = \text{const}\ \mu \vec{U}$
- should depend on a as well — —

$[\mu] = [\rho\nu] = \frac{kg}{m^3}\ \frac{m^2}{s}$

$[U] = \frac{m}{s}$

$[\mu U] = \frac{kg}{s^2}$ → to get force, need m

$[a \mu \vec{U}] = \frac{kg \cdot m}{s^2}$ ✓

So → $\vec{F} = \underbrace{(\text{const})}_{\text{dimless}}\ \mu a \vec{U}$; we are after dimless #.

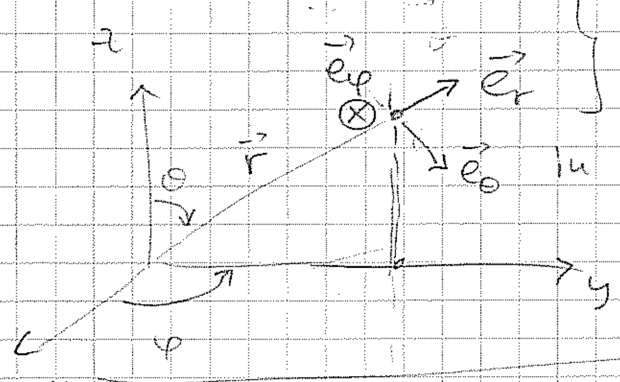
Answer is $\vec{F} = \underbrace{6\pi}\mu a \vec{U}$

- not just for the fun of solving diff eqns!
- low-R drag force useful to measuring μ for various substances! (so 6π matters, really!)

So let's get it!

- let's use $\vec{x} = (r, \theta, \varphi)$ coordinates
- b.c. are φ independent, so expect soltn to only depend on r & θ $\to$ "2 dimensionality"

$\to \vec{u} = \vec{e}_r u_r + \vec{e}_\theta u_\theta$, $u_\varphi = 0$



} unit vectors in picture here ($\varphi = \pi/2$ shown)

Gradient in spherical coordinates: $\vec{\nabla} F = \vec{e}_r \frac{\partial F}{\partial r} + \vec{e}_\theta \frac{1}{r}\frac{\partial F}{\partial \theta} + \vec{e}_\varphi \frac{1}{r\sin\theta}\frac{\partial F}{\partial \varphi}$

divergence in spherical coordinates: $\vec{\nabla}\cdot\vec{F} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 F_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta F_\theta) + \frac{1}{r\sin\theta}\frac{\partial F_\varphi}{\partial \varphi}$

Laplacian — " — :

$$\Delta F = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial F}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial F}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 F}{\partial\varphi^2}$$

curl: $\overline{\nabla}\times\overline{F} = \vec{e}_r\left(\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, F_\varphi) - \frac{\partial F_\theta}{\partial\varphi}\right)$

$+\ \vec{e}_\theta\left(\frac{1}{\sin\theta}\frac{\partial F_r}{\partial\varphi} - \frac{\partial}{\partial r}(r F_\varphi)\right)\frac{1}{r}$

$+\ \overline{e}_\varphi\left(\frac{\partial}{\partial r}(r F_\theta) - \frac{\partial F_r}{\partial\theta}\right)\frac{1}{r}$

——— u ——— will use as we need

→ 1st: solve $\overline{\nabla}\cdot\vec{u} = 0$ by symmetry $u_\varphi = 0$; u_r & $u_\theta(r,\theta)$ only.

$\vec{u} = \overline{e}_r u_r + \overline{e}_\theta u_\theta \Rightarrow$ so

$\overline{\nabla}\cdot\overline{u} = 0 \Rightarrow\quad 0 = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 u_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, u_\theta)$

$\Downarrow \times$ (by r^2 & $\sin\theta$)

$0 = \frac{\partial}{\partial r}(r^2\sin\theta\, u_r) + \frac{\partial}{\partial\theta}(r\sin\theta\, u_\theta)$

(recall $0 = \frac{\partial}{\partial x}u_x + \frac{\partial}{\partial y}u_y \Rightarrow \begin{array}{l} u_x = \partial_y\psi \\ u_y = -\partial_x\psi \end{array}$)

so $r^2\sin\theta\, u_r = \partial_\theta\,\psi(r,\theta)$

$r\sin\theta\, u_\theta = -\partial_r\,\psi(r,\theta)$

hence taking $u_r = \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\psi$

& $u_\theta = -\frac{1}{r\sin\theta}\frac{\partial}{\partial r}\psi$ solves axisymm $\overline{\nabla}\cdot\overline{u} = 0$

