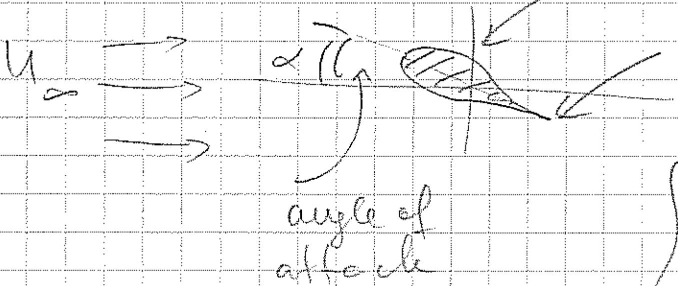
Now, we will not have time to study these conformal mappings & the flow past a wing in detail (see e.g. Acheson Ch 4 where detail's spelled out — and you are prepared to understand it now!).

Idea goes as follows: Use conformal mapping to map flow past cylinder w/ some circulation Γ (flow uniform @ ∞) to the flow past a symmetrical airfoil w/ a sharp trailing edge.



It turns out that there's a unique value of Γ, for a given α for which the velocity field is free of singularities ((Γ depends on α & shape & thickness of airfoil, really))

the "Kutta-Zhukovski hypothesis" says that the irrotational, incompressible inviscid flow that is obtained for this value of Γ (w/out singularities) corresponds to the flow that is actually observed — apart from a thin layer around the wing where velocity adjusts from the "slip"-inviscid value to the "no-slip" value of 0, appropriate for a viscous fluid.

$$\Gamma = -\pi U L \sin\alpha \iff$$ for an airfoil which is thin, symmetrical, length L, angle of attack α.

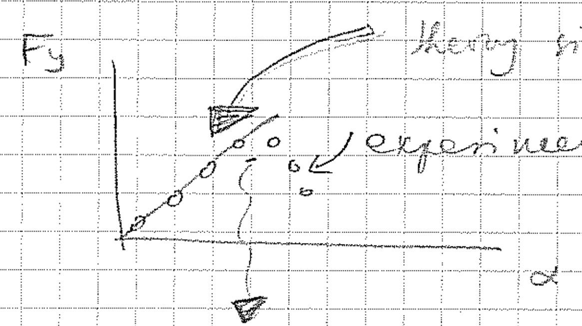
(Γ: critical value (= nonsingular velocity))

The lift force, as per the theorem is, then

$$F_y = -\rho U \Gamma = \pi \rho U^2 L \sin\alpha$$

Claim: It works -- if α is a few degrees (only)

theory $\sin\alpha \simeq \alpha$, $\alpha \ll 1$



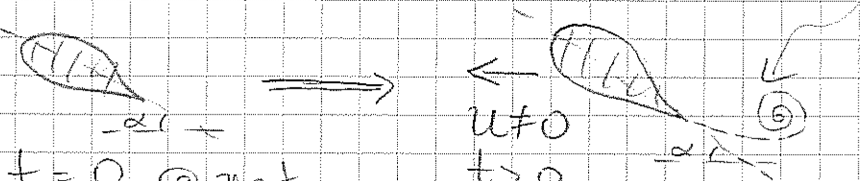
$> 6^\circ - 10^\circ$ or so !!

This IS a practically relevant result of ideal-flow theory !!

Now, how does an airfoil get to the situation where $\Gamma \neq 0$, in particular $\Gamma = \Gamma_{critical}$?

- turns out, if you start a wing w/ a sharp trailing edge from start:

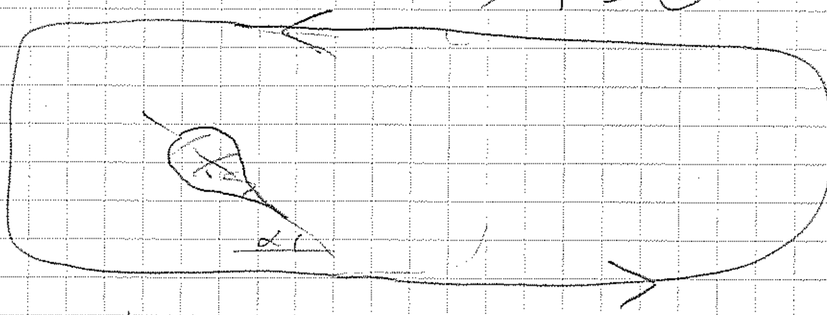
put ink in water - will see pathline like this.



"trailing vortex"

flow is that of ideal fluid, inviscid everywhere - - except a small boundary layer initially we have

