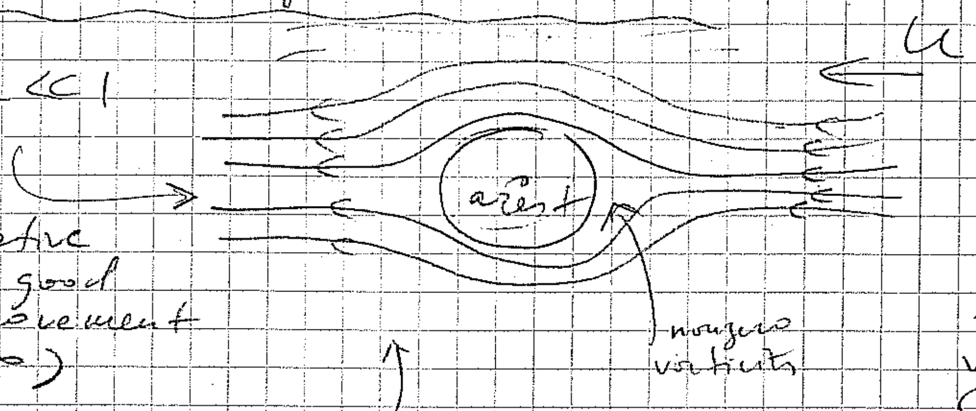
What happens to the flow as $R \nearrow$?

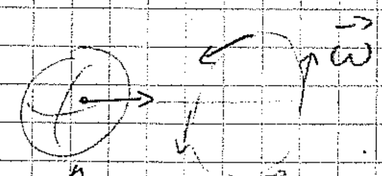
* $R \ll 1$



qualitative picture good (and improvement as $r \to \infty$)

* remember this flow was described by

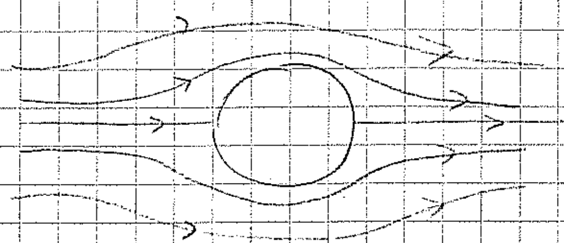
$$\Delta \vec{\omega} = 0 \ , \quad \vec{\omega} = \omega_\varphi \vec{e}_\varphi$$



$$\omega_\varphi = \frac{U \sin\theta}{r^2}$$

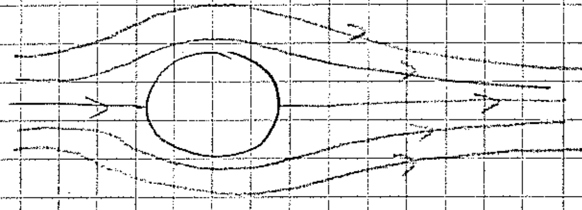
* vorticity is generated at surface of the body by the dragging of the fluid and then diffuses away in all directions as if from a stationary source
* the flow, in the Stokes' eqns ($R \ll 1$) approx is fore- and after- symmetric
* in improved approximation, vorticity diffuses from a moving source, breaking the before-after symmetry, far away from the body (where Stokes' breaks down)

this suggests that as $R \uparrow$ asymmetry before-after will be more pronounced for a cylinder [experiments / numerical simulations]



$R = 0.25$

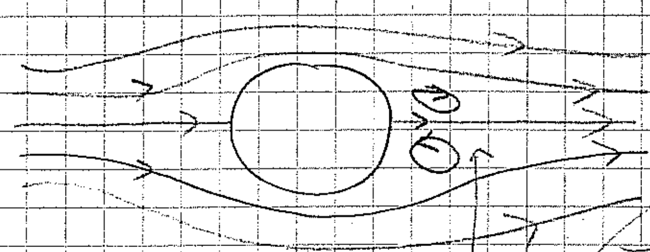
($\approx$ fore/aft-symmetry)



$R \approx 3.5$

(fore-aft-symmetry gone)

(1st at $R = 6$, apparently)



$R \sim 9$

"standing eddies"

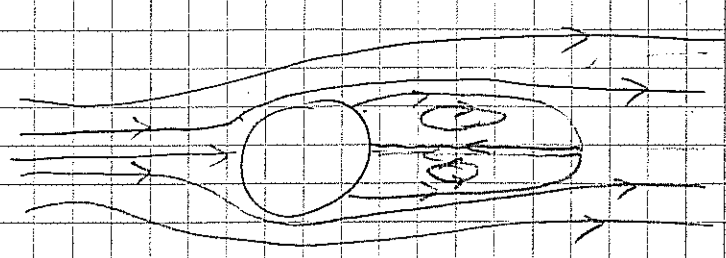
as R increases this region w/ closed streamlines behind cylinder becomes longer & wider