Stokes' eqn $\rightarrow$ $\mu \Delta \vec{u} = \vec{\nabla} p$ imply $(\vec{\nabla} \times \ldots)$

$$\Delta(\vec{\nabla} \times \vec{u}) = 0$$

$\vec{\nabla} \times \vec{u}$, for $\vec{u}$ given ito Ψ, is $= ?$

$$\vec{\omega} = \vec{\nabla} \times \vec{u} = (\text{for } \bar{u} = u_r, u_\theta, \ \varphi\text{-indep.})$$

$$= \vec{e}_\varphi \frac{1}{r}\left(\frac{\partial}{\partial r}(r\, u_\theta) - \frac{\partial}{\partial \theta}(u_r)\right)$$

- vorticity in this axisymmetric flow is $\sim \vec{e}_\varphi$ <u>only</u>

so: $\omega_\varphi = \frac{1}{r}\frac{\partial}{\partial r}(r\, u_\theta) - \frac{1}{r}\frac{\partial}{\partial \theta} u_r =$ sub u_r, u_θ thru Ψ

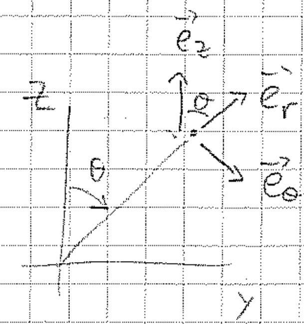
$$= \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{-1}{\sin\theta}\frac{\partial}{\partial r}\Psi\right) - \frac{1}{r}\frac{\partial}{\partial \theta}\left(\frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\Psi\right)$$

$$= -\frac{1}{r \sin\theta}\frac{\partial^2 \Psi}{\partial r^2} - \frac{1}{r^3}\frac{\partial}{\partial \theta}\left(\frac{1}{\sin\theta}\frac{\partial \Psi}{\partial \theta}\right).$$

and we're told $\Delta \omega_\varphi = 0 \Leftarrow$ (Stokes' eqn.)

+ b.c. @ $r = a$ are $\vec{u}\Big|_{r=a} = \vec{e}_z U$

$$\vec{u}\Big|_{r=a} \cdot \vec{e}_r \equiv u_r\Big|_{r=a} = U\, \bar{e}_z \cdot \bar{e}_r = U \cos\theta$$



$$\bar{u}\Big|_{r=a} \cdot \bar{e}_\theta \equiv u_\theta\Big|_{r=a} = U\, \bar{e}_z \cdot \bar{e}_\theta = U \cos\left(\theta + \frac{\pi}{2}\right)$$
$$= -U \sin\theta$$

so

$$u_r\Big|_{r=a} = U \cos\theta = \frac{1}{a^2 \sin\theta}\frac{\partial \Psi}{\partial \theta}\Big|_{r=a}$$

$$u_\theta\Big|_{r=a} = -U \sin\theta = -\frac{1}{r \sin\theta}\frac{\partial \Psi}{\partial r}\Big|_{r=a}$$

So we have

$$\begin{cases} \left.\dfrac{\partial \psi}{\partial \theta}\right|_{r=a} = \dfrac{1}{2} U a^2 \sin 2\theta \\ \left.\dfrac{\partial \psi}{\partial r}\right|_{r=a} = U a \sin^2\theta \end{cases}$$

+ Now we must solve $\Delta \omega_\varphi = 0 \Leftarrow\!\!\not\Rightarrow (\Delta \vec{\omega} = 0)$
subject to above b.c. for ψ @ $r=a$.

+ away from sphere, as $r \to \infty$, it must be that
$\vec{u}|_{fluid} \to 0$, so we're looking for a "dying" soln @ ∞

+ <u>recall simple facts</u>: "Gauß' law" $\Delta \varphi = 4\pi \rho$

φ: el.-static potential; ρ: charge density.

$\rho = Q\,\delta^{(3)}(\vec{x})$ — point charge @ $\vec{x}=0$: field is $\varphi \sim \dfrac{1}{r}$

$Q \overset{\vec{\ell}}{\longleftrightarrow} -Q$ $\quad \vec{d} = e\vec{\ell}$: dipole : field is $\varphi \sim \dfrac{\vec{d}\cdot\vec{r}}{r^3}$

these solve Laplace $=0$ away from $\vec{r}=0$

charge $\varphi \sim \dfrac{1}{r}$ so $\displaystyle\oint_{S^2_\infty} \varphi \sim Q$

dipole : $\varphi \sim \dfrac{1}{r^2}$ so $\displaystyle\oint_{\infty} = 0$ (no charge inside)