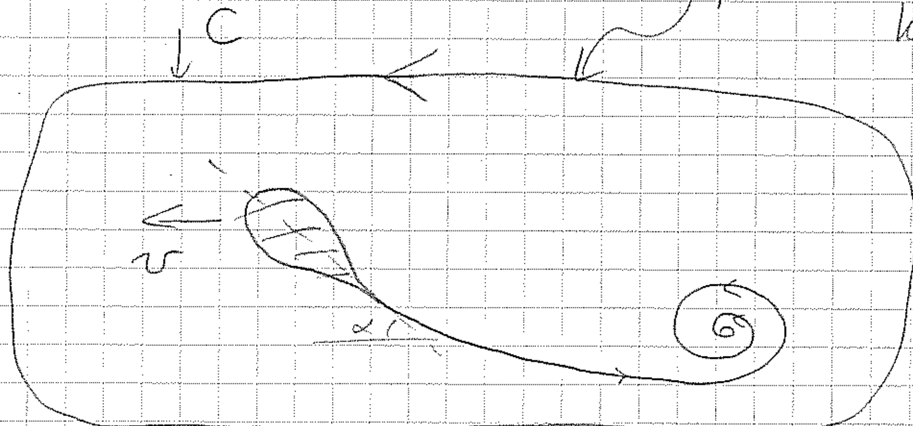
$\Gamma = 0$



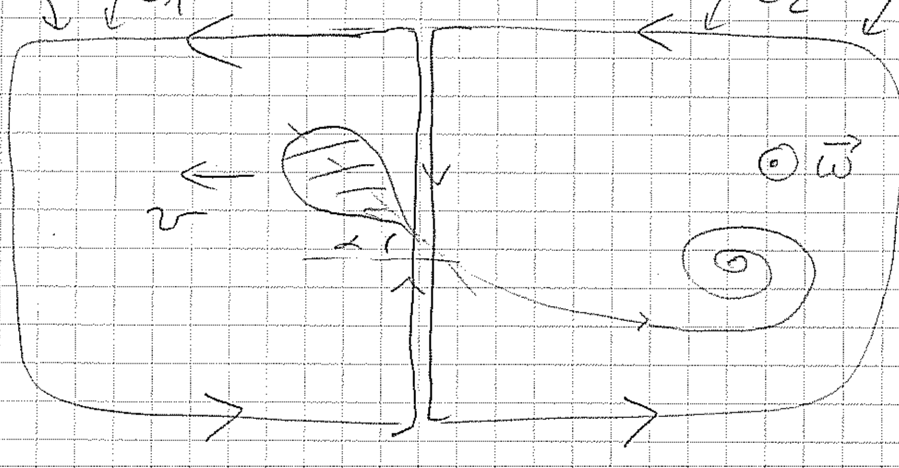
$t = 0$

at some $t > 0$:

$\Gamma = 0$, still, by Kelvin's theorem – – viscosity does not matter for this contour (take it big enough)



in more detail: Γ_1



$$\Gamma = \Gamma_1 + \Gamma_2 = 0 \quad \text{but} \quad \Gamma_2 \neq 0 \rightarrow \text{in fact}$$

$$\Gamma_2 = \oint_{C_2} \vec{u}\, d\vec{l} = \int_{S_2} \vec{\omega}\, d\vec{S} > 0. \rightarrow$$ positive vorticity shed off via trailing vortex

$\partial S_2 = C_2$

→ Therefore, Γ_1 must be < 0 ($\equiv -\Gamma_2$)

→ shedding of vortex continues until Γ_1 reaches its value relevant for the given constant $\vec{v}$, α, & shape, such that flow is free of singularities, i.e. $\Gamma_1 < 0$ has the final Kutta-Zhukovski critical value — (see photographs in Batchelor, for example).

[→ see also the pictures of the flying of "Encarsia formosa" wasp in Acheson]

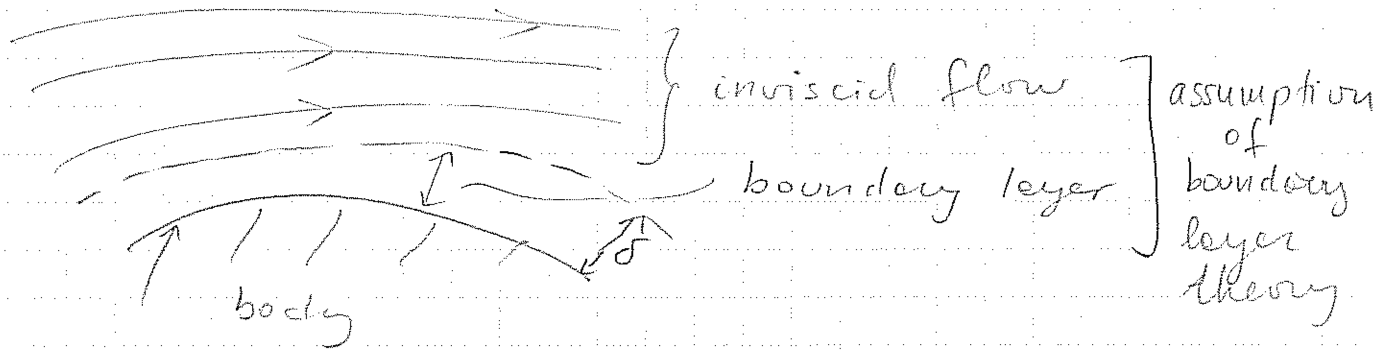
So, all this works OK for small angles of attack (α). However at larger values of α, flow looks different from that predicted by inviscid theory (with only a small boundary layer where $\vec{u} \to 0$ at the body's surface). This is because of

<u>Boundary Layer Separation</u> —

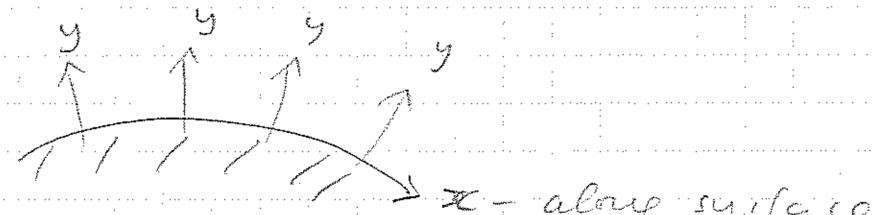
– which causes the flow to be quite different from the ideal one also in the bulk of the fluid

the boundary layer equations

- (steady) 2d flow (for simplicity)



→ body is curved, so one may consider general curvilinear coordinates (x, y)



where $\hat{y}$ is everywhere $\perp$ to surface

→ we won't do this - instead we'll look at a flat boundary - (BUT see where some fun geometry may creep in on us.)

