Inviscid flows — good approx. away from boundaries (when separation of boundary layer does not occur)

"$R \to \infty$" — much simpler to analyze

even when boundary layer separation occurs, inviscid flow describes well large parts of the flow (e.g. outside wake)

+

Incompressible

↳ (all conditions of incompressibility should apply)

+

"Irrotational" $\equiv$ $\vec{\omega} = \nabla \times \vec{u} = 0$

↳ due to "$\dot{\Gamma} = 0$"

remember, really $\underbrace{\dot{\Gamma} = -\nu \int_{C(t)} \nabla \times \bar{\omega} \cdot d\bar{l}}$

creation of circulation important @ small ν in regions near boundarie

& $\Gamma = \int_{C(t)} \bar{u} \cdot d\bar{l} = \int_{S(t)} \bar{\omega} \cdot d\bar{S}$ $(\partial S = C)$

- if $\Gamma = 0$ at some moment of time (e.g $t = 0$) $\forall C \Rightarrow \bar{\omega} = 0$

$\Rightarrow$ flow is irrotational at $t = 0$

- since $\dot{\Gamma} = 0 \Rightarrow$ flow remains irrotational @ $t > 0$
- $\vec{\omega}$ may be created near boundaries ... separate treatment

$\Rightarrow$ so, it's useful to consider inviscid, irrotational & incompressible flows

$\overline{\nabla}\cdot\overline{u} = 0$ — inviscid

$\vec{\nabla}\times\vec{u} = 0$ — irrotational

Claim: $\vec{\nabla}\times\vec{u} = 0 \Rightarrow \vec{u} = \vec{\nabla}\phi$

(Poincaré's lemma, again)

(Proof — electrostatics: $\left|\begin{array}{l}\vec{\nabla}\cdot\vec{E} = 4\pi\rho \\ \vec{\nabla}\times\vec{E} = 0 \Leftrightarrow \overline{E} = \overline{\nabla}\phi \Rightarrow \Delta\phi = 4\pi\rho\end{array}\right.$)

"ϕ" s.t. $\overline{u} = \vec{\nabla}\phi$ is called "velocity potential"

$+ \quad \overline{\nabla}\cdot\overline{u} = 0 \Rightarrow \Delta\phi = 0 \Rightarrow$ velocity potential is harmonic

irrotational, incompressible, inviscid flow

$\parallel\!\parallel$

Laplace eqn.

all this is true in 3d (general)

Now, let's consider 2d flows $\Rightarrow$ basically we let $\vec{u}$ have (u_x, u_y) only & $u_x(x,y)$, $u_y(x,y)$.

It's still true (in 2d) that $\vec{u} = \vec{\nabla}\phi \Rightarrow u_x = \partial_x\phi$, $u_y = \partial_y\phi$

$\phi = \phi(x,y)$

But $\overline{\nabla}\cdot\overline{u} = 0$ in 2d means also $u_x = \partial_y\psi$ & $u_y = -\partial_x\psi$

So we have that 2d incompressible, inviscid flow is described by

$$u_x = \partial_x \phi = \partial_y \psi$$
$$u_y = \partial_y \phi = -\partial_x \psi$$

which obey this relation!

potential $\phi(x,y)$
+
stream f-n $\psi(x,y)$

<u>Reminder</u>: $z = x + iy$ (z: $\mathbb{C}$ - variable)

$f(z)$ - analytic f-n ($\mathbb{C} \xrightarrow{f} \mathbb{C}$)

depends on z **NOT** z^*

$$f(z) = \operatorname{Re} f(z) + i \operatorname{Im} f(z)$$

$$\frac{\partial}{\partial z^*} f(z) = 0 \iff \text{statement of analyticity}$$

$$z = x + iy$$
$$z^* = x - iy$$

$$\frac{\partial}{\partial z} = \frac{1}{2}(\partial_x - i\partial_y) \qquad \frac{\partial}{\partial z} z = \frac{1}{2}(\partial_x - i\partial_y)(x+iy)$$
$$= \frac{1}{2}(1+1) = 1$$

$$\frac{\partial}{\partial z} z^* = \frac{1}{2}(\partial_x - i\partial_y)(x-iy)$$

$$\frac{\partial}{\partial z^*} = \frac{1}{2}(\partial_x + i\partial_y) \qquad = \frac{1}{2}(1-1) = 0$$

$$\frac{\partial}{\partial z^*} f(z) = 0 \iff \frac{1}{2}(\partial_x + i\partial_y)(\operatorname{Re} f + i \operatorname{Im} f) = 0 \iff$$

$$\iff \partial_x(\operatorname{Re} f) + i\,\partial_x(\operatorname{Im} f) + i(\partial_y \operatorname{Re} f) - \partial_y(\operatorname{Im} f) = 0 \iff$$