Sol'n of Δ(anything) $= 0$ $\Longleftrightarrow$ expansion in multipoles

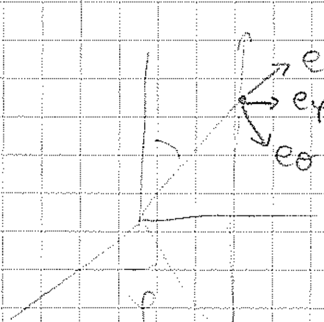
- monopole $\sim 1/r$
- dipole $\sim \frac{1}{r^2}$
- quadrupole $\sim \frac{1}{r^3}$ — —

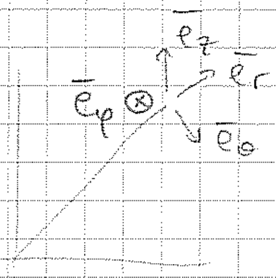
———— " ————

Here $\Delta \vec{\omega} = 0 \Rightarrow \bar{\omega}$ is a vector

dipole term : $\vec{\omega} = \text{const}\ \dfrac{\vec{U} \times \vec{r}}{r^3}$ is leading (no monopole possible, as no scalar)

Of course, $\bar{U} \times \bar{r}$, w/ $\bar{U} \parallel \hat{z}$ can only have nonzero φ.





$\bar{e}_z \times \bar{e}_r = \vec{e}_\varphi \sin\theta$ & $\bar{e}_z \times \bar{e}_\varphi = \emptyset$.

$\bar{e}_z \times \bar{e}_\theta = -\vec{e}_\varphi \cos\theta$

So $\vec{U} \times \bar{r} = U \vec{e}_z \times (r \vec{e}_r) = U \vec{e}_\varphi \sin\theta\, r$

So $\omega_\varphi = \dfrac{U \sin\theta\, r}{r^3} \times \text{const.} \Longleftarrow$ determined by solving $\Delta \vec{\omega} = 0$ & taking dipole term Ansatz (consistent w/ $\bar{\omega} \to 0$ as $\vec{r} \to \infty$

On the other hand

$$\omega_\varphi = -\frac{1}{r\sin\theta}\frac{\partial^2\psi}{\partial r^2} - \frac{1}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial\psi}{\partial\theta}\right)$$

p (126).

by def. of $\bar{\omega} = \bar{\nabla}\times\bar{u}$ & soltn of $\bar{\nabla}\cdot\bar{u} = 0$.

So, equate the two:

$$-\frac{1}{r\sin\theta}\frac{\partial^2\psi}{\partial r^2} - \frac{1}{r^3}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial\psi}{\partial\theta}\right) = C\,\frac{U\sin\theta}{r^2}$$

or

$$\frac{\partial^2\psi}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial\psi}{\partial\theta}\right) = -C\,\frac{U\sin^2\theta}{r}$$

$$+\quad \left.\frac{\partial\psi}{\partial r}\right|_{r=a} = Ua\sin^2\theta$$

$$\left.\frac{\partial\psi}{\partial\theta}\right|_{r=a} = \frac{1}{2}Ua^2\sin 2\theta = \frac{1}{2}Ua^2\frac{\partial}{\partial\theta}(\sin^2\theta)$$

staring @ b.c. : take $\underline{\psi \sim \sin^2\theta}$.

hence take $\psi = f(r)\,U\sin^2\theta$. as a try.

b.c. says $f'(r) \to a$ as $r\to a$

& $f(r) \to \frac{a^2}{2}$ as $r \to a$.

w/ $\psi = f(r)\, U \sin^2\theta$, $f(a) = \frac{a^2}{2}$

$f'(a) = a$

eqn for f is

$$- f''(r)\, U + \sin\theta\, \frac{f(r)\, U}{r^2} \underbrace{\partial_\theta\left(\frac{1}{\sin\theta}\partial_\theta \sin^2\theta\right)} = -C\, \frac{U \sin^2\theta}{r}$$

$$\partial_\theta \frac{2\sin\theta\cos\theta}{\cancel{\sin\theta}} = -2\sin\theta$$

$$f'' U \sin^2\theta - 2\frac{f}{r^2} U \sin^2\theta = -C\frac{U}{r}\sin^2\theta$$

so $$f'' - \frac{2f}{r^2} = -\frac{C}{r}$$

General solution: $f = \beta r^\alpha$

$$\alpha(\alpha-1)\beta r^{\alpha-2} - 2\beta r^{\alpha-2} = -\frac{C}{r}$$

$$(\alpha(\alpha-1) - 2)\, r^{\alpha-2}\beta = -\frac{C}{r}$$

$\Rightarrow$ $\alpha = 1$, $\beta = \frac{1}{2}$ solves inhomogeneous eqn.