Exact 2D flow eqns are:

$$
\left[\begin{aligned}
&(a)\quad u_x \partial_x u_x + u_y \partial_y u_x = -\frac{1}{\rho}\partial_x p + \nu(\partial_x^2 u_x + \partial_y^2 u_x)\\
&(b)\quad u_x \partial_x u_y + u_y \partial_y u_y = -\frac{1}{\rho}\partial_y p + \nu(\partial_x^2 u_y + \partial_y^2 u_y)\\
&\qquad \partial_x u_x + \partial_y u_y = 0
\end{aligned}\right.
$$

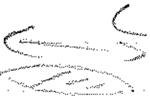
in boundary layer u_x & u_y vary much more rapidly w/ y than w/ x (you are familiar w/ this from "lubrication theory" discussion)

in other words,

$$\left|\frac{\partial u_x}{\partial y}\right| \gg \left|\frac{\partial u_x}{\partial x}\right| \qquad (*)$$

let $u_x \sim U_0$ - basically value of u_x predicted by inviscid flow theory @ the outer boundary of the boundary layer

let thickness be δ

let scale of variation of velocities in $\vec{x}$ be $\sim L$ (i.e. size of wing )

then $(*) \Rightarrow$ $\frac{U_0}{\delta} \gg \frac{U_0}{L}$ or $L \gg \delta$.

From the <u>exact</u> continuity eqn. $\Rightarrow$

$$\Rightarrow \left|\frac{\partial}{\partial y} u_y\right| = \left|\frac{\partial}{\partial x} u_x\right| \sim \frac{U_0}{L}$$

continuity

$$\Rightarrow \left|\frac{\partial u_y}{\partial y}\right| \sim \frac{U_0}{L} \Rightarrow u_y \sim \frac{U_0}{L} y + \underbrace{u_y(y=0)}_{=0}$$

$$\Rightarrow u_y \lesssim \frac{U_0}{L}\delta \ll U_0 \sim u_x$$

(since $y < \delta$ inside layer)

$\Rightarrow$ so $u_y \ll u_x$ —

So, then we can do what we did when we a-dimensionalyzed NS eqn, but here we use <u>different</u> scaling for x, u_x, y, u_y:

let
$$u_x = U_0 u_x'$$
$$u_y = U_0 \frac{\delta}{L} u_y'$$
$$x = L x'$$
$$y = \delta y'$$
x', y', u_x', u_y' are dim-less.

then: (x-component of NS)

eqn. (a) from p(179)

$$u_x \partial_x u_x + u_y \partial_y u_x = -\frac{1}{\rho}\partial_x p + \nu \partial_x^2 u_x + \nu \partial_y^2 u_x$$

becomes:

$$\frac{U_0^2}{L} u_x' \partial_{x'} u_x' + \frac{U_0^2}{L} u_y' \partial_{y'} u_x' =$$
$$= -\frac{1}{L\rho}\partial_x' p + \frac{\nu U_0}{L^2}\partial_{x'}^2 u_x' + \frac{\nu U_0}{\delta^2}\partial_{y'}^2 u_x'$$

$\times \frac{L}{U_0^2}$

$$(a')\quad u_x' \partial_{x'} u_x' + u_{y'} \partial_{y'} u_{x'} = -\frac{1}{U_0^2 \rho}\partial_x' p + \left(\frac{\nu}{U_0 L}\right)\partial_{x'}^2 u_{x'} + \frac{L^2}{\delta^2}\left(\frac{\nu}{U_0 L}\right)\partial_{y'}^2 u_{x'}$$

Similarly eqn(b) becomes:

$$\left(\frac{\delta}{L}\right)\frac{U_0^2}{L} u_x' \partial_{x'} u_y' + \underbrace{\left(\frac{\delta}{L}\right)\frac{U_0^2}{L}}_{(\times\, u_y' \partial_y' u_y')} = -\frac{1}{\delta\rho}\partial_y' p + \frac{\nu U_0}{L^2}\frac{\delta}{L}\partial_{x'}^2 u_y' +$$
$$+ \frac{\nu U_0}{L^2}\frac{L}{\delta}\partial_{y'}^2 u_y'$$

× some $\dfrac{L}{u_0^2}$ $\Rightarrow$

$$(b'):\quad \left(\frac{\delta}{L}\right) u_x' \,\partial_{x'} u_y' + \left(\frac{\delta}{L}\right) u_y' \,\partial_{y'} u_y' = -\left(\frac{L}{\delta}\right)\frac{1}{u_0^2\rho}\,\partial_{y'} p$$
$$+\left(\frac{\nu}{u_0 L}\right)\left(\frac{\delta}{L}\right)\partial_{x'}^2 u_{y'}$$
$$+\left(\frac{\nu}{u_0 L}\right)\left(\frac{L}{\delta}\right)\partial_{y'}^2 u_{y'}$$