$\Leftrightarrow$ since Re & Im of expression should both vanish $\Leftrightarrow$

$$\partial_x (\mathrm{Re} f) - \partial_y (\mathrm{Im} f) = 0$$

$$\partial_x (\mathrm{Im} f) + \partial_y (\mathrm{Re} f) = 0$$

$$\Leftrightarrow \quad \partial_x (\mathrm{Re} f) = \partial_y (\mathrm{Im} f) \quad \text{(Cauchy-Riemann conditions)}$$

$$\partial_x (\mathrm{Im} f) = -\partial_y (\mathrm{Re} f)$$

$\Updownarrow$ c f

$\exists$ another choice, clearly, taking $\Phi = \mathrm{Im} f$, $\Psi = -\mathrm{Re} f$ also works

$$\partial_x \phi = \partial_y \Psi$$

$$\partial_y \phi = -\partial_x \Psi$$

$\Rightarrow$ $\Phi = \mathrm{Re}$(of analytic f-n)

$\Psi = \mathrm{Im}$(of an analytic f-n)

$\Rightarrow$ introduce $\| w(z) = \phi + i\Psi \Rightarrow$ "<u>complex potential</u>"

$\| z = x + iy$, ϕ - potential

Ψ - stream f-n

Ok, but what eq-ns do ϕ & Ψ obey?

* we already showed $\Delta \phi = 0$ (irrotational + incompressible)

* Ψ is the stream function; it is also harmonic for irrot. flow

<u>Proof</u>: $u_x = \partial_y \Psi$, $u_y = -\partial_x \Psi$ $\to$ but (u_x, u_y) is <u>irrotational</u>, i.e.

$$\partial_y u_x - \partial_x u_y = 0 \Rightarrow$$

$$\Rightarrow \partial_y^2 \Psi - \partial_x(-\partial_x \Psi) = 0 \Rightarrow \underline{\Delta \Psi = 0}$$

? about Euler's equ –

– recall p. 146 – 147

– we took NS

– we took $\vec{\nabla}\times$ NS

obtained $\dot{\vec{\omega}} = \vec{\nabla}\times(\vec{u}\times\vec{\omega}) + \nu\Delta\vec{\omega}$

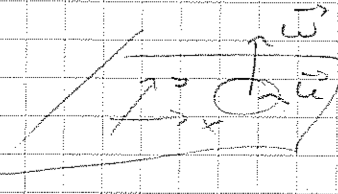
(0, here inviscid)

Now since we're in 2d

$\omega^i = \epsilon^{ijk}\partial_j u_k$

$\downarrow$ only 3-component of $\vec{\omega}$ survives

$\omega^3 = \partial_x u_y - \partial_y u_x$



but $u_y = -\partial_x \psi$, $u_x = \partial_y \psi$

so $\omega^3 = -\Delta\psi$

but ψ is harmonic $\Rightarrow$ $\omega^3 = 0$

(irrotational)

$\Rightarrow$ so this equ. is satisfied by harmonic ψ

ie. Euler equ. $\Rightarrow$ $\dot{\vec{\omega}} = \vec{\nabla}\times(\vec{u}\times\vec{\omega})$, which is automatic, since ω=

p – the pressure field is to be determined after finding $\vec{u}$ – field: from p. 147 Euler (NS w/ $\nu=0$) equ

gave $\dot{\vec{u}} + \vec{u}\times\vec{\omega} = \frac{\vec{F}}{\rho} - \vec{\nabla}\left(\frac{p}{\rho} + \frac{1}{2}\vec{u}^2\right)$

$\vec{u} = \vec{\nabla}\phi$ (irrotational) 0 > assume $=0$, in our application

then $\frac{\partial}{\partial t}\vec{\nabla}\phi = -\vec{\nabla}\left(\frac{p}{\rho} + \frac{1}{2}\vec{u}^2\right)$

$\Rightarrow \vec{\nabla}\left(\dot{\phi} + \frac{p}{\rho} + \frac{1}{2}\vec{u}^2\right) = 0 \Rightarrow \dot{\phi} + \frac{p}{\rho} + \frac{1}{2}(\vec{\nabla}\phi)^2 = \text{const}$

$\Rightarrow$ know $\phi(x,y,t)$ $\Rightarrow$ know $p(x,y,t) = \text{const} - \rho\dot{\phi} - \frac{\rho}{2}(\vec{\nabla}\phi)^2$.