drag at higher R

$$\sim \underbrace{\pi a^2}_{\text{x-section}} \underbrace{\frac{\rho U^2}{2}}_{\text{variation of pressure}} f(R) \sim a^2 \rho U^2$$

i.e. $f(R) \sim \frac{1}{R}$ small R, $f(R) \sim$ const. larger R

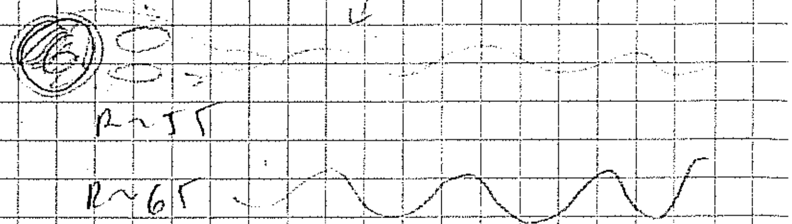
e.g.



$R = 40$ etc.

wake instability

In addition, far away



$R \sim 55$

$R \sim 65$

etc.

So we're led to considering
<u>flows at large(r) R</u>

it would appear that, as we did at small-R = drop inertia, keep $\nu \Delta \vec{u}$
here, we could $\underbrace{\text{keep inertia \& drop } \nu\Delta\vec{u}}$ -term

unfortunately, this does not agree w/ what's observed

(this would imply that large-R flows $\equiv$ ideal fluid flows)

- main difference lies in effect of boundaries, where ν becomes relevant
- as we saw in sphere case, nontrivial $\vec{\omega}$ (vorticity) is generated @ boundaries & diffuses away
turns out that vorticity $\vec{\omega}$ can not be created or destroyed in interior of homogeneous fluid & is produced @ boundaries.

So we begin <u>by studying EOM for $\vec{\omega}$:</u>

start w/ NS:

$$\rho\left(\frac{\partial \vec{u}}{\partial t} + (\vec{u}\,\vec{\nabla})\vec{u}\right) = \vec{F} - \vec{\nabla} p + \mu \Delta \vec{u}$$

let $\rho = \text{const}$, $\vec{\nabla}\cdot\vec{u} = 0$

so
$$\frac{\partial \vec{u}}{\partial t} + (\vec{u}\vec{\nabla})u = \frac{\vec{F}}{\rho} - \vec{\nabla}\left(\frac{p}{\rho}\right) + \nu \Delta \vec{u}$$

some tricks — (GOAL — get eqn for $\vec{\omega}$)

- $$\vec{u} \times (\vec{\nabla} \times \vec{u}) = \frac{1}{2} \vec{\nabla}(\vec{u} \cdot \vec{u}) - (\vec{u} \cdot \vec{\nabla}) \vec{u}$$

Proof: $\epsilon^{ijk} u^j (\bar{\nabla} \times \bar{u})^k = \epsilon^{ijk} u^j \epsilon^{klm} \partial^l u^m =$

$$\epsilon^{ijk} \epsilon^{klm} = \delta^{il} \delta^{jm} - \delta^{im} \delta^{jl}$$

$$= (\delta^{il} \delta^{jm} - \delta^{im} \delta^{jl})(u^j \partial^l u^m) =$$

$$= u^m \partial^i u^m - u^j \partial^j u^i$$

$$= \frac{1}{2} \partial^i (u^m u^m) - (u^j \partial^j) u^i \quad \boxtimes$$

hence $(\bar{u} \cdot \bar{\nabla}) \bar{u} = -\bar{u} \times \bar{\omega} + \frac{1}{2} \vec{\nabla}(\vec{u}^2)$

plug into NS:

$$\frac{\partial \bar{u}}{\partial t} - \bar{u} \times \bar{\omega} = \frac{\bar{F}}{\rho} - \bar{\nabla}\left(\frac{p}{\rho} + \frac{1}{2} \vec{u}^2\right) + \nu \Delta \vec{u}$$

take $\bar{\nabla} \times (\_) \Rightarrow$

assume $\bar{F} = \bar{\nabla} \psi$

$$\frac{\partial \vec{\omega}}{\partial t} - \vec{\nabla} \times (\vec{u} \times \vec{\omega}) = \cancel{\vec{\nabla} \times \left(\frac{\bar{F}}{\rho}\right)}^{\,\to\, \emptyset} - \cancel{\vec{\nabla} \times \vec{\nabla}}^{\,\to\, \emptyset}\left(\frac{p}{\rho} + \frac{1}{2} \vec{u}^2\right) + \nu \Delta (\vec{\nabla} \times \vec{u})$$

$$\frac{\partial \bar{\omega}}{\partial t} - \underline{\bar{\nabla} \times (\bar{u} \times \bar{\omega})} = \nu \Delta \vec{\omega}$$