& $\underbrace{\alpha(\alpha-1) = 2}$ solves homogeneous

$\alpha = 2$

$\alpha = -1$

<u>Generally</u>: $f = \frac{1}{2} C r + B r^2 + D \frac{1}{r}$

Now as $r \to \infty$, $f/r^2 \to 0$ $\left(\frac{1}{r} \partial f / \partial r \to \phi \right)$

Velocity field following from $\psi = U f(r) \sin^2\theta$ is

D14 $$\vec{u} = \vec{U} \frac{1}{r} \frac{\partial f}{\partial r} + \vec{r} \frac{\vec{r} \cdot \vec{U}}{r^2} \left(\frac{2f}{r^2} - \frac{1}{r} \frac{\partial f}{\partial r} \right)$$

$\left(\vec{U} = U \vec{e_z} \right)$

hence $B = 0$ & $f = \frac{1}{2} C r + \frac{D}{r}$

$$f' = \frac{1}{2} C - \frac{D}{r^2}$$

$$f(a) = \frac{a^2}{2} = \frac{C}{2} a + \frac{D}{a} \quad \Rightarrow \quad C = \frac{3}{2} a$$

$$f'(a) = a = \frac{C}{2} - \frac{D}{a^2} \qquad D = -\frac{a^3}{4}$$

(check: $\frac{3}{4} a^2 - \frac{1}{4} a^2 = \frac{a^2}{2}$

$\frac{3}{4} a + \frac{a^3}{4a^2} = a$ ✓)

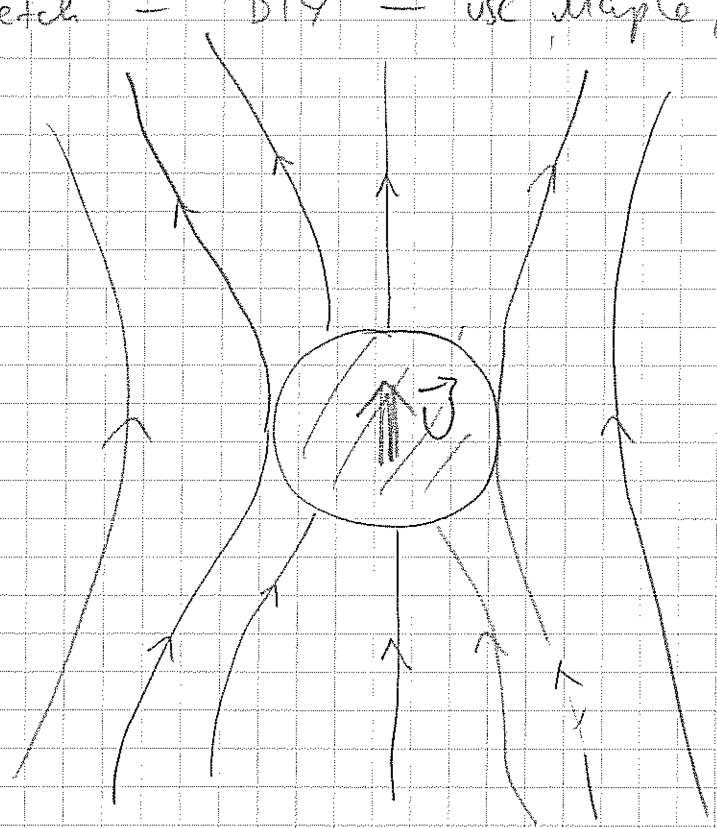
So $$\psi = U r^2 \sin^2\theta \left(\frac{3}{4} \frac{a}{r} - \frac{a^3}{4r^3} \right)$$

$\psi(r, \theta)$ - streamfunction

recall $u_r = \frac{1}{r^2 \sin\theta} \partial_\theta \psi$

$u_\theta = -\frac{1}{r \sin\theta} \partial_r \psi$

Sketch – DIY – use Maple, Mathematica, whatever



- this is much more symmetric than I plotted
 (axisymmetric, really)

- velocity falls off like $\frac{1}{r}$ as $r \to \infty$

$$\vec{u} \approx \vec{U} \frac{3a \sin^2\theta \ldots}{4r} + \ldots$$

so $\mu \Delta u \sim \frac{\mu a}{r^3} U$ far away

while $\rho(\vec{u}\cdot\vec{\nabla})\vec{u} \approx \rho \; \vec{U}\cdot\vec{\nabla}\vec{U} \sim \rho \frac{1}{r^2} a U^2$

relative to sphere velocity

depending on how small R breaks down at sufficiently large r

$$\frac{\text{inertia}}{\text{viscosity}} \sim \frac{\rho a U^2 / r^2}{U\mu a / r^3} \sim \frac{rU}{\nu} \sim \frac{r}{a}\left(\frac{Ua}{\nu}\right) \sim \left(\frac{r}{a}\right)R$$

– not so good far away!