So we have: $(\phi, \psi)(x,y)$ (a) both harmonic

(b) $w(z) = \phi + i\psi$

$\Rightarrow$ (c) $\Delta w(z) = 0$

$\Rightarrow w(z)$ is also harmonic

BUT $\Delta = \partial_x^2 + \partial_y^2 = 4\partial_z \partial_{\bar{z}}$

Proof: $\frac{1}{4}(\partial_x - i\partial_y)(\partial_x + i\partial_y) = \frac{1}{4}(\partial_x^2 + \partial_y^2)$ ▫

Now every $w(z)$ is harmonic, apart from singularities

Now, u_x & u_y are easy to get from w:

$$\frac{\partial}{\partial z} w(z) = \frac{1}{2}(\partial_x - i\partial_y) w(z) =$$

$$= \frac{1}{2}(\partial_x - i\partial_y)(\phi + i\psi)$$

$$= \frac{1}{2}\Big(\underbrace{\partial_x \phi + \partial_y \psi}_{\partial_x \phi} - i(\underbrace{\partial_y \phi - \partial_x \psi}_{-\partial_y \phi})\Big)$$

$$\frac{\partial w}{\partial z} = \partial_x \phi - i\,\partial_y \phi$$

$$\Rightarrow \frac{\partial w}{\partial z} = u_x - i u_y$$

$\Rightarrow$ $\mathrm{Re}\,\frac{\partial w}{\partial z} \Leftrightarrow u_x$ $\quad$ $\mathrm{Im}\,\frac{\partial w}{\partial z} \Leftrightarrow -u_y$

So: $w(z)$ — analytic f-n = complex potential of flow

$z \in$ region on x-y plane

$\frac{\partial w(z)}{\partial z}$ — "complex velocity"

(N.B.: every $w(z)$ gives two flows actually ...

— w/ $\phi \to \psi$

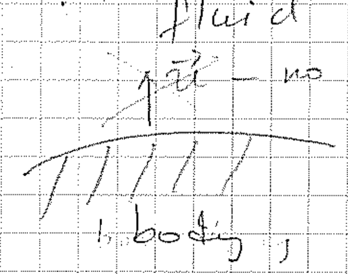
$\psi \to -\phi$)

Moral: $w(z)$ - analytic in $\mathcal{D} \in \mathbb{C}$ $\Leftrightarrow$ flow in $\mathcal{D} \in \mathbb{C}$

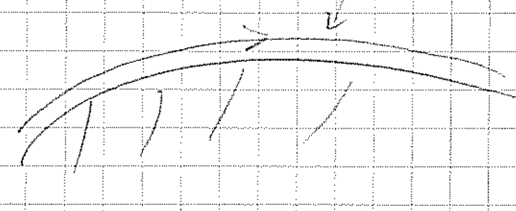
(164)

? boundary conditions — ??

— in inviscid flow theory



so instead: b.c.



$\Rightarrow$ <u>boundaries of bodies must be streamlines</u>

$\Rightarrow$

$$w(z) = \phi(x,y) + i\,\psi(x,y)$$

$$\frac{dw}{dz} = u_x(x,y) - i\,u_y(x,y)$$

at boundaries

$$\psi(x,y)\Big|_{x,y\,\in\,\text{boundary}} = \text{const} \quad \left(\text{i.e. boundaries are streamlines}\right)$$

$\rightarrow$ so we solve ideal irrotational flow eqns by simply giving $w(z)$ — all info's in it

(pressure found then thru plugging $\vec{u}(x,y,t)$ in Euler eqn)

<u>MORAL</u>: • 2d irrotational inviscid flows <u>are special</u>

• no work required to find soltns of Euler eqns — fixed by analyticity properties!!

(a dream for a theorist — $\exists\ \infty\ \#\ w(z) \Rightarrow \infty\ \#$ flows solved!!

