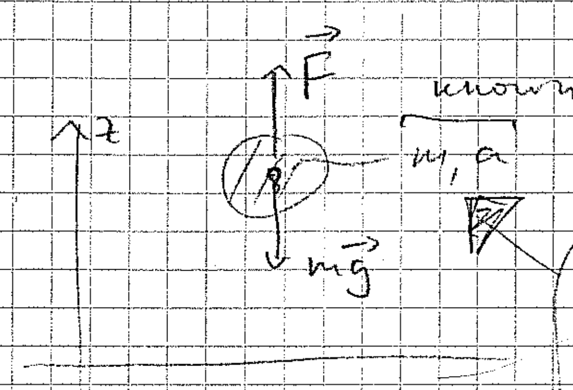
Since $\vec{F} = -6\pi\mu a \vec{U}$

Use to measure μ

$$m\ddot{z} = -mg + 6\pi\mu a \dot{z}$$



Neglected buoyancy, trivial to include in eqn.

$\Downarrow$

limiting velocity is

rhs = 0

$$v_{lim.} = \dot{z} = \frac{mg}{6\pi\mu a} \Rightarrow \boxed{\mu = \frac{mg}{6\pi a v_{lim}}}$$

(very good @ $R \ll 1$)

(up to $R \sim 0.3$ or so)

Can also solve rigid ellipsoid (DIY) — :)

For a body of arbitrary shape $F_i = 4\pi\mu \underbrace{P_{ij}}_{} U_j d$

tensor depending on shape of body

& flow field at large distances is like that for a sphere, i.e. axisymmetric w.r.t. $P_{ij} U_j$ - axis.

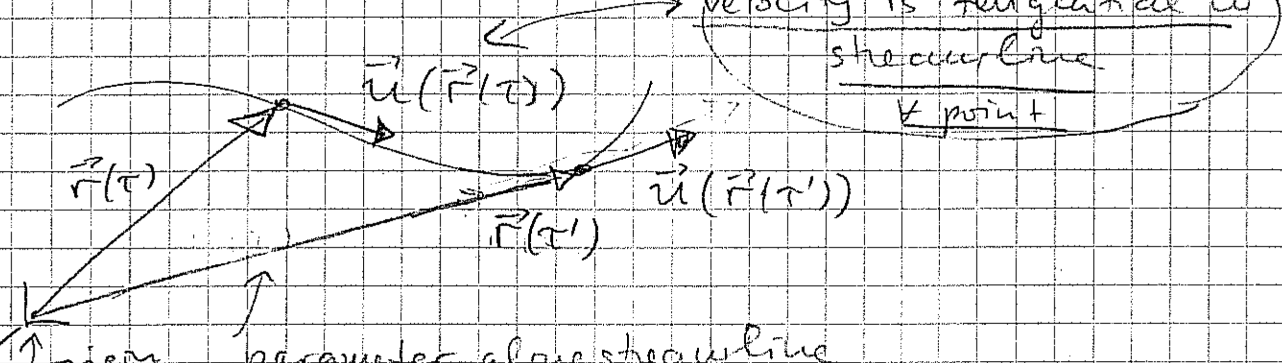
No general useful recipe for P_{ij}, however...

$\longrightarrow$ measure direction of force - find $P_{ij} U_j$ direction —

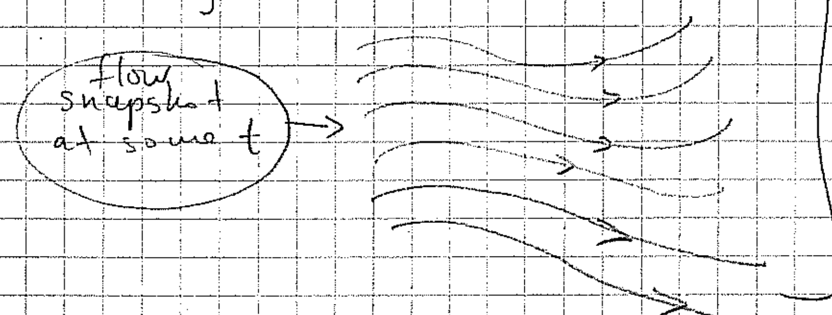
$\longrightarrow$ far away flow field is $\simeq$ " stokeslet at origin " picture on p. 132 $\longrightarrow$ "not so good" (interpret??)

