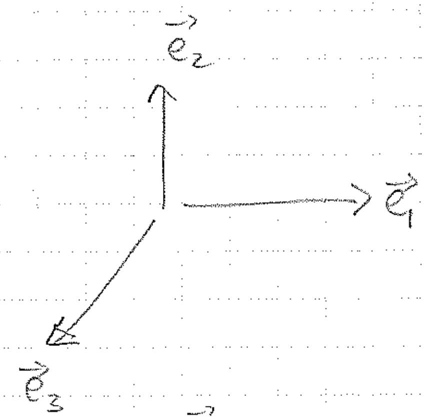
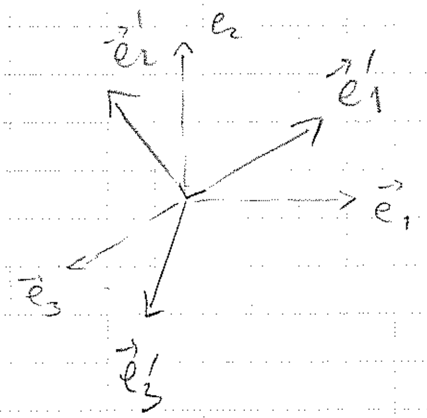
An interlude about coordinate frames & tensors



an orthonormal basis

$$\vec{e}_i \cdot \vec{e}_j = \delta_{ij} \quad , i, j = 1, 2, 3$$

$i \neq j$: orthogonal
$i = j$: unit norm
$\equiv$ ortho-normal



another orthonormal basis

$$\vec{e}_i' \cdot \vec{e}_j' = \delta_{ij} \quad , \text{ too}$$

is always a linear comb. of $\{\vec{e}_i\}$

i.e. $\vec{e}_j' = \underbrace{O_{ji} \, \vec{e}_i}$

(remember $\sum_{i=1}^{3}$ understood !)

j-th vector of primed basis =

= orthogonal linear comb of vectors of unprimed basis (e_i)

where $\|O\|_{ij} = O_{ij}$ is orthogonal

$O^T = O^{-1}$, in other words , $\det O = 1$ if orientation is preserved, i.e. both bases are r.h. (right-handed)

$$(O^T)_{ij} = O_{ji}$$

so in this notation $O^T \cdot O = \mathbb{1}$ means

$$\underbrace{(O^T)_{ij} O_{je}}_{\left(\sum_{j=1}^{3}\right)} = \delta_{je} \quad \text{or} \quad O_{ji} O_{je} \equiv \delta_{ie}$$

Similarly $\vec{e}_i{}' = O_{ij} \vec{e}_j$

$\Updownarrow$ since $O^T = O^{-1}$

$$\vec{e}_i = (O^T)_{ij} \vec{e}_j{}' \equiv \vec{e}_j{}' O_{ji}$$

Also, it is clear that $O_{ij} \equiv \vec{e}_i{}' \cdot \vec{e}_j$

(follows from $\vec{e}_i{}' = O_{ij} \vec{e}_j \quad / \cdot \vec{e}_k$

$$\vec{e}_i{}' \cdot \vec{e}_k = O_{ij} \vec{e}_j \cdot \vec{e}_k = O_{ij} \delta_{jk} = O_{ik}$$)

An arbitrary vector $\vec{u}$ can be defined by its components in any basis

$$\vec{u} = u^i \vec{e}_i = u^{i\prime} \vec{e}_i{}'$$

but $u^i \vec{e}_i = u^{i\prime} O_{ij} \vec{e}_j \quad / \times \vec{e}_k$

$$u^k = u^{i\prime} O_{ik}$$

N.B. we do not distinguish upper/lower indices !!

components of $\vec{u}$ in unprimed

components of $\vec{u}$ in primed

$\not{c}$ v.v. $\quad u^{i'} \mathcal{O}_{ik} = u_k \;/\; \times (\mathcal{O}^T)_{k\ell}$

$\sum_\ell$

$$u^{i'} \underbrace{\mathcal{O}_{ik} (\mathcal{O}^T)_{k\ell}} = u_k (\mathcal{O}^T)_{k\ell}$$

$$u^{i'} \delta_{i\ell} = u_k (\mathcal{O}^T)_{k\ell}$$

$$\shortparallel$$

$$u^{\ell'} = u_k (\mathcal{O}^T)_{k\ell} = \mathcal{O}_{\ell k} u_k$$

(the inverse of $\quad u_k = u^{i'} \mathcal{O}_{ik}$)

A vector $\vec{u}$ or $\{ u^i \}$ is a 1-index tensor.

An n-index tensor is defined by its components in an arbitrary coordinate system: $\quad \{ t^{i_1 \cdots i_n} \}$

Examples w/ $n = 2$: $\quad u^i u^j$ is a 2-index (2nd rank) tensor

(σ^{ij} is, too)

$n = 3$: $\quad \epsilon^{ijk}$ is a 3-index tensor (totally antisymmetric)

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A tensor's components transform like products of vectors upon changes of basis

i.e.
$$u^i u^j = u^{k'} u^{l'} \mathcal{O}_{ki} \mathcal{O}_{lj}$$

$$\sigma^{ij} = \sigma^{kl'} \mathcal{O}_{ki} \mathcal{O}_{lj}$$

components in unprimed frame

components of stress tensor in primed frame

$$\sigma^{ij\,\prime} = \mathcal{O}_{ik} \mathcal{O}_{jl} \sigma_{kl}$$

thinking of σ^{ij} as a 3x3 m-x w/ components σ^{ij} we can write

$$\sigma^{ij\,\prime} = \mathcal{O}_{ik} \sigma_{kl} \mathcal{O}_{jl}$$
$$= \mathcal{O}_{ik} \sigma_{kl} (\mathcal{O}^T)_{lj} \qquad (\mathcal{O}_{jl} = (\mathcal{O}^T)_{lj})$$

or $$\sigma' = \mathcal{O} \sigma \mathcal{O}^T$$

in matrix notation

or $$\sigma = \mathcal{O}^T \sigma' \mathcal{O}$$ useful!!