— of course, question that remains is whether any of them are interesting — (clearly, yes, wouldn't be talking about it —)

<u>a few simple ex's:</u>

(1) uniform flow w/ $u_x, u_y = \text{const}$ (indep of x, y)

$$w(z) = (u_x - i u_y) z$$

$$\frac{dw}{dz} = u_x - i u_y$$

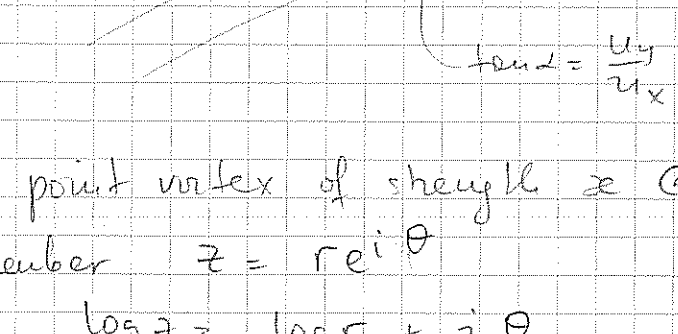
$$\psi(x,y) = ? \quad -- \quad w(z) = (u_x - i u_y)(x + iy)$$

$$\psi(x,y) = \mathrm{Im}\, w(z) = u_x y - u_y x$$

$\psi(x,y) = \text{const}$

along straight lines $u_x y - u_y x = c$

$$y = \frac{u_y}{u_x} x + c$$

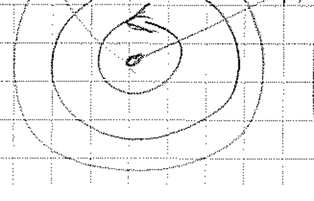


(2) point vortex of strength $\varkappa$ @ $z = 0$: $w = -\frac{i\varkappa}{2\pi} \log z$ — remember $z = re^{i\theta}$

$\log z = \log r + i\theta$

$$w = i\left(-\frac{\varkappa}{2\pi} \log r\right) + \left(\frac{\varkappa}{2\pi}\theta\right)$$

$\psi(r) = \text{const}$ @ $r = \text{const}$ — streamlines



<u>N.B.</u>: flow is <u>irrotational</u> apart from $r = 0$!!

$$\frac{dw}{dz} = -\frac{i\varkappa}{2\pi}\frac{1}{z} = -\frac{i\varkappa}{2\pi}\frac{x-iy}{(x+iy)(x-iy)}$$

$$= -\frac{i\varkappa}{2\pi}\left(\frac{x}{r^2} - i\frac{y}{r^2}\right) = \underbrace{-\frac{\varkappa}{2\pi}\frac{y}{r^2}}_{u_x} \underbrace{- i\frac{\varkappa}{2\pi}\frac{x}{r^2}}_{u_y}$$

<u>flow is irrotational</u> since $\Delta w = 4\partial_z\partial_{\bar z}\, w(z) = 0$ everywhere but @ origin - where w is singular, so it's not part of flow of fluid

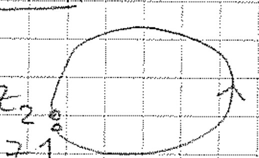
"$\varkappa$" = strength of vortex, also $\equiv$ circulation around any contour that goes thru origin

recall generally $\frac{dw}{dz} = u_x - iu_y$

& $dz = dx + i\,dy$

$$\mathrm{Re}\frac{dw}{dz}dz = \mathrm{Re}(u_x - iu_y)(dx + i\,dy) =$$
$$= u_x\,dx + u_y\,dy$$

So $\underbrace{\mathrm{Re}\oint_C \frac{dw(z)}{dz}dz \equiv \Gamma_C}$ $\Longleftrightarrow$ circulation $= \mathrm{Re}\int_C \frac{dw}{dz}dz$

But: $\oint_C \frac{dw}{dz}dz =$  $= w(z_2) - w(z_1)\Big|_{z_2 \to z_1}$

$\Rightarrow$ only non-single valued analytic $w(z)$ can give rise to circulation

($-\ln z$ has a branch cut @ $z=0$, hence $\Gamma \neq 0$.)