* let's find streamlines of Stokes' flow

recall def. from 1st class:



[entire flow is "foliated" by streamlines e.g.



]

i.e. "E.O.M." for streamlines:

$$\frac{d\vec{r}(\tau)}{d\tau} = \vec{u}(\vec{r}(\tau), t)$$

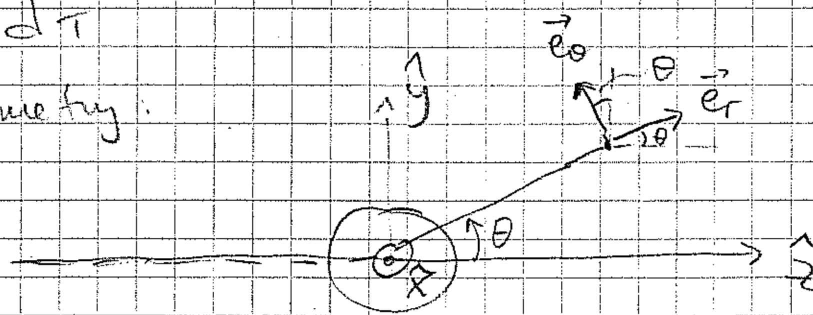
if flow NOT stationary, streamlines "wiggle" - different ∀ t

fluid flow field

our Stokes' flow is stationary, so streamlines do not "wiggle"!

$$\frac{d\vec{r}(\tau)}{d\tau} = \vec{u}(\vec{r}(\tau)).$$

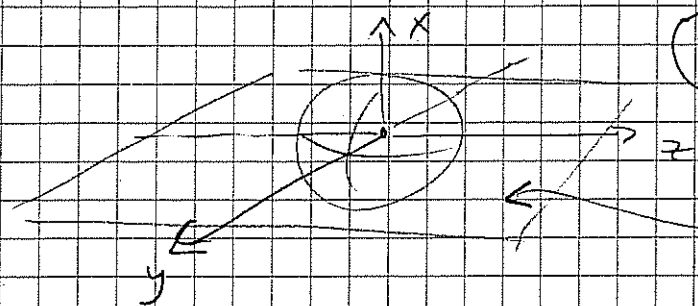
We have axisymmetry:



(137)

so we'll take a slice @ $x=0$:

(of flow)



we're looking @ this place

Furthermore, we have $y \Leftrightarrow -y$ symmetry, so

Ok to look @ $y>0$ only

velocity vector has u_z & u_y only &

streamlines are found by solving:

$$\frac{d\,y(\tau)}{d\tau} = u_y\big(y(\tau), z(\tau)\big)$$

$$\frac{d\,z(\tau)}{d\tau} = u_z\big(y(\tau), z(\tau)\big)$$

Now recall in 2d incompressible flow we had

introduced stream function:

$$\partial_x u_x + \partial_y u_y = 0 \Rightarrow u_x = \partial_y \Psi(x,y),\ u_y = -\partial_x \Psi(x,y)$$

so equ. for streamlines $(x(\tau), y(\tau))$:

$$\frac{dx(\tau)}{d\tau} = u_x(x(\tau), y(\tau)) = \frac{\partial}{\partial y}\Psi(x(\tau), y(\tau))$$

$$\frac{d\,y(\tau)}{d\tau} = u_y(x(\tau), y(\tau)) = -\frac{\partial}{\partial x}\Psi(x(\tau), y(\tau))$$

$$\frac{dx}{d\tau} = \frac{\partial\Psi}{\partial y},\quad \frac{dy}{d\tau} = -\frac{\partial\Psi}{\partial x} \Rightarrow d\tau = \frac{dx}{\partial\Psi/\partial y} = -\frac{dy}{\partial\Psi/\partial x}.$$