$$\epsilon^{ijk} \partial^j (\bar{u} \times \bar{\omega})^k = \epsilon^{ijk} \partial^j \epsilon^{klm} u^l \omega^m =$$

$$= (\delta^{il} \delta^{jm} - \delta^{im} \delta^{jl}) \partial^j (u^l \omega^m) = \partial^m (u^i \omega^m) - \partial^m (u^m \omega^i)$$

$$= (\partial^m u^i) \omega^m + \underbrace{u^i \partial^m \omega^m}_{0} - \underbrace{(\partial^m u^m)}_{0 \text{ incomp}} \omega^i - u^m \partial^m \omega^i = \omega^m \partial^m u^i - u^m \partial^m \omega^i$$

$$(\omega^m = \epsilon^{mpq} \partial^p u^q)$$

or, in short $\bar{\nabla} \times (\bar{u} \times \bar{\omega}) = (\bar{\omega}\,\bar{\nabla})\,\bar{u} - (\bar{u}\,\bar{\nabla})\,\bar{\omega}$

$$\Rightarrow \boxed{\frac{\partial \bar{\omega}}{\partial t} + (\bar{u}\,\bar{\nabla})\,\bar{\omega} = (\bar{\omega}\,\bar{\nabla})\,\bar{u} + \nu \Delta \bar{\omega}}$$

or
$$\boxed{\frac{D\vec{\omega}}{Dt} = \nu \Delta \vec{\omega} + (\vec{\omega}\,\vec{\nabla})\,\vec{u}}$$

- this is an interesting eqn:

for sphere's Stokes' flow

$$\bar{\omega} = \omega_\varphi \, \bar{e}_\varphi$$

& $\bar{u}$ is φ-independent

hence $(\bar{\omega}\,\bar{\nabla})\,\bar{u} = 0$: { for 2d flows / for unidirectional flow

or

$$\underbrace{\frac{D\vec{\omega}}{Dt} = \nu \Delta \vec{\omega}}$$

this is really a "diffusion eqn" for vorticity!!

$$\left(\frac{\partial n}{\partial t} = \underset{\uparrow}{D}\, \Delta n \right)$$

diffusion coefft

- written as

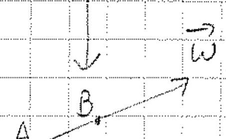
$$\dot{\vec{\omega}} = -\underset{\nearrow}{(\bar{u}\,\bar{\nabla})}\,\bar{\omega} + \underbrace{(\bar{\omega}\cdot\bar{\nabla})\,\bar{u}} + \nu \underset{\uparrow}{\Delta \bar{\omega}}$$

same as $(\bar{u}\,\bar{\nabla})\,\bar{u}$ for velocity "convective" term

$\rightarrow \dot{\omega} \neq 0$ 'cause of nonuniformity of ω along the flow

"diffusion" of $\underbrace{\bar{\omega}}$ (like "diffusion" of $\vec{u}$)

(same coefft. ν)



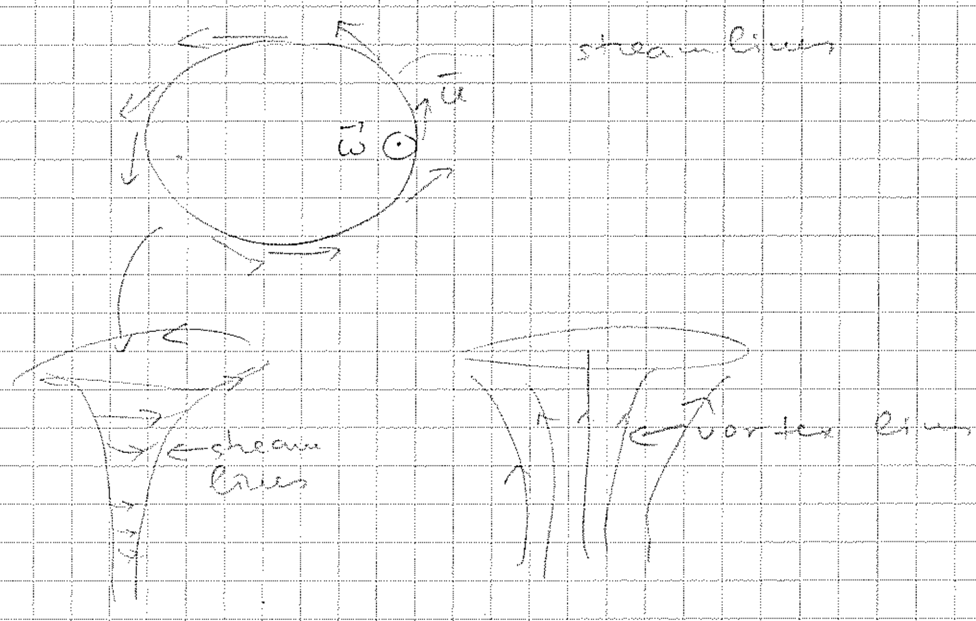
$\vec{\omega} \rightarrow$ local dirn of $\bar{\omega}$