(3) flow past cylinder

$$w(z) = U\left(z + \frac{a^2}{z}\right)$$

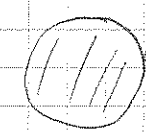
as $z \to \infty$ $w \to Uz$, our example 1 – uniform flow along x axis w/ U

What's ψ? $\psi = \operatorname{Im} w = U \operatorname{Im}\left(re^{i\theta} + \frac{a^2}{r}e^{-i\theta}\right)$

$$= U\left(r - \frac{a^2}{r}\right)\sin\theta$$

streamlines $U\left(r - \frac{a^2}{r}\right)\sin\theta = \text{const}$

note $r = a$ is a streamline! – see plot → 167.1

 $\psi = 0$ here & $\theta = 0, \pi$ are streamlines, too

Can add circulation! $w(z) = U\left(z + \frac{a^2}{z}\right) + \frac{i\Gamma}{2\pi}\log z$

Streamlines? $\Rightarrow$ $\psi = U\left(r - \frac{a^2}{r}\right)\sin\theta + \frac{\Gamma}{2\pi}\log r$

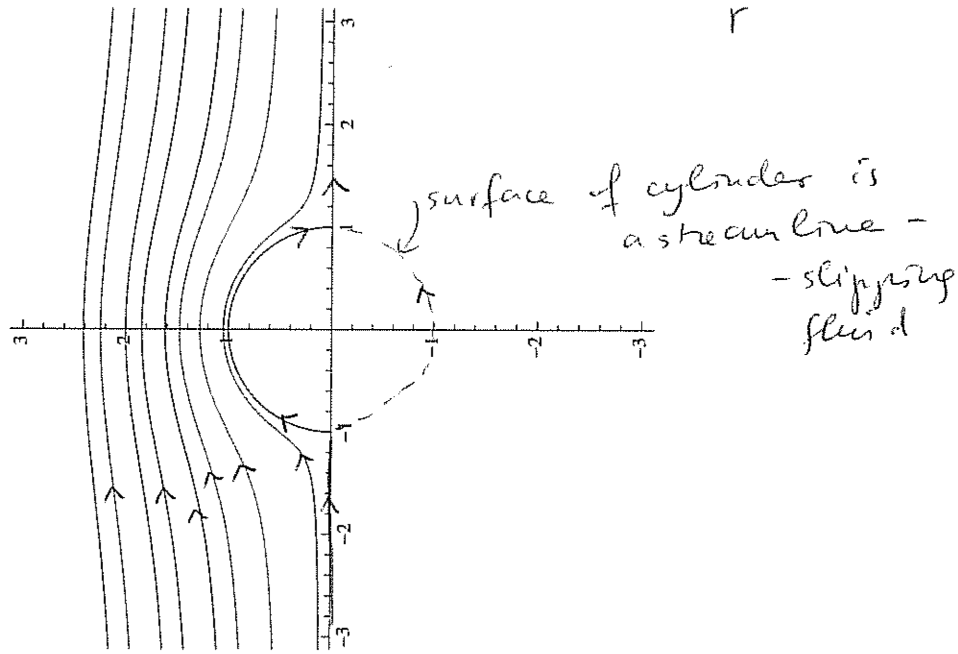
Γ is the circulation, since only $\log z$ part gives one around any contour that encloses cylinder

Flows w/ $\Gamma \neq 0$ cause a lift force on cylinder!

We'll show that in a bit – but 1st will consider a more general statement →

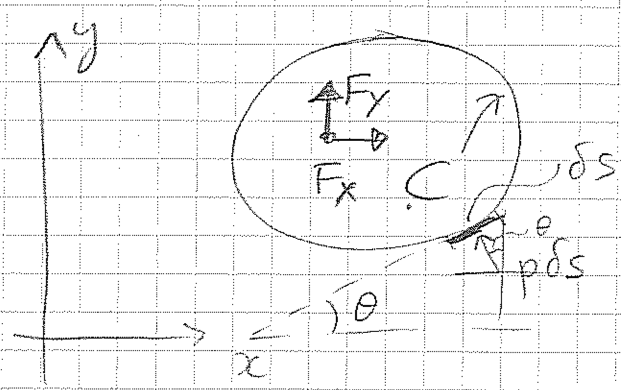
half of the flow w/

$$U\left(r - \frac{1}{r}\right)\sin\theta = \text{const}$$



2d, cylinder, uniform @ ∞, $\Gamma = 0$

incompressible inviscid



— let C – boundary of body.