$\Rightarrow$ streamline equ:

$$\frac{dx}{\frac{\partial \psi}{\partial y}} = -\frac{dy}{\frac{\partial \psi}{\partial x}}$$

or

$$\frac{\partial \psi}{\partial x}\,dx = -\frac{\partial \psi}{\partial y}\,dy$$

or

$$\frac{\partial \psi}{\partial x}\,dx + \frac{\partial \psi}{\partial y}\,dy = 0$$

or $\quad d\left(\psi(x,y)\right) = 0$

$\Rightarrow$ streamline in x,y-plane

$\Updownarrow$

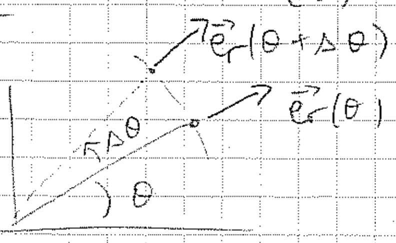
lines where $\psi(x,y) = \text{const.}$
(since $d\psi = 0$ on streamlines)

MORAL: If $\psi(x,y)$ – stream function
then: curves of constant stream f'n $\equiv$ streamlines

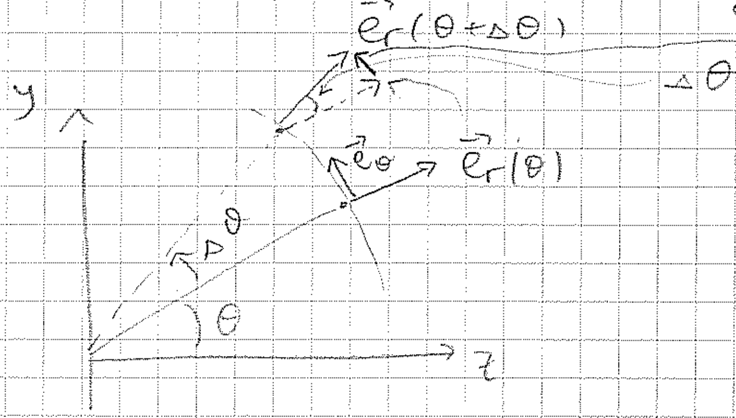
What about our (r,θ) - streamlines?

a slight subtlety: $(r(\tau), \theta(\tau)) \equiv$ streamline, clearly

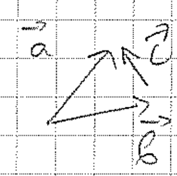
Equ? $\quad \vec{r}(\tau) = r(\tau)\,\vec{e}_r(\theta(\tau))$: 'cause $\vec{e}_r$ depends on θ



$$\frac{\vec{e}_r(\theta+\Delta\theta)-\vec{e}_r(\theta)}{\Delta\theta} = \frac{d\,\vec{e}_r}{d\theta}$$



$$\vec{e}_r(\theta+\Delta\theta) - \vec{e}_r(\theta) = \Delta\theta\,\vec{e}_\theta$$



$$\vec{b} + \vec{c} = \vec{a}$$

$$\vec{c} = \vec{a} - \vec{b}$$

$$\therefore \quad \frac{d\,\vec{e}_r}{d\theta} = \vec{e}_\theta$$

so; in (r,θ) we have

$$\frac{d\,\vec{r}(\tau)}{d\tau} = \frac{dr}{d\tau}\vec{e}_r + r\frac{d\,\vec{e}_r}{d\theta}\frac{d\theta}{d\tau}$$