Finally – the force!

Need

normal to sphere

$$\sigma_{ij}\Big|_{r=a} n_j =$$

$$= \left(-p\,\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\right)\Bigg|_{r=a} n_j = (*)$$

DIY: use velocity field $\vec{u} = \vec{U}\,\frac{1}{r} f' + \vec{r}\,\frac{\vec{r}\cdot\vec{U}}{r^2}\left(\frac{2f}{r^2} - \frac{1}{r} f'\right)$

to show that

$$(*) = \Big(-p n_i + \mu n_i\, \vec{U}\cdot\vec{n}\left(-\frac{f''}{r} + \frac{6f'}{r^2} - \frac{10 f}{r^3}\right)$$

$$+ \mu U_i \left(\frac{f''}{r} - \frac{2f'}{r^2} + \frac{2f}{r^3}\right)\Big)_{r=a} = (**)$$

to find $p \longrightarrow$ need to use $\vec{\nabla} p = \mu \Delta \vec{u}$ or, just a bit smarter, since $\bar{\omega} = \bar{\nabla}\times\bar{u}$, note that

$$\bar{\nabla}\times\bar{\omega} = \bar{\nabla}\times(\bar{\nabla}\times\bar{u}) = -(\bar{\nabla}\cdot\bar{\nabla})\,\bar{u} = -\Delta\bar{u}$$

$$\Rightarrow \bar{\nabla} p = \mu\, \bar{\nabla}\times\bar{\omega} \quad ; \quad \text{also recall } \Delta p = 0 \text{, too:}$$

pressure @ ∞

but $\vec{\omega} = C\,\dfrac{\vec{U}\times\vec{r}}{r^3}$, hence $p = \mu\, C\,\dfrac{\bar{U}\cdot\bar{x}}{r^3} + p_0$

same coeff fixed by $\bar{\nabla} p = \mu\, \bar{\nabla}\times\bar{\omega}$.

→ then put p & f & C into (**)

to find

$$\sigma_{ij}\Big|_{r=a} n_j = -p_0 n_i - \frac{3\mu U_i}{2a}$$

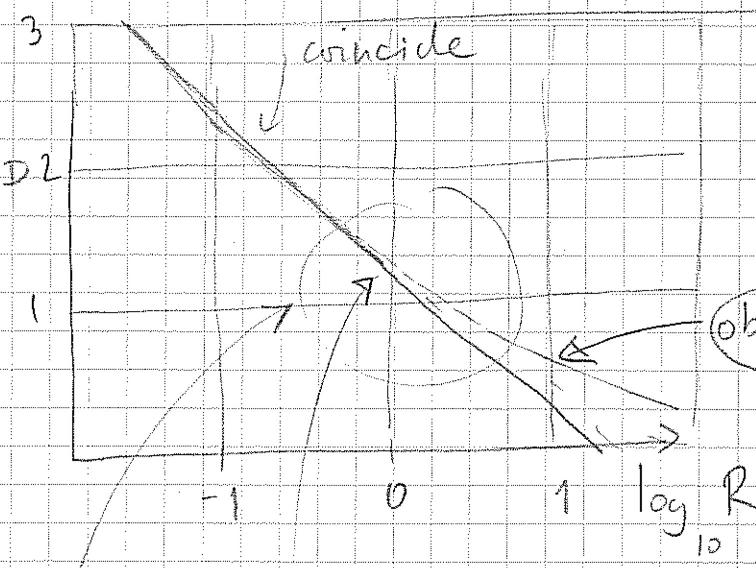
this doesnot contribute
after $a^2\int d\varphi\, d\theta \sin\theta\, n^i = 0$

but
since $U_i = U\delta_{i3}$

we have $\underbrace{a^2\int d\varphi\, d\theta \sin\theta}_{4\pi a^2}\ \frac{3\mu U_i}{2a} = 6\pi a \mu U_i$

So force on sphere is

$$\vec{F} = -\vec{U}\, 6\pi a \mu$$



Stokes' law
(an 2nd order improvement)

drag

$$C_D = \frac{|\vec{F}|}{\left(\frac{1}{2}\rho U^2\right)(\pi a^2)} = \frac{24}{R}$$

x-section

$$R = 2\frac{aU}{\nu}$$

(2 is historical, here)