— let $w(z)$ be a steady flow around body

<u>Blasius' theorem</u>: $F_x - i F_y = \frac{1}{2} i \rho \oint_C \left(\frac{dw}{dz}\right)^2 dz$

Proof's very easy!

recall that $p = -\frac{1}{2}\rho \vec{u}^2 + \underset{\text{constant (will drop out)}}{p_0}$

by Bernoulli's law (steady incompressible ideal flow)

- then $p\,\delta S$ is the force $\perp$ surface at δS. From picture:

$$\Rightarrow \begin{cases} \delta F_x = -p\,\delta S \sin\theta \\ \delta F_y = p\,\delta S \cos\theta \end{cases} \Rightarrow \delta F_x - i\delta F_y = -p\,\delta S(\sin\theta + i\cos\theta) =$$

$$= -p\,\delta S\, i(\cos\theta - i\sin\theta) = -i p\, e^{-i\theta}\delta S$$

Then, since the boundary C is a streamline, the velocity is tangential to C: $u_x = |\vec{u}|\cos\theta$, $u_y = |\vec{u}|\sin\theta$

hence $\left(\frac{dw}{dz}\right)_{\text{on } C} = u_x - i u_y \Big|_{\text{on } C} = |\vec{u}|(\cos\theta - i\sin\theta) =$

$= |\vec{u}|e^{-i\theta}$ or $|\vec{u}|_{\text{on } C} = \left(\frac{dw}{dz}\right)_{\text{on } C} e^{i\theta}$

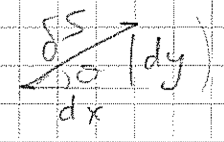
Since $p = -\frac{1}{2}\rho|\vec{u}|^2 + p_0$ we have

$\delta F_x - i\delta F_y = \left(\frac{1}{2}\rho|\vec{u}|^2 - p_0\right) i e^{-i\theta}\delta S = \longrightarrow$

$$= \frac{i}{2}\rho \underbrace{|\vec{u}|^2}_{= \left(\frac{dw}{dz}\right)^2 e^{2i\theta}} e^{-i\theta}\,\delta S - i p_0 e^{-i\theta}\,\delta S$$

$$= \rho\,\frac{i}{2}\left(\frac{dw}{dz}\right)^2 e^{i\theta}\,\delta S - i p_0 e^{-i\theta}\,\delta S$$

$F_x - iF_y$ next $\int$ over C

note that $e^{i\theta}\,\delta S = \Big($  $\Big) =$

$= \delta S\cos\theta + i\,\delta S\sin\theta$

$= dx + i\,dy$

$= dz$ on C.

$e^{-i\theta}\,\delta S = dx - i\,dy$

$$\text{so } \delta F_x - i\,\delta F_y = \rho\,\frac{i}{2}\left(\frac{dw}{dz}\right)^2 dz \underbrace{- i p_0 (dx - i\,dy)}_{\text{this vanishes upon } \oint_C}$$

$$\Rightarrow F_x - iF_y = \int_C (\delta F_x - i\,\delta F_y)$$

$$F_x - iF_y = \frac{i\rho}{2}\oint_C \left(\frac{dw}{dz}\right)^2 dz$$

$\boxtimes$

Apply immediately to our cylindrical flow w/ Γ!

HW4

$F_x = 0$

$F_y = -\rho U\Gamma$ ⟸ as per Zhukovski's lift theorem

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→ from Blasius' th-m, it's 2 steps to get the <u>Zhukovski lift theorem</u>

If <u>flow is uniform // x-axis @ ∞, w/ $U>0$</u>

then $F_x = 0, \quad F_y = -\rho U \Gamma$, Γ – any contour surrounding body anticlockwise

i.e. $\begin{cases} \text{– no drag} \\ \text{– lift} \sim -\Gamma \end{cases}$

<u>Proof</u>: • Choose origin $z=0$ inside body

• flow is $w(z)$ – analytic, so is $\frac{dw}{dz}$

$\downarrow$

= <u>Laurent expansion at $z=0$</u>

for $\mathbb{C}$-velocity: $\frac{dw(z)}{dz} = \underbrace{z^k + z^{k-1} + \dots}_{k>0} \; z^0 + z^{-1} + z^{-2} + \dots$

$k>0$ NOT allowed as flow is uniform @ ∞

$\rightarrow \underbrace{U z^0}$ coeff't fixed.

(expansion converges outside body, at least until nearest other boundary reached –!)