$$\frac{d\,\bar{r}}{d\tau} = \frac{dr}{d\tau}\bar{e}_r + r\,\bar{e}_\theta\frac{d\theta}{d\tau} \Longleftarrow \left(\begin{array}{c}\text{well known: velocity =} \\ \text{= radial + angular} \\ (\dot{r}) \qquad (r\dot{\theta})\end{array}\right)$$

$$\frac{d\,\bar{r}}{d\tau} = \vec{u} = u_r\,\vec{e}_r + u_\theta\,\vec{e}_\theta$$

hence: $\dfrac{dr}{d\tau} = u_r(r,\theta)$ $\quad\Rightarrow\quad \dfrac{dr}{u_r} = r\dfrac{d\theta}{u_\theta}$

$$r\frac{d\theta}{d\tau} = u_\theta(r,\theta)$$

Now $\quad u_r = \dfrac{1}{r^2\sin\theta}\dfrac{\partial\psi}{\partial\theta}, \quad u_\theta = -\dfrac{1}{r\sin\theta}\dfrac{\partial\psi}{\partial r}$

hence streamline eqn is $\left(\frac{dr}{u_r} = r \frac{d\theta}{u_\theta} \right)$.

$$\frac{dr}{u_r} = \frac{dr}{\frac{1}{r^2 \sin\theta} \frac{\partial \psi}{\partial \theta}} = r^2 \sin\theta \frac{dr}{\left(\frac{\partial \psi}{\partial \theta}\right)}$$

$$\parallel$$

$$r \frac{d\theta}{u_\theta} = r \frac{d\theta}{-\frac{1}{r \sin\theta} \frac{\partial \psi}{\partial r}} = -r^2 \sin\theta \frac{d\theta}{\left(\frac{\partial \psi}{\partial r}\right)}$$

or $$r^2 \sin\theta \frac{dr}{\frac{\partial \psi}{\partial \theta}} = -r^2 \sin\theta \frac{d\theta}{\frac{\partial \psi}{\partial r}}$$

$$\Rightarrow \quad \frac{\partial \psi}{\partial r} dr = -\frac{\partial \psi}{\partial \theta} d\theta \quad \Rightarrow$$

$$\Rightarrow \quad \frac{\partial \psi}{\partial r} dr + \frac{\partial \psi}{\partial \theta} d\theta = 0$$

or $$\underline{d(\psi(r,\theta)) = 0.}$$

$\Rightarrow$ streamlines $\equiv$ lines of $\psi(r,\theta) = $ const.

$$\psi(r,\theta) = \frac{U r^2 \sin^2\theta}{4} \left(3\frac{a}{r} - \left(\frac{a}{r}\right)^3 \right)$$

p131

$$\psi = \text{const} \equiv \frac{U}{4} r^2 \sin^2\theta \frac{a}{r} \left(3 - \left(\frac{a}{r}\right)^2 \right)$$

or $$\boxed{r \left(3 - \left(\frac{a}{r}\right)^2 \right) \sin^2\theta = \text{const}.}$$

$$\psi = \frac{U}{4} \; r\left(3 - \frac{1}{r^2}\right) \sin^2\theta = \text{cst}$$

positive $(0 - \infty)$

$(a = 1)$

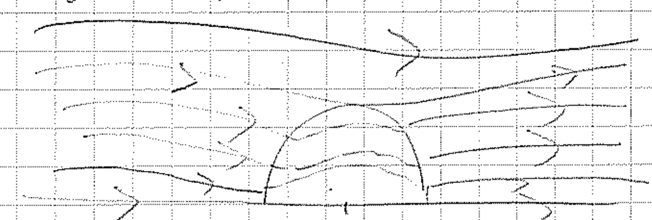
$$\left(3r - \frac{1}{r}\right)\sin^2\theta = c$$

$$3r^2\sin^2\theta - \sin^2\theta = c\,r$$

$$3\sin^2\theta\, r - c\,r - \sin^2\theta = 0$$

$$r = \frac{1}{6}\frac{1}{\sin^2\theta}\left(c + \sqrt{c^2 + 12\sin^4\theta}\right)$$