It is natural to rescale p by $u_0^2\rho$ – – as outside of boundary layer, this is the expected value of $p - p_\infty$, by Bernoulli's law for an ideal fluid, so then

we say $p - p_\infty = u_0^2 \rho\, p'$ & we have, in terms of dim-less variables (note p_∞ = const, so $\partial_x p_\infty = \partial_y p_\infty = 0$), $R \equiv u_0 L/\nu$:

$$(a'):\quad \partial_x' p' = -u_{x'}\,\partial_{x'} u_{x'} - u_{y'}\partial_{y'} u_{x'} + \frac{1}{R}\,\partial_{x'}^2 u_{x'} + \left(\frac{L}{\delta}\right)^2 \frac{1}{R}\,\partial_{y'}^2 u_{x'}$$

$$(b')\quad \partial_{y}' p' = -\left(\frac{\delta}{L}\right)^2 u_{x'}\,\partial_{x'} u_{y'} - \left(\frac{\delta}{L}\right)^2 u_{y'}\,\partial_{y'} u_y'$$
$$+\left(\frac{\delta}{L}\right)^2 \frac{1}{R}\,\partial_{x'}^2 u_{y'} + \frac{1}{R}\,\partial_{y'}^2 u_{y'}$$

- There are some interesting eqns!

note we have two small parameters :

$$\frac{1}{R} \quad \text{(remember, } R \to \infty \text{ is ideal)}$$

&

$$\frac{\delta}{L} \ll 1$$

terms on the r.h.s. of $\partial_x' p'$ & $\partial_y' p'$ have dimensionless coeffts, which are of :

$$\underline{O}(1) \longrightarrow \text{keep}$$

$$\underline{O}\left(\frac{1}{R}\right) \ll 1$$

$$\underline{O}\left(\frac{\delta}{L}\right) \ll 1$$

$$\underline{O}\left(\left(\frac{\delta}{L}\right)^2 \frac{1}{R}\right) \ll 1$$

$$\& \quad \underline{O}\left(\left(\frac{L}{\delta}\right)^2 \frac{1}{R}\right) \longrightarrow \text{keep.}$$

So; (a', b') become, in this limit :

$$\partial_{x'} p' = -u_x' \partial_x' u_x' - u_y' \partial_y' u_x' + \left(\frac{L}{\delta}\right)^2 \frac{1}{R} \partial_{y'}^2 u_x'$$

$$\partial_{y'} p' = 0.$$

pressure doesn't vary inside B.L. (boundary layer) — it is basically equal to that of the ideal flow @ $y = \delta$.

Finally, we (Prandtl, really, ~1907)

make the assumption that viscosity effects (i.e last term in $\partial_{x'} p' = \dots\dots + \left(\frac{L}{\delta}\right)^2 \frac{1}{R} \partial_{y'}^2 u_{x'}$) should be such to make $\partial_y u_x$ non-negligible – i.e.

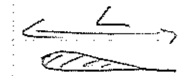
$$\frac{L}{\delta} \sim \sqrt{R} \text{ or}$$

$$\underbrace{\frac{\delta}{L} \sim \frac{1}{\sqrt{R}}}_{\text{– scaling for B.L thickness}}$$

→ despite small value of ν, B.L. is just thin enough to make viscous term as important as the rest of the terms ($\vec{\nabla} p$, $D\vec{u}/Dt$) inside B.L.

airplane wing (B-747) ~ cm or so

(?) → $L \sim 10\,m$

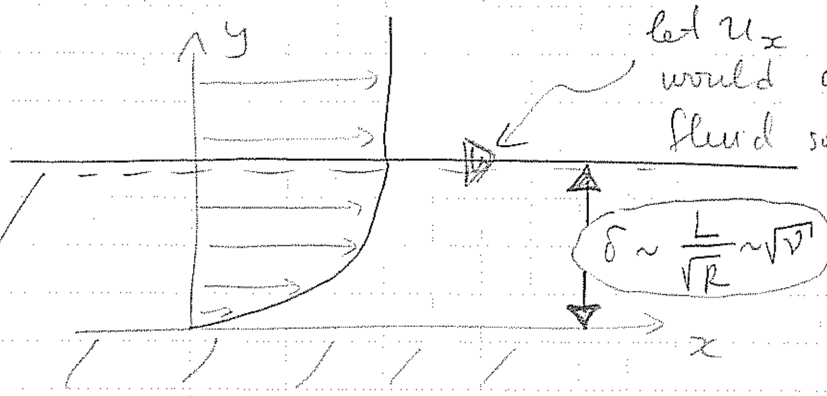


$$R \sim \frac{1000\,\frac{km}{h} \cdot 10\,m}{1.5\,10^{-5}\,\frac{m^2}{s}} \sim \frac{280\,\frac{m}{s}\,10\,m}{10^{-4}\,\frac{m^2}{s}} \sim 2\times 10^{8},\ \sqrt{R} \sim 10^{4}$$

$$\delta \sim \frac{10\,m}{\sqrt{R}} \sim 10^{-3}\,m \iff \text{tiny}$$

– yet all the "action" is there!