$\Rightarrow \left.\frac{dw}{dz}\right|_{z\to\infty} = U \Rightarrow$

$\Rightarrow \frac{dw}{dz} = U + \frac{a_1}{z} + \frac{a_2}{z^2} + \dots$, Now by Blasius:

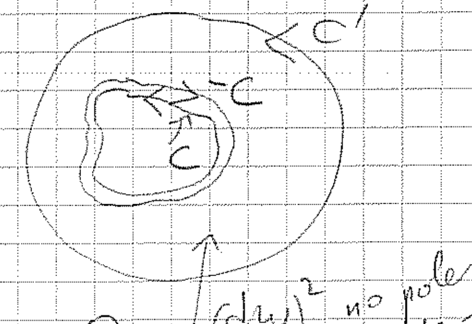
$F_x - iF_y = \frac{i\rho}{2} \oint_C \left(\frac{dw}{dz}\right)^2 dz$ $\Rightarrow$ but

$\downarrow$ this was taken over boundary of body

$\Rightarrow \oint_{C'-C} \left(\frac{dw}{dz}\right)^2 dz = 0$

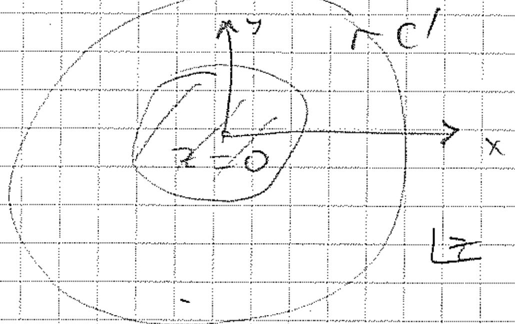
$\downarrow$

can take any C'



$\left(\frac{dw}{dz}\right)^2$ no poles inside fluid flow

$\oint \;\rightarrow\; F_x - i F_y = \frac{i\rho}{2} \oint\limits_{C'} \left(U_i + \frac{a_1}{z} + \frac{a_2}{z^2} + \ldots \right)^2 dz =$



$$= \frac{i\rho}{2} \left(\oint\limits_{C'} U^2 dz + \oint\limits_{C'} \frac{2Ua_1}{z} dz + \oint\limits_{C'} \frac{2Ua_1}{z^2} dz + \ldots \right)$$

$\oint\limits_{C'} \frac{dz}{z^n} = 0 \quad \forall n \neq -1$

- take C' a circle - say smallest circle enclosing body
- then $dz = Re^{i\theta}$

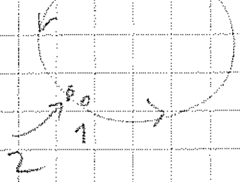
$z = Re^{i\theta}$

$$\oint\limits_{C'} \frac{dz}{z^n} = \frac{R}{R^n} \int_0^{2\pi} i\, d\theta \frac{e^{i\theta}}{e^{in\theta}} = \frac{2\pi i}{R^{n-1}} \delta_{n,1}$$

$\rightarrow$ only $\frac{1}{z}$ term contributes

$$F_x - i F_y = i\rho U a_1 \overbrace{\oint\limits_{C'} \frac{dz}{z}}^{2\pi i} = -\rho U a_1 2\pi$$

Now, $\underbrace{\oint\limits_{C'} \frac{dw}{dz} dz}_{\equiv \Gamma, \text{ in fact}} = \oint\limits_{C_1} a_1 \frac{dz}{z} = 2\pi i a_1$ (we said $\mathrm{Re} \oint \frac{dw}{dz} dz = \Gamma$ - but it's got NO Im part $\longrightarrow$

as $\oint_{C} \frac{dw}{dz}\, dz =$  $= w[2] - w[1]$

$= \phi[2] - \phi[1] + i\,(\underbrace{\psi[2] - \psi[1]}_{})$

can take contour @ surface

ψ is constant on the surface so this $\equiv 0$.

this is Γ, as we've already shown.