This gives plot in Batchelor copied on (p 132):

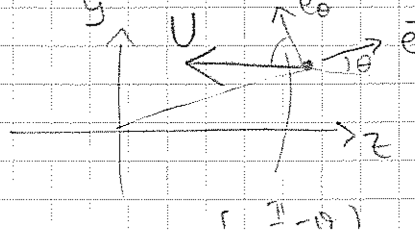


$\Longrightarrow$ misleading – frame where sphere instantaneously @ rest ...

$\Longrightarrow$ go to frame where sphere is @ rest

$$\vec{u}(r,\theta) \longrightarrow \underbrace{-U\vec{e}_z}_{\delta\psi = ?} + \underbrace{\vec{u}(r,\theta)}_{\text{we know } \psi \text{ for this}}$$

$$-U\vec{e}_z = -U\cos\theta\,\vec{e}_r + U\sin\theta\,\vec{e}_\theta \;\Longrightarrow\; \begin{cases} -U\cos\theta = \dfrac{1}{r^2\sin\theta}\,\partial_\theta\,\delta\psi \\ U\sin\theta = -\dfrac{1}{r\sin\theta}\,\partial_r\,\delta\psi \end{cases}$$



$\delta\psi$ = streamf-n for $-U\vec{e}_z$ flow.

$$\partial_\theta \delta\psi = -U r^2 \sin\theta\cos\theta$$

$$\partial_r \delta\psi = -U r \sin^2\theta$$

$$\Rightarrow \quad \delta\psi = -\frac{U r^2}{2}\sin^2\theta$$

check $\quad \partial_\theta \delta\psi = -\frac{Ur^2}{2}\partial_\theta \sin^2\theta = -\frac{Ur^2}{2} 2\sin\theta\cos\theta$

$$\partial_r \delta\psi = -U r \sin^2\theta$$

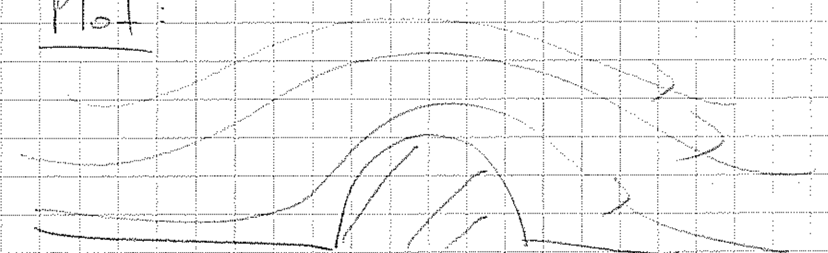
so we have

$$\psi + \delta\psi = \frac{U}{4} r\left(3 - \frac{1}{r^2}\right)\sin^2\theta - \frac{Ur^2}{2}\sin^2\theta$$

$$= \frac{U}{4}\underbrace{\left[\overbrace{\left(-2r^2 + 3r - \frac{1}{r}\right)}^{=0 \text{ @ } r=1,\ <0 \text{ @ } r>1}\sin^2\theta\right]}_{\text{streamlines = contours of this.}}$$

so $\quad \underline{\left(2r^2 - 3r + \frac{1}{r}\right)\sin^2\theta} = \text{const.}$

<u>Plot:</u>



* see Mathematica plot, sphere of unit radius
– streamlines for two cases.