So pictorially, we have



let U_x be the velocity that would arise from ideal fluid solution @ $y = 0$ — remember boundary is a streamline ("slip")

$$\left\{ \begin{array}{l} (u_x \partial_x + u_y \partial_y) u_x = -\frac{1}{\rho} \partial_x p + \nu \partial_y^2 u_x \\ \partial_x u_x + \partial_y u_y = 0 \\ \partial_y p = 0 \\ u_x = u_y = 0 \text{ @ } y = 0 \end{array} \right. \quad (**)$$

B.L. equs. back in dimensionful variables

by Bernoulli $p + \frac{1}{2} \rho U_x^2 = $ const along streamline at the edge of B.L.

then we have $-\frac{1}{\rho} \frac{dp}{dx} = U_x \frac{\partial}{\partial x} U_x$ just at the edge of B.L. $\rightarrow$ so if $U_x(x) \uparrow$ $p(x) \downarrow$

& $U_x(x) \downarrow$ $p(x) \uparrow$.

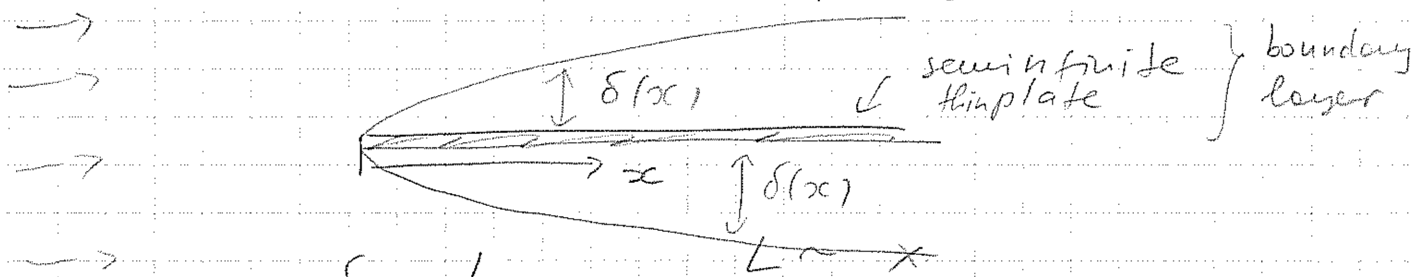
B.L. equs (**) are solved w/b.c. that

$u_x \rightarrow U_x(x)$ ideal flow soltn

as $\frac{y}{\delta} \rightarrow \infty$ ($\delta \sim \nu^{1/2}$, remember)

"matching" of B.L & inviscid flow

$\delta(x) \sim \sqrt{x \nu / U}$

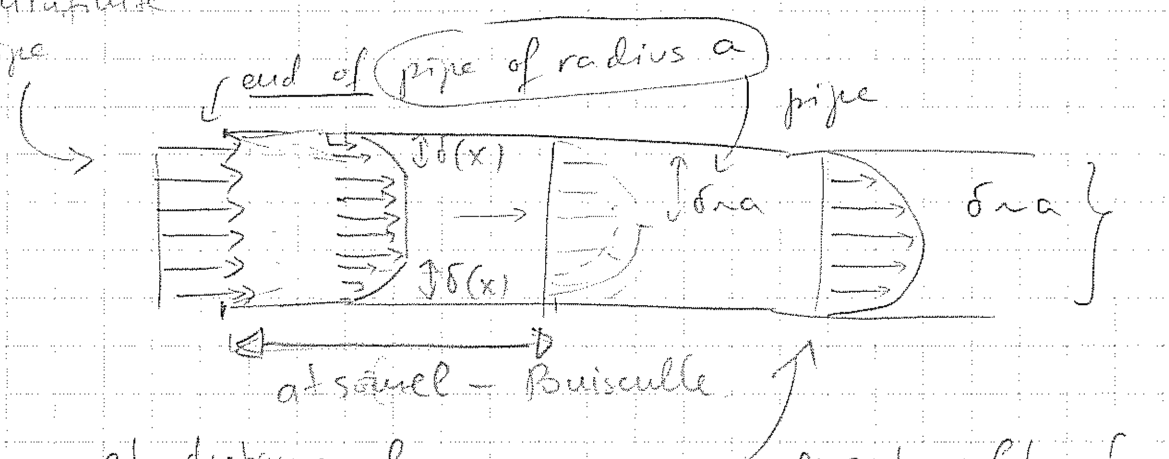


$$\delta \sim \frac{L}{\sqrt{R}} \qquad R = \frac{UL}{\nu} = \frac{Ux}{\nu}$$

$$\delta \sim \frac{x}{\sqrt{\frac{Ux}{\nu}}} = \sqrt{\frac{x\nu}{U}}$$

→ as per "exact" solution

or semiinfinite pipe



at distance l
δ becomes $\sim a$

use $\delta \sim \sqrt{\frac{x\nu}{U}}$

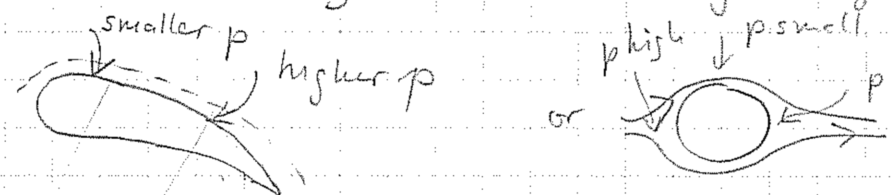
$\delta \sim a$ when $x \sim l$: $\sqrt{\frac{l\nu}{U}} \sim a \Rightarrow l \sim \frac{Ua^2}{\nu}$