$\Rightarrow \quad 2\pi i a_1 = \Gamma, \quad 2\pi a_1 = \frac{\Gamma}{i} = -i\Gamma$

$\Rightarrow \quad F_x - iF_y = -\rho U 2\pi a_1 = -\rho U(-i\Gamma) = +i\rho U\Gamma$

no drag

"d'Alembert" paradox

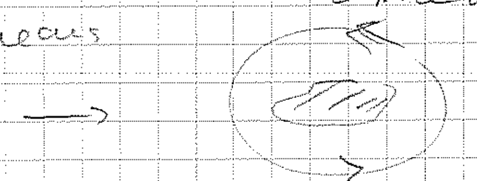
$\longrightarrow \quad F_x = 0$

$F_y = -\rho U \Gamma$

so lift ~ circulation × (−1) $\quad F_x > 0$ if $\Gamma < 0$

$\Gamma < 0$ means

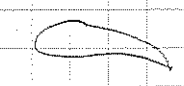
↓ more velocity here (this part gives negative contrib to Γ)



less velocity here (this gives > 0 contrib to Γ)

U

→ faster (p - less)

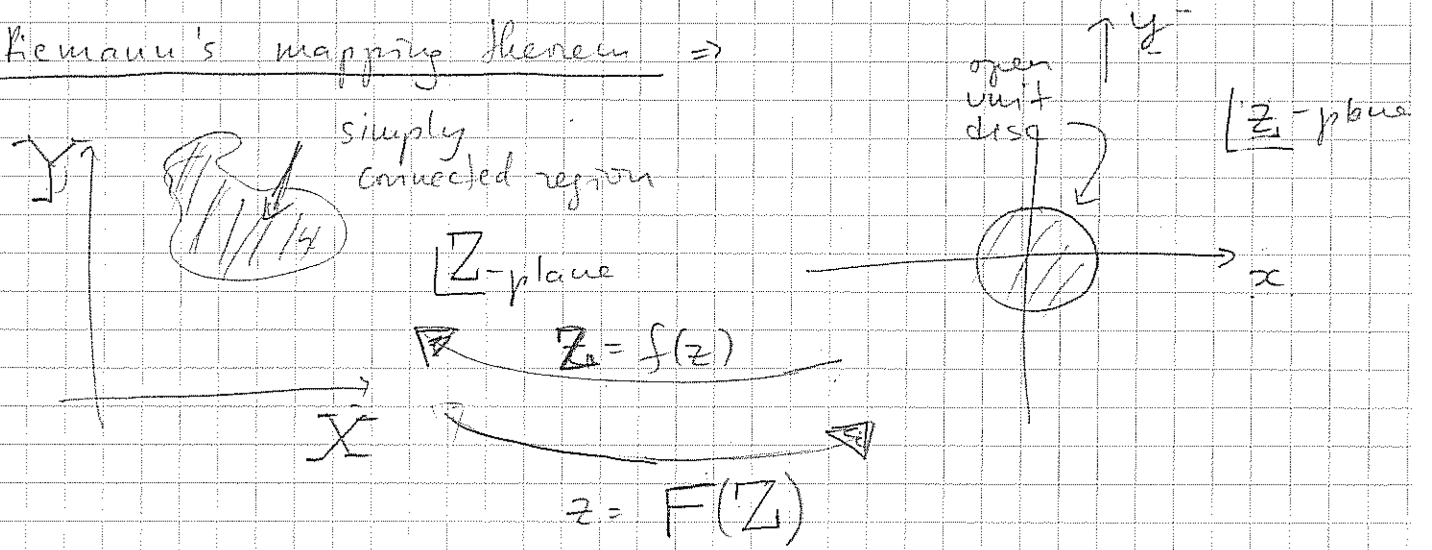


↑ ← lift up

→ slower (p - more)

It may seem that this flow past cylinder w/ $\Gamma \neq 0$ added by hand is "frivolous" $\longrightarrow$ but not – the theory of $\mathbb{C}$ analytic functions comes to help here [very powerful!]

Riemann's mapping theorem $\Rightarrow$



(& $f(z)$ & $F(Z)$ are both analytic

under such a map $w(z) \longrightarrow W(Z) = w(z) = w(F(Z))$

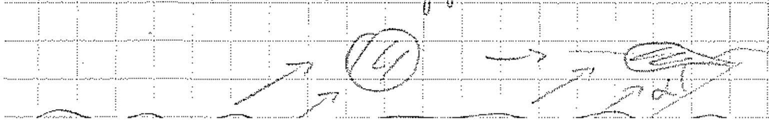
and a flow is mapped to a flow

around   around any shape body

In particular — let ("Zhukovski transformation")

$$Z = f(z) = z + \frac{1}{z} \quad \& \quad z = \frac{1}{2} Z + \frac{1}{4}\left(Z^2 - 1\right)^{1/2}$$

unit disc is mapped to? $\Rightarrow$ can map to airfoil



angle of attack   etc.

DIY