$(\bar{\omega}\cdot\bar{\nabla})\,\bar{u} = |\omega| \left.\frac{\delta \vec{u}}{|AB|}\right|$, $\delta \bar{u} = \bar{u}_B - \bar{u}_A$

just like <u>streamlines</u> = lines s.t. tangent vectors $\underbrace{\text{ }}_{\text{at } \bar{x},\, t}$ are $= \vec{u}(\bar{x}, t)$

one can introduce <u>vortex lines</u> = lines s.t. tangent vector @ $\bar{x}, t$ $= \vec{\omega}(\bar{x}, t)$

e.g. for your hurricane problem of hw. 2:



(coming up)

Concepts of <u>vorticity</u> & <u>circulation</u> play important role in inviscid fluid flows & airlift theory...

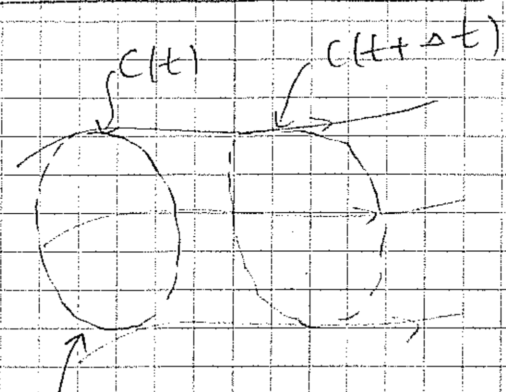
<u>Our plan:</u> – high-R ⇒ might as well take $\nu = 0$

– study simple 2d flow & <u>Zhukowski's lift theorem</u>

– implies paradoxes ⇒ boundary layers... (~ν important)

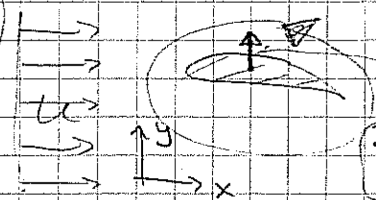
((that'll be all!))

Circulation & vorticity:



importance of Γ:

"Zhukovski lift th...."



$$\boxed{F_y = -\rho U \Gamma}$$

material curve, closed ← every point move w/ fluid ($\in C(t)$) according to:

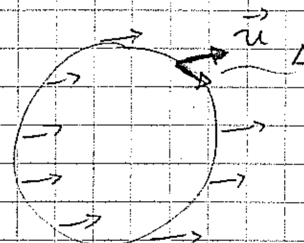
$$\frac{d\vec{x}(t)}{dt} = \vec{u}(\vec{x}(t), t)$$

not necessarily stationary flow

"circulation"

$$\Gamma(t) = \oint_{C(t)} \vec{u} \cdot d\vec{l}$$

e.g.



$$\boxed{\Gamma = \sum_{C(t)} \vec{u} \cdot \Delta\vec{l}}$$

Γ depends on time because (a) $\vec{u} = \vec{u}(\vec{x}, t)$

(b) points on contour move w/ flow

$$\frac{d\Gamma(t)}{dt} = \frac{d}{dt} \oint_{C(t)} \underbrace{\vec{u}(\vec{x}(t), t)}_{\vec{x} \in C(t)} \cdot \underbrace{d\vec{l}(t)}_{\text{tangent vector to } C(t) \text{ @ } \vec{x} \in C(t)} =$$

$$= \oint_{C(t)} \frac{d}{dt}\left(\vec{u}(\vec{x}(t), t)\right) \cdot d\vec{l}(t) + \oint_{C(t)} \vec{u}(\vec{x}(t), t) \frac{d}{dt}\left(d\vec{l}(t)\right)$$

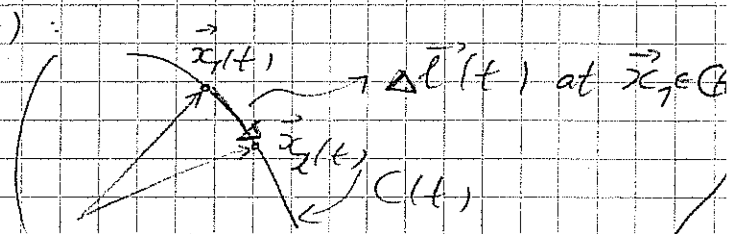
$$\frac{d}{dt}\, u^i(\bar{x}(t), t) = \frac{\partial u^i}{\partial x^j} \underbrace{\frac{\partial x^j}{\partial t}}_{u^j} + \frac{\partial u^i}{\partial t} =$$