or $l \sim aR$

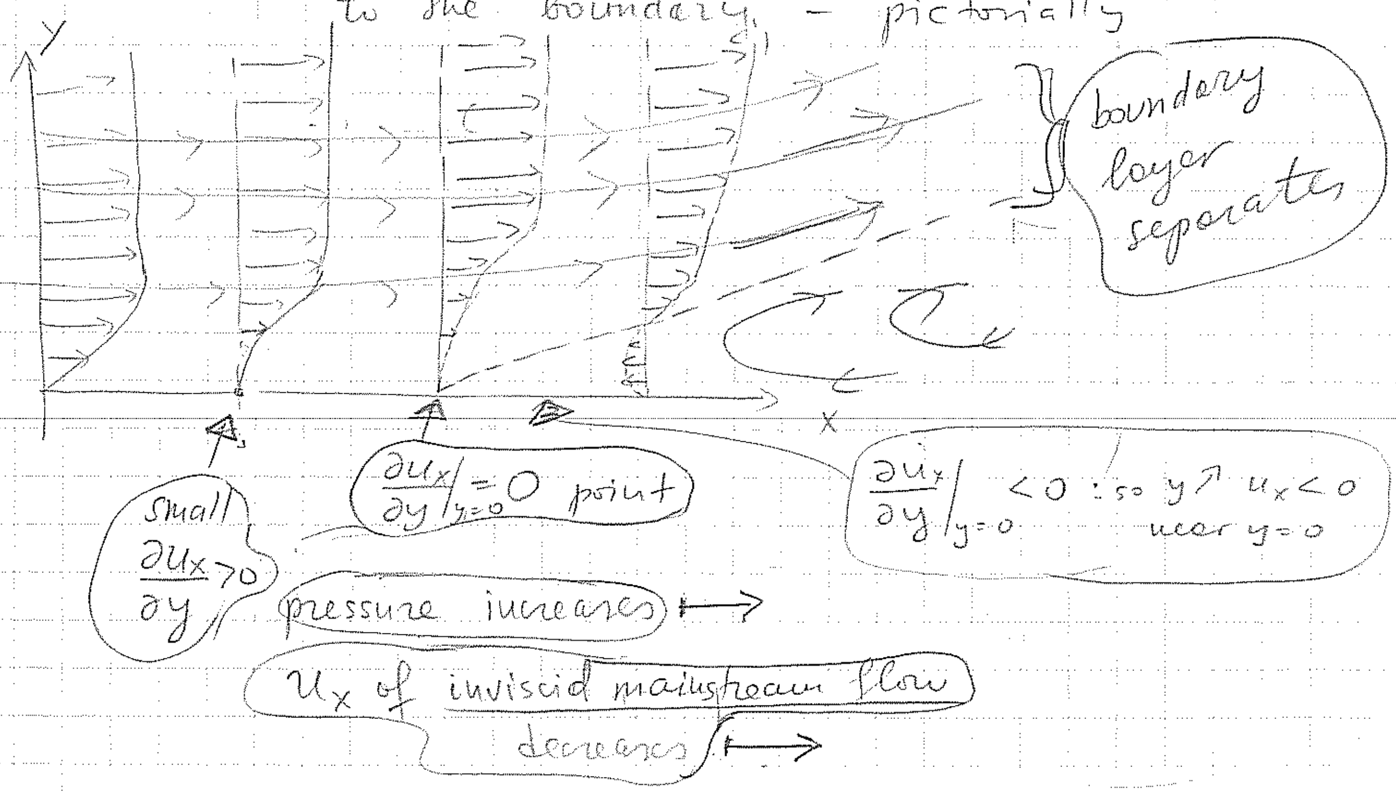
we won't dwell on solving these equs (time's out). } see pic. on 184.1

But we'll use them to understand the phenomenon of <u>boundary layer separation</u> in <u>qualitative</u> <u>terms</u> -

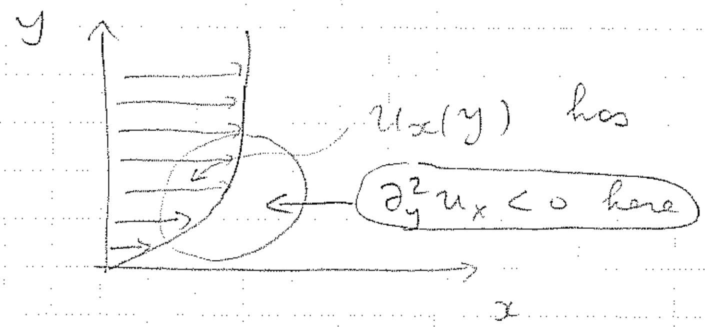
- in a few words what can happen is that if the pressure increases along the boundary layer

e.g. 

adverse pressure gradient, if sufficiently large, can cause a reversed flow close to the boundary - pictorially



When does the B. layer seperation occur?



$u_x(y)$ has $\frac{\partial^2 u_x}{\partial y^2} < 0$ at $y \lesssim \delta$ — as is clear from picture — to match the mainstream flow @ edge of B.L.

on the other hand, B.L. eqns @ $y = 0$ (where $u_x = u_y = 0$) say that

$$-\frac{1}{\rho}\partial_x p + \nu\,\partial_y^2 u_x = 0 \quad \text{at } y=0$$

or

$$\partial_y^2 u_x = \left(\frac{1}{\rho\nu}\right)\underline{\partial_x p} \quad \text{at } y=0$$

(a) if $\partial_x p < 0$ i.e. pressure decreases in the dir'n of mainstream flow

$$\partial_y^2 u_x\big|_{y=0} < 0 \Longleftarrow$$

$\downarrow$

so $\partial_y^2 u_x < 0$ at $y=0$ and at $y \lesssim \delta$

(b) if $\partial_x p > 0$, i.e. if pressure increases in the dir'n of mainstream flow

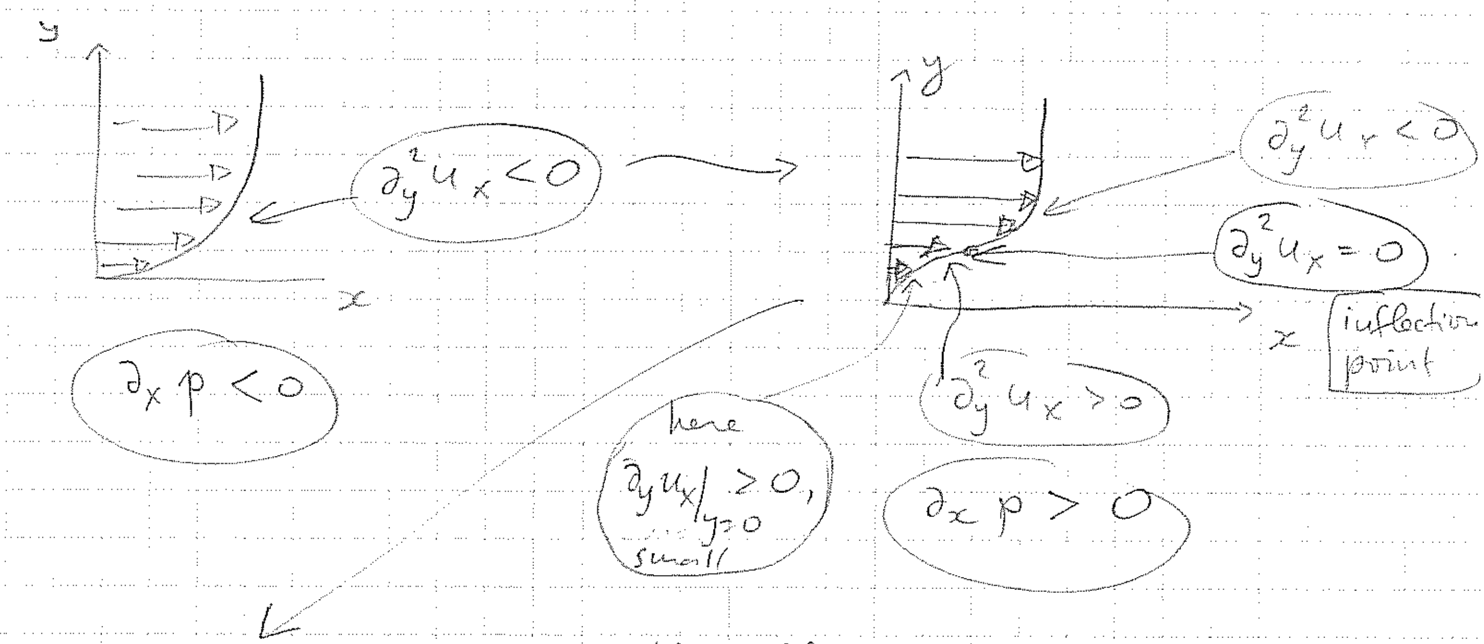
$$\partial_y^2 u_x\big|_{y=0} > 0 \Longleftarrow$$

— but we also know $\partial_y^2 u_x\big|_{y\lesssim\delta} < 0$, so there must be a point where $\partial_y^2 u_x = 0$ for (some) $y \in (0, \delta)$

In other words, for $\overbrace{\partial_x p > 0}^{\text{(mainstream flow)}}$ case