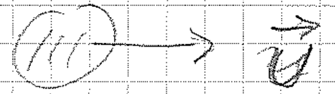
MORAL: • care is always needed

• only for stationary flows

velocity field
pressure field — indep. of t

$$\bar{u} = \bar{u}(\bar{x})$$
$$p = p(\bar{x})$$

are streamlines $\equiv$ lines which physical fluid elements follow



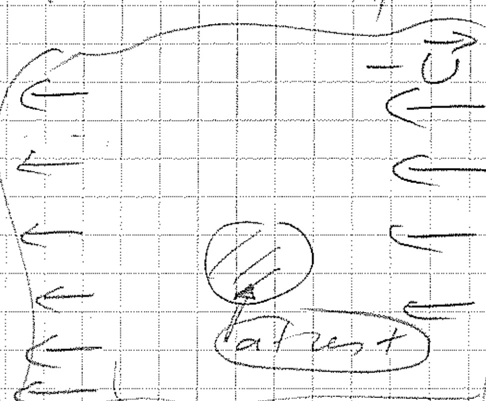
sphere moving thru fluid w/ $\vec{u}$

at this point $\bar{u}(\bar{x}, t)$

clearly depends on t.

NOT a stationary flow

VS.



$-\vec{U}$

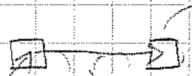
at rest

fluid incident on sphere w/ $-\vec{U}$

$\Rightarrow$ stationary flow.

$\vec{u}(\bar{x}, t)$ — t-independent!

recall for physical fluid elements



$$\bar{x}(t+\Delta t) = \bar{x}(t) + \bar{u}(x(t), t)\,\Delta t$$

$x(t)$ $\quad$ $\vec{u}(x(t), t)$

$$\Rightarrow \boxed{\frac{d\vec{x}(t)}{dt} = \bar{u}(\vec{x}(t), t)}$$

"pathline": particle trajectory

! DIFFERENCE
same if STEADY

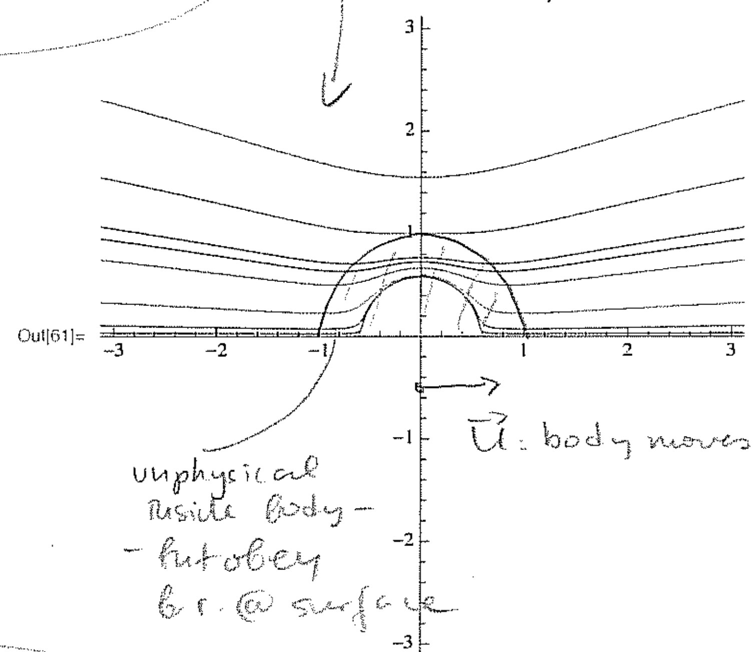
while streamlines: snapshot of $\bar{u}(\bar{x}, t)$

t: at given t: $\bar{x}(\tau)$: $\dfrac{d\bar{x}(\tau)}{d\tau} = \bar{u}(\bar{x}(\tau), t)$

streamlines

$$\left(3r - \frac{1}{r}\right)\sin^2\theta = \text{const.}$$

unsteady flow
(moves w/ body)
streamlines $\neq$ pathlines



streamlines

$$\text{const} = \left(2r^2 - 3r + \frac{1}{r}\right)\sin^2\theta$$

steady flow:
streamlines =
= pathlines

$\vec{U}_{fluid}$