u^j, by e.o.m. of physical fluid element

$$= u^j \partial_j u^i + \dot{u}^i = \frac{D u^i}{D t}$$

$$?\ \frac{d}{dt}\, d\vec{l}(t)\ ?$$

@ t: $\Delta \vec{l}(t)$: $\lim\limits_{x_2 \to x_1}$



$$\Delta \vec{l}(t+\Delta t) = \bar{x}_2(t+\Delta t) - \bar{x}_1(t+\Delta t)$$

$$= \bar{x}_2(t) + \Delta t\, \vec{u}(x_2(t), t)$$

$$- \bar{x}_1(t) - \Delta t\, \vec{u}(x_1(t), t)$$

$$\Delta \vec{l}(t+\Delta t) = \Delta \vec{l}(t) + \Delta t \left(\vec{u}(x_2(t), t) - \vec{u}(x_1(t), t \right.$$

$$\Delta l^i(t+\Delta t) - \Delta l^i(t) = \Delta t \left(u^i(\vec{x}_2(t), t) - u^i(\vec{x}_1(t), t) \right)$$

when $x_2 \to x_1$

$$= \Delta t \Big(u^i(\bar{x}_1(t) + \Delta \vec{l}(t), t)$$

$$- u^i(\bar{x}_1(t), t) \Big) =$$

$$= \Delta t\, \frac{\partial u^i}{\partial x^j}(\vec{x}_1(t), t)\, \Delta l^j(t)$$

so

$$\frac{\Delta \vec{l}}{\Delta t} = (\Delta \vec{l} \cdot \vec{\nabla})\, \vec{u}$$

So —

$$\frac{1}{2}\partial_j(\vec{u}^2)\, d\ell^j = \frac{1}{2}\, d\vec{\ell}\,\vec{\nabla}\left(\frac{\vec{u}^2}{2}\right)$$

$$\frac{d\Gamma}{dt} = \oint_{C(t)} \underbrace{\frac{D\vec{u}}{Dt}}_{\downarrow\, NS}\cdot d\vec{\ell} + \oint_{C(t)} u^i\,\partial_j u^i\, d\ell^j$$

$$= \underbrace{\oint_{C(t)} \frac{\vec{F}}{\rho}\cdot d\vec{\ell}}_{} - \oint_{C(t)} \frac{1}{\rho}\vec{\nabla}p\cdot d\vec{\ell} + \oint_{C(t)} \nu\,\Delta\vec{u}\cdot d\vec{\ell}$$

let $\vec{F} = -\vec{\nabla}\psi$
+ $\rho = \text{const}$

$$+ \oint_{C(t)} \vec{\nabla}\left(\frac{1}{2}\vec{u}^2\right) d\vec{\ell}$$

$$\oint_{C(t)} \vec{\nabla}\psi\, d\vec{\ell} = \phi$$

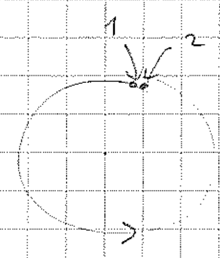
↓ single valued

$$\mathring{\Gamma} = -\int_{C(t)} \underbrace{\vec{\nabla}\left(\frac{p}{\rho} + \frac{1}{2}\vec{u}^2\right)\cdot d\vec{\ell}}_{} + \int_{C(t)} \nu\,\Delta\vec{u}\cdot d\vec{\ell}$$

if p & $\vec{u}^2$
are single valued
around $C(t)$

$\equiv 0.$

$$\left(\int_C \vec{\nabla} f\, d\vec{\ell} = \right.$$



$$= f(2) - f(1) = 0$$

since $1 = 2$
closed contour)

relation to $\vec{\omega}$: $\vec{\nabla}\times\vec{\omega} = \vec{\nabla}\times(\vec{\nabla}\times\vec{u}) = -\Delta\vec{u}$

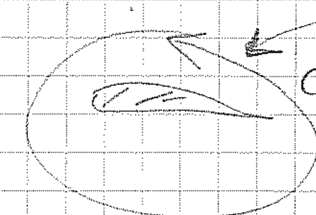
so
$$\boxed{\mathring{\Gamma} = -\nu\int_{C(t)} (\vec{\nabla}\times\vec{\omega})\cdot d\vec{\ell}}$$

very general, $\forall\, C(t)$ – closed
s.t. ψ, p, $\vec{u}^2$ are single valued

Moral $\rightarrow$ $\dot{\Gamma} = 0$ for inviscid fluid

- circulation around closed contours of material fluid elements is conserved (contour moves w/ fluid)
- again – single valuedness of ψ ($\vec{F} = -\vec{\nabla}\psi$)
  $p, \vec{u}^2$ used – but not simple connectedness of $C(t)$!