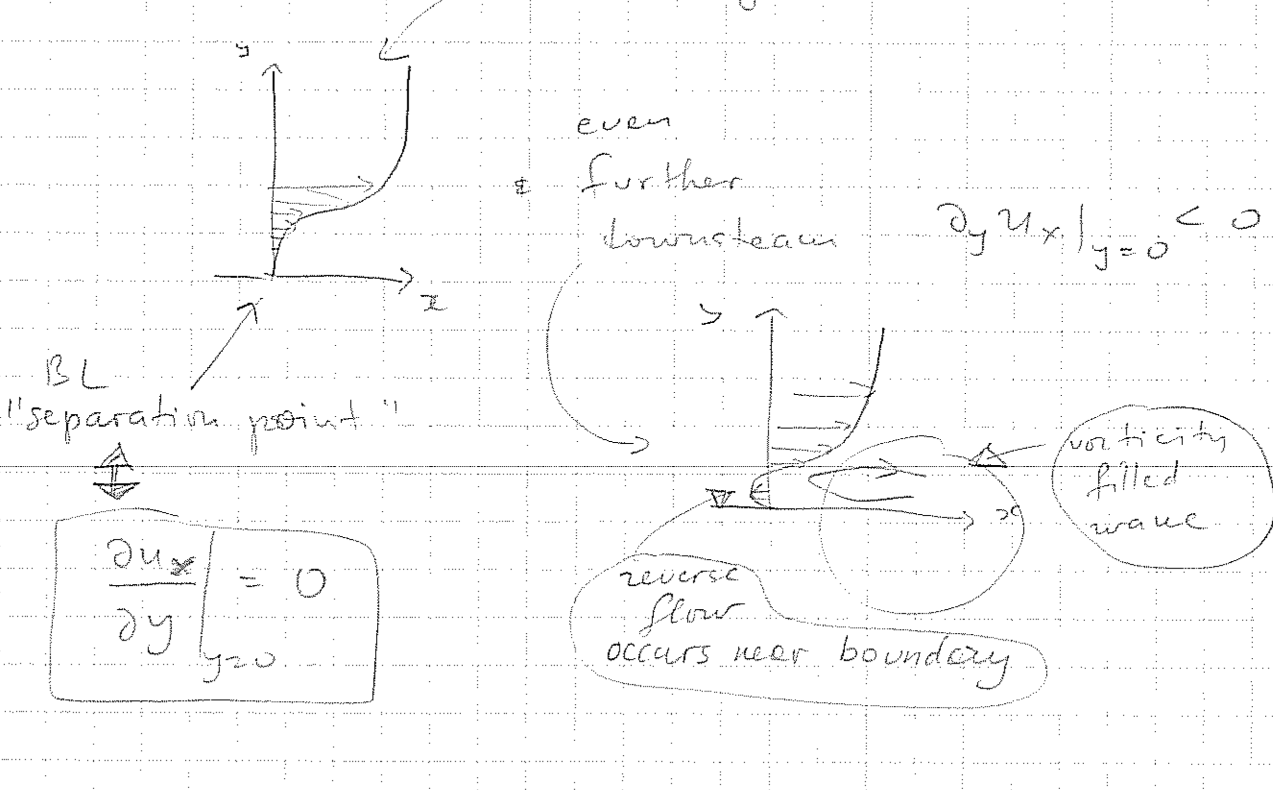
(as in the ~~aft~~ part of a cylinder), there must be

an "inflection point" where $\partial_y^2 u_x$ changes sign, i.e.



as one proceeds down the flow, one can reach

a point where $\partial_y u_x|_{y=0} = 0$



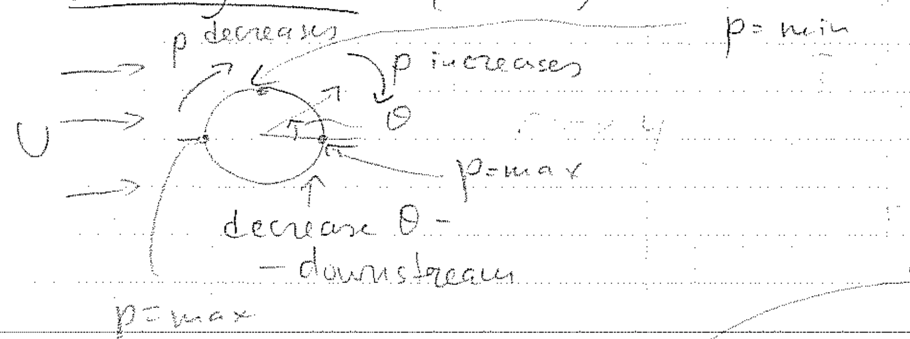
Note:

A study of BL separation can't be done using BL eqns alone — since in the wake of BL we have $u_y \sim u_x$ rather than $u_y/u_x \sim 1/\sqrt{R}$ → so from the point of view of BL eqns ($R \to \infty$ limit) u_y must grow to ∞ (in dim-less variables, it's clear). So point of BL separation is a singular point of BL eqns...

— to study separation, one has to go beyond BL eqns (where mainstream flow is regarded as given) — since separation affects the mainstream —

(— complicated analysis, only done ~1980 or so!!)

On cylinder (hw 4)



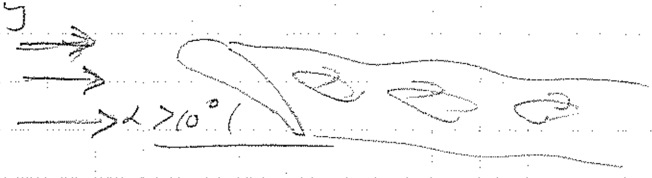
separation point occurs in the back of the cylinder.

On the Zhukovski wing:

BL separation only occurs near wing tip. (trailing vortex)

for small α increase of pressure is slow & smooth — if the wing is engineered right → reason why Zhukovski th-m works for $\alpha \ll 1$.

for $\alpha \gtrsim 10^\circ$, separation of BL occurs early



∄ theory that does not take into account BL separation & turbulent wake fails ...

no good theory for this exists to date ...

— " — THE END — " —