"Kelvin's circulation theorem"



not simply connected

$C(t)$

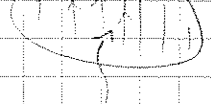
only $\nu \neq 0$ changes Γ

Once again:

$$\dot{\Gamma}(t) = \frac{d}{dt}\left(\int_{C(t)} \vec{u}\cdot d\vec{l}\right) = -\nu \int_{C(t)} (\vec{\nabla}\times\vec{\omega})\cdot d\vec{l}$$

$$\Gamma = \int_{C(t)} \vec{u}\cdot d\vec{l} = \int_{S(t)} \vec{\nabla}\times\vec{u}\cdot d\vec{S} = \int_{S(t)} \vec{\omega}\cdot d\vec{S}$$

vorticity



$C(t)$

$S(t)$

Γ = total vorticity inside any fluid surface w/ boundary $C(t)$

- $\dot{\Gamma}$ only depends on $\vec{\nabla}\times\vec{\omega}$ near $C(t)$ (i.e. locally)
- even for viscous fluid, if ν is small (high R) & $\vec{\nabla}\times\vec{\omega}$ is small: $\dot{\Gamma} \approx 0$

generally $\dot{\Gamma}$ will be appreciable only where $\vec{\nabla}\times\vec{\omega}$ is substantial (to overcome $\nu \rightarrow 0$)

$\Rightarrow$ near boundaries! $\Leftrightarrow$ where $\vec{\omega}$ is created

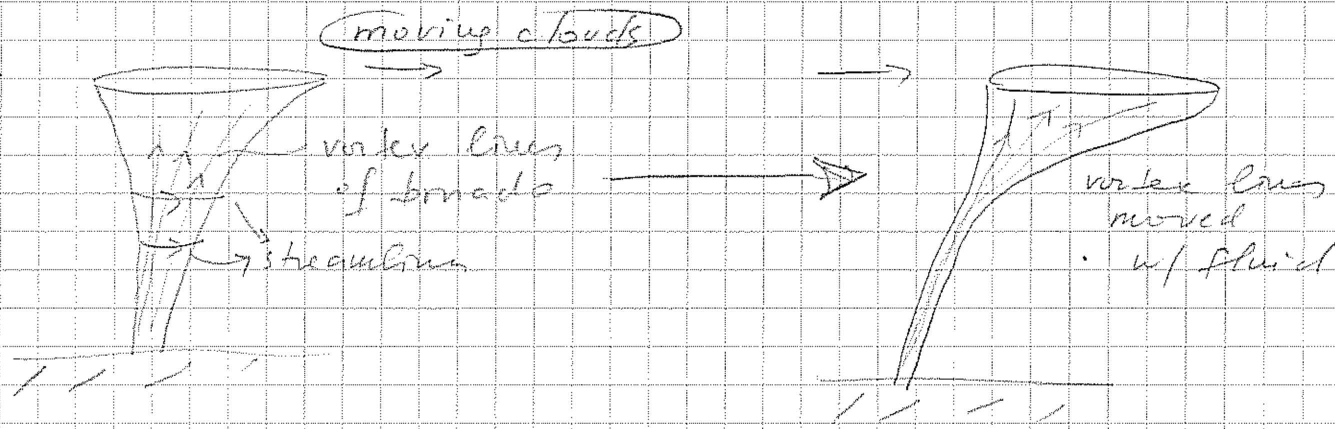
Ampere's law $\oint_C \vec{A}\,d\vec{l} = \int_S \vec{B}\,d\vec{S}$

the conservation of circulation of an inviscid incompressible fluid combined w/ the fact that "vortex lines move w/ the fluid" has some powerful consequences

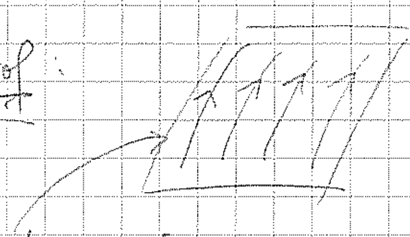
↳ a.k.a Helmholtz vortex theorem

what it says is that fluid elements that lie in a vortex line continue to do so

e.g.

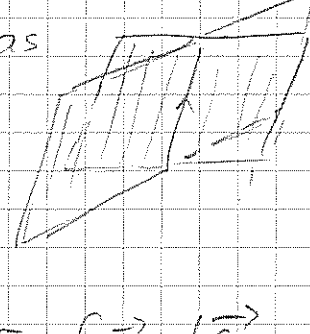


→ proof:



vortex surface: every element is a vortex line

always can view vortex line as intersection of two vortex surfaces



let $C(t)$ - a contour lying on a vortex surface

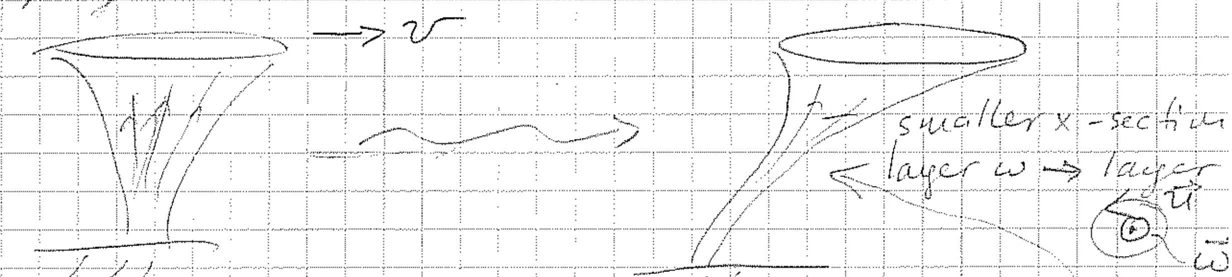
$\Gamma(t) = 0$ since $\Gamma = \int \vec{\omega} \cdot d\vec{S} = 0$ $(\underbrace{\vec{\omega}}_{\parallel \text{surface}} \perp \underbrace{d\vec{S}}_{\perp \text{surface}})$

now since $\dot{\Gamma} = 0$, & $\Gamma = 0$ $\forall\, C(t) \in$ (vortex surface)

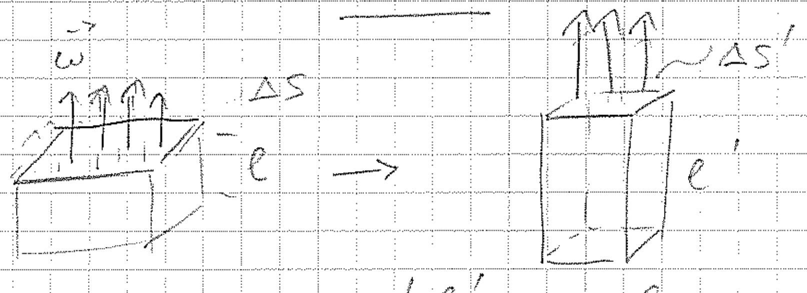
$\Rightarrow$ $\vec{\omega} \perp d\vec{S}$ $\forall t$ $\Rightarrow$ a vortex surface remains a vortex surface as time goes on → & this holds for any vortex surface

Hence, the intersection of two vortex surfaces (i.e. our vortex line) will remain the intersection of two vortex surfaces ⇒ "vortex line remains a vortex line" or "physical fluid element that lie on a vortex line at some t remain on the vortex line at later t's" — so "vortex lines move w/ flow of fluid" (incompressible, inviscid)

Now back to our tornado — since v-x lines move w/ fluid



motion of clouds stretches the vortex



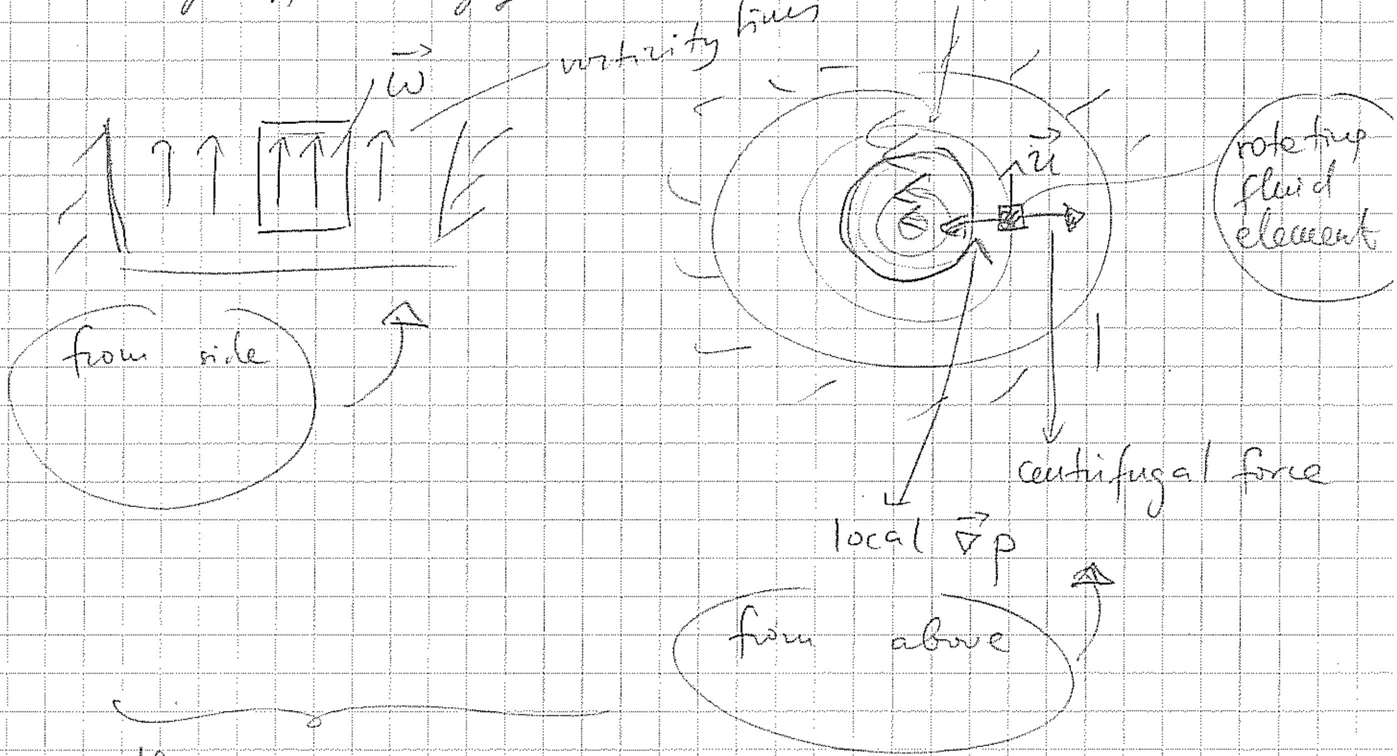
$$\Delta S' l' = \Delta S \cdot l \Rightarrow l \uparrow \rightarrow \Delta S \downarrow$$

$$\text{but } \Gamma \sim \omega \Delta S \Rightarrow \Delta S \downarrow \rightarrow \omega \uparrow$$

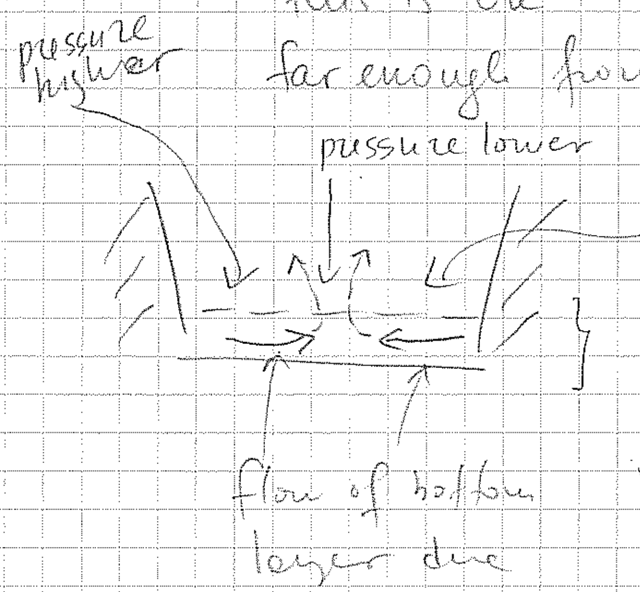
⇒ stretching a vortex ⇒ increased $\vec{\omega}$

or v.v. compressing a vortex ⇒ decreased $\vec{\omega}$

the latter case ("compressing" or "shortening" of vortex) is responsible for spin-down in a stirred cup of coffee:

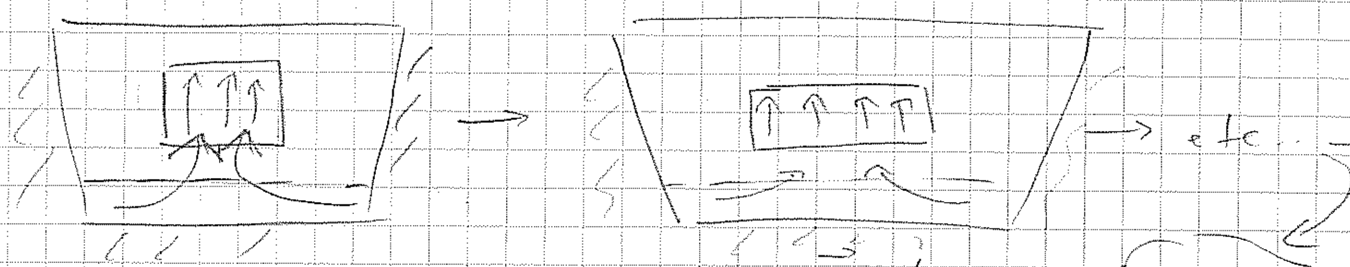


this is OK
far enough from boundaries, where ν is important



} balance between $\vec{\nabla} p$ & centrifugal
worse near bottom: $\vec{\nabla} p$ pushes bottom layer inwards
is inward & upwards @ center bottom — these carry much less vorticity (as bottom viscous layer rotates more slowly) so vorticity near bottom is decreased $\longrightarrow$

In effect, vortex lines become shorter



so $\vec{\omega} \downarrow$, & ~~tea~~ spins down.

This story can be made more precise by the study of boundary layers (we'll do a bit of qualitative analysis later).

Before that, let's do some classic FM "stuff"

2D inviscid, incompressible & irrotational flow & the Zhukovski lift theorem

- we've almost done all preparatory work, since we already introduced the